Markov Decision Process Routing Games

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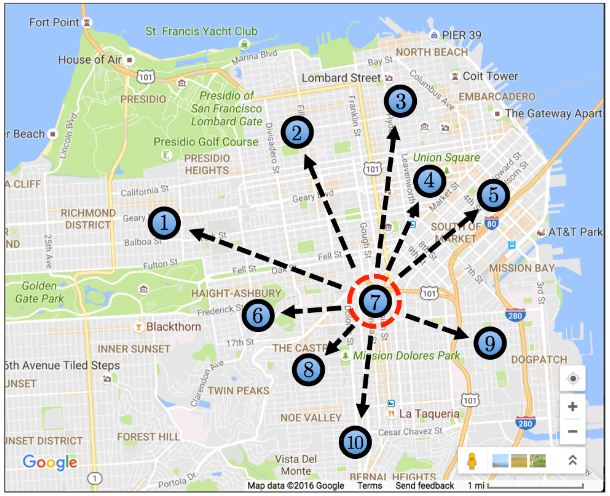
Oct 17, 2017

Competition in Smart Cities

Outline

- Review: Non-atomic routing games
- Cyclic routing games
- Markov decision process (MDP) routing game
- Example: ride-sharing game, street parking



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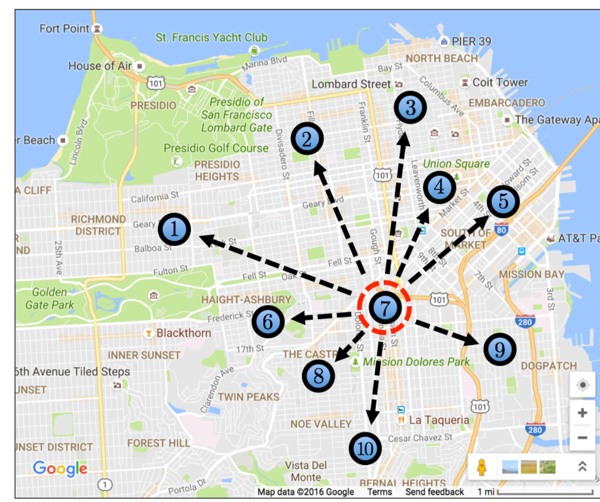
Literature

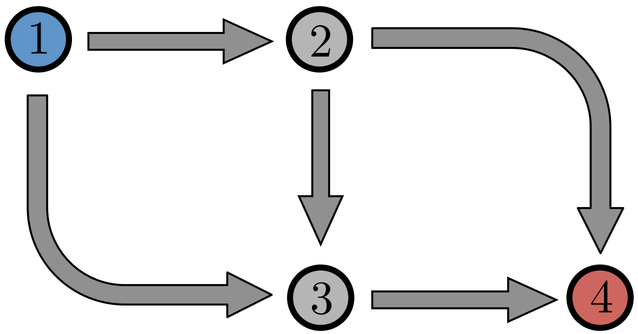
Continuous Population Stochastic Games -- infinitesimal agents solve an MDP

- **Anonymous sequential games**
  [Jovanovic & Rosenthal, 88], [Bergin, et al. 91,95], [Wiecek, et al. 09,11,15]
- **Mean-field games**
  [Lasry & Lions, 06,07],[Caines, 2015], [Gomes, et al. 10],[Gueant, 11]



Classic Routing Game

$\mathcal{G} = (\mathcal{N}, \mathcal{E})$



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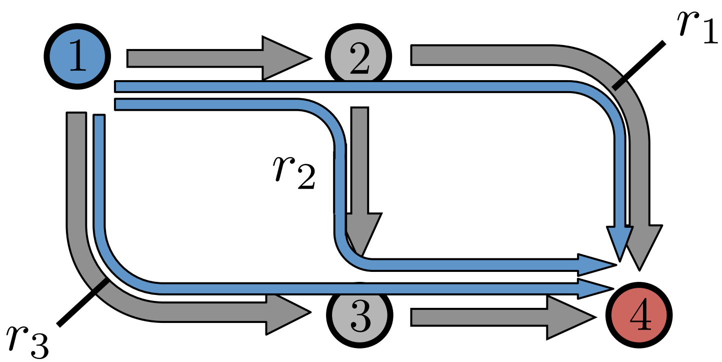
$\mathcal{G} = (\mathcal{N}, \mathcal{E})$ $\mathcal{R}$: routes



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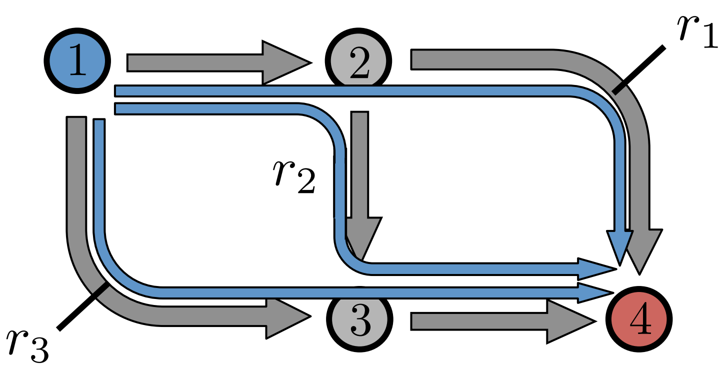

$$x = \mathbf{R}z$$

$z \in \mathbb{R}_+^{|\mathcal{R}|}$: Mass on routes

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$$\mathbf{R} = \begin{array}{cc} & \text{routes} \\ & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{edges} \end{array}$$

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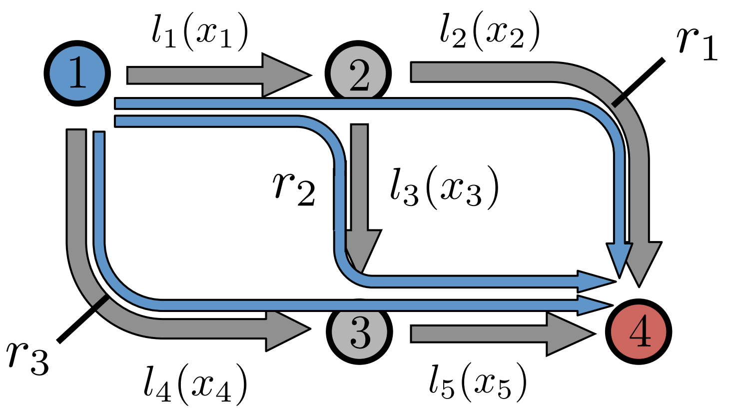

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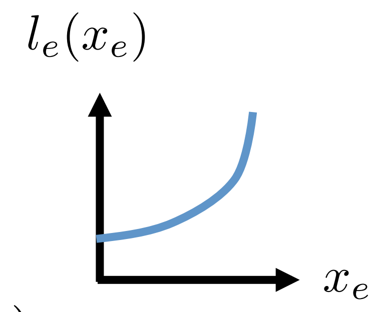

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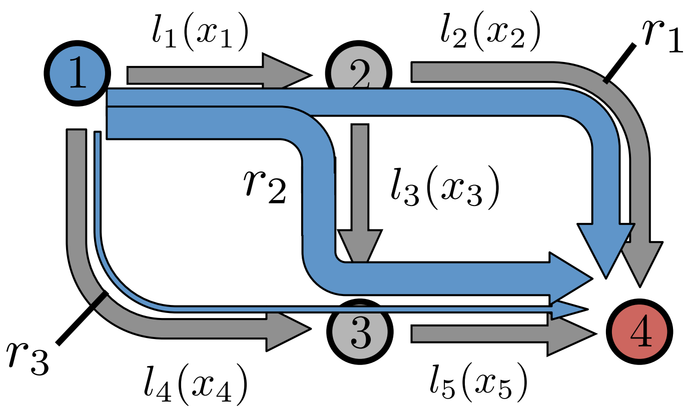

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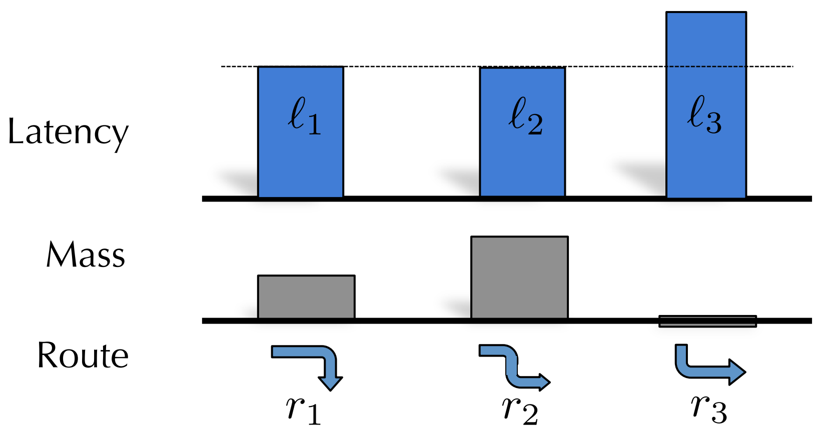

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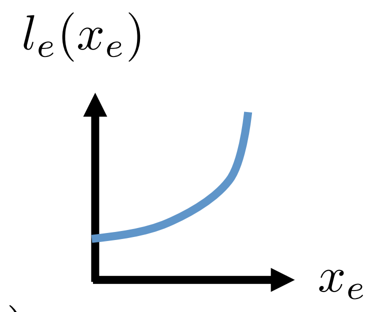

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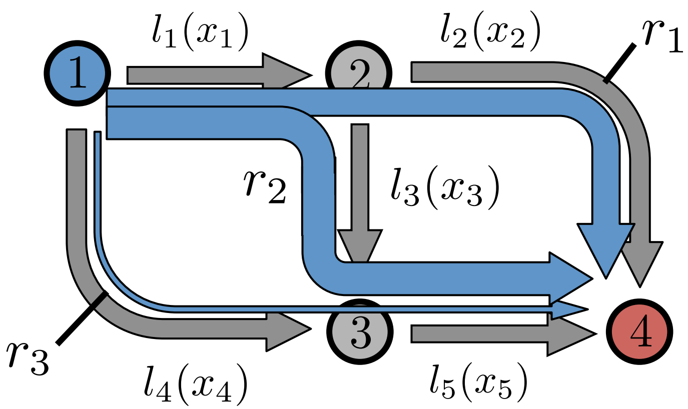

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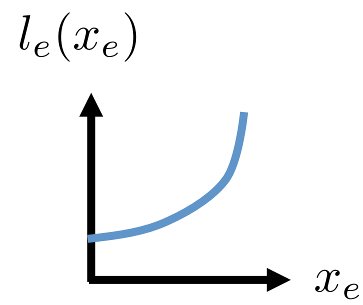

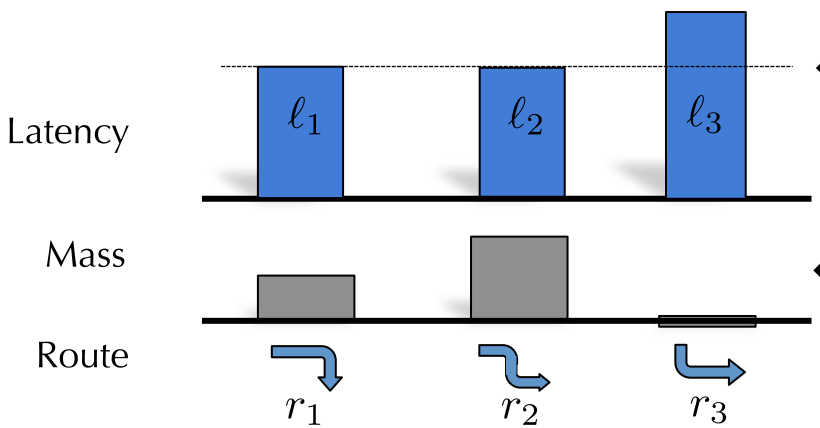

Equilibrium Condition

Wardrop Equilibrium

[Wardrop, 52]

[Sandholm, 01]

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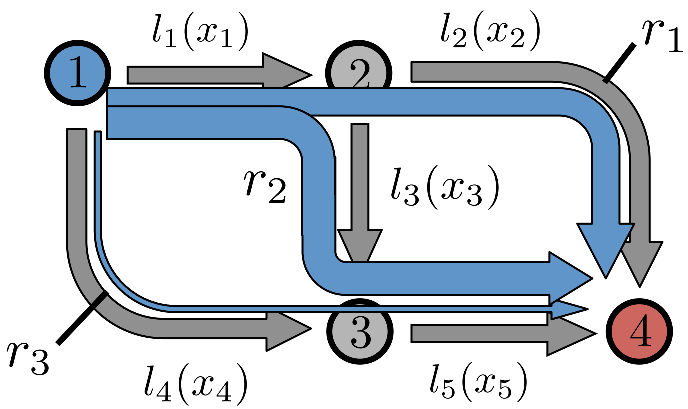

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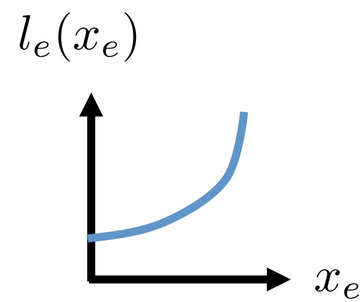
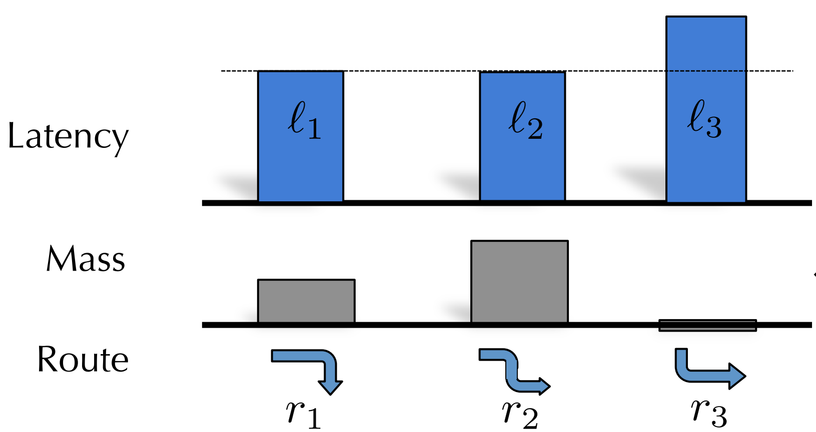


Potential Function $F(x) = \sum_e \int_0^{x_e} l_e(u)\, du$

[Beckmann, et al. 56]



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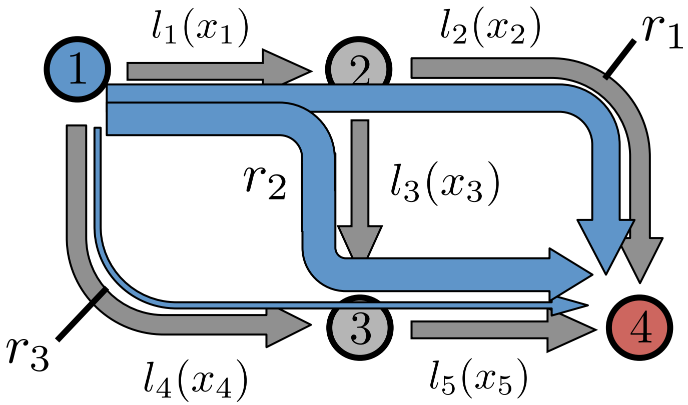

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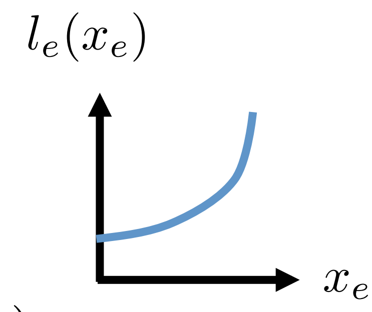

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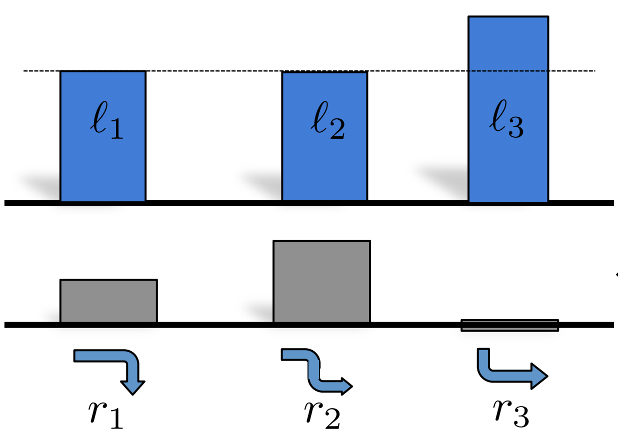
Latency

Mass

Route



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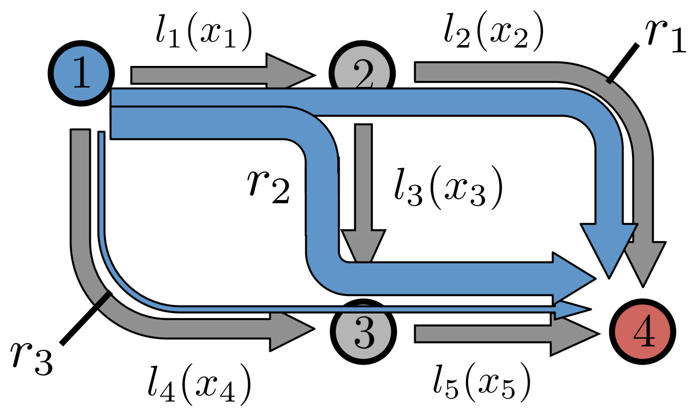

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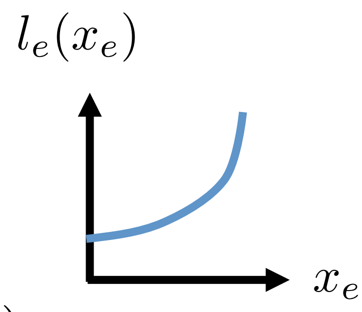

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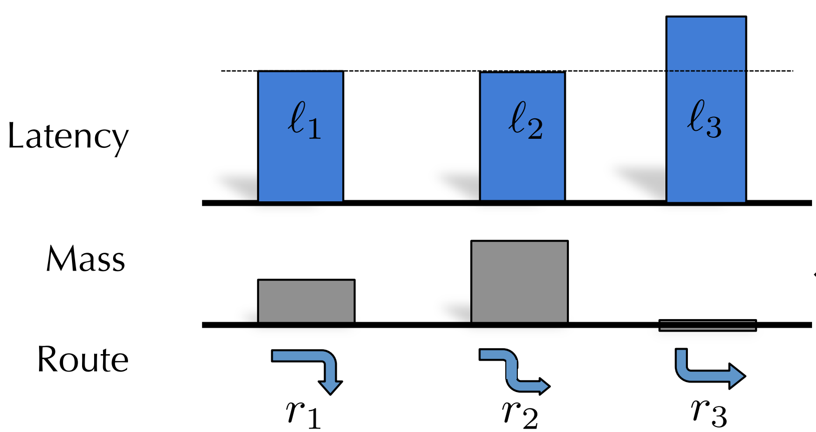

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Social Cost $\quad J(x) = \sum_e x_e l_e(x_e)$

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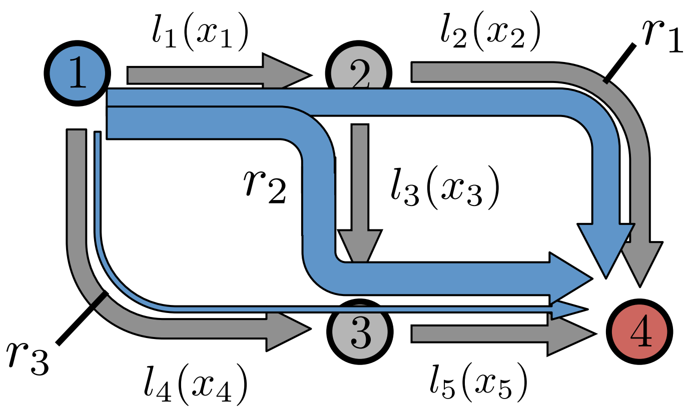

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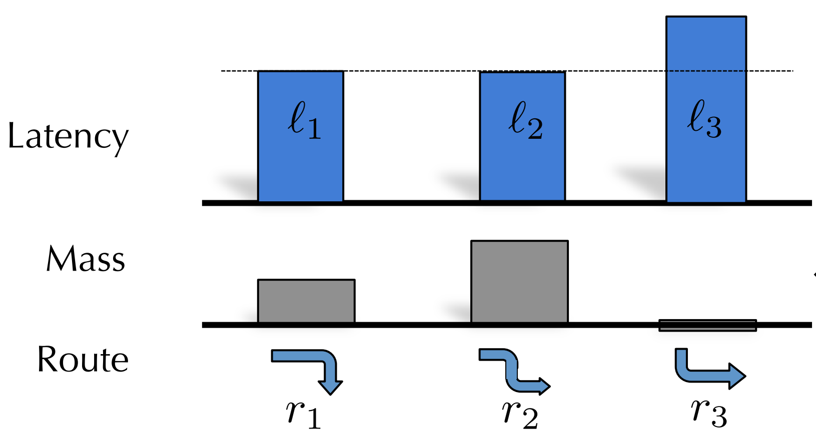

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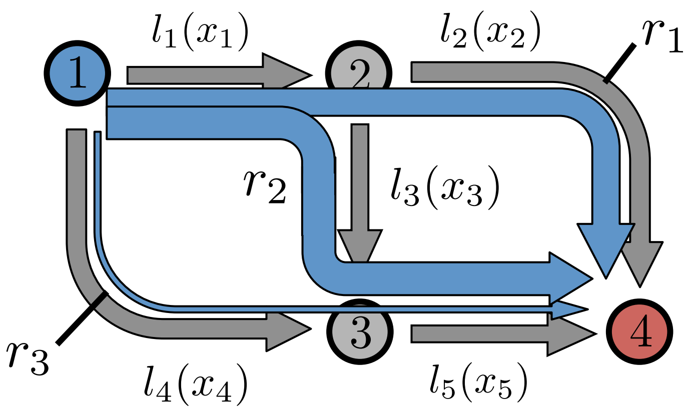

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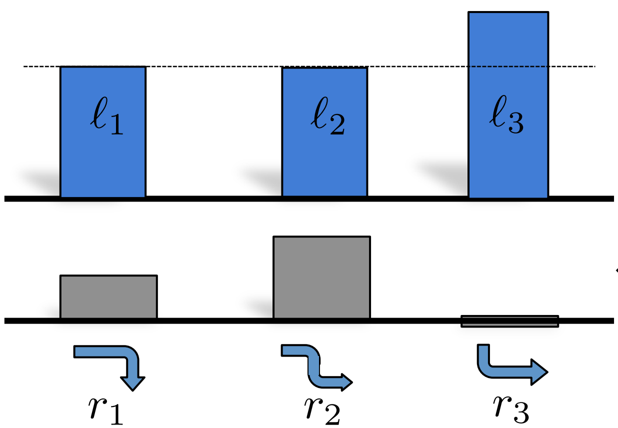

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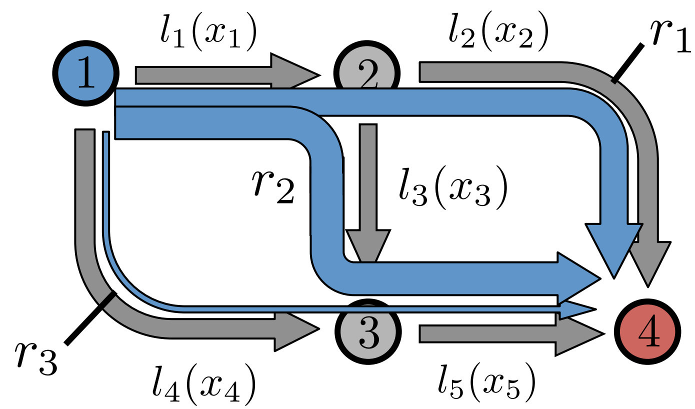

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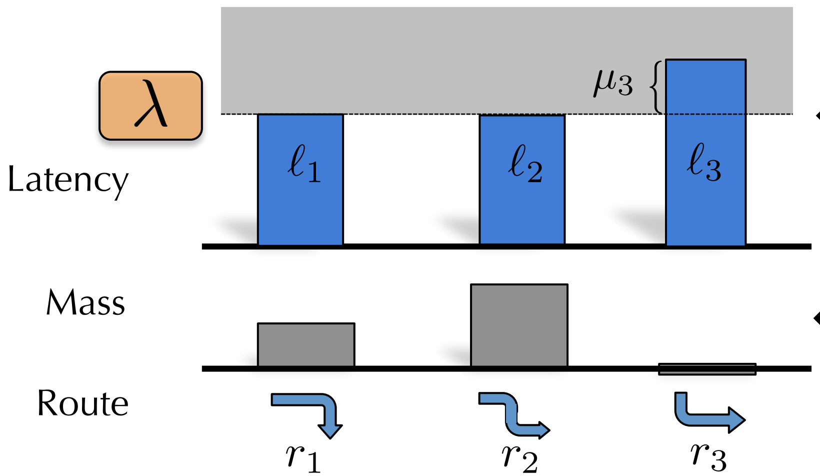

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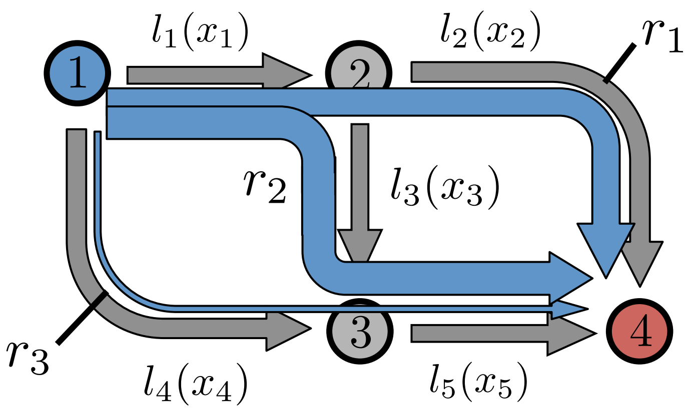

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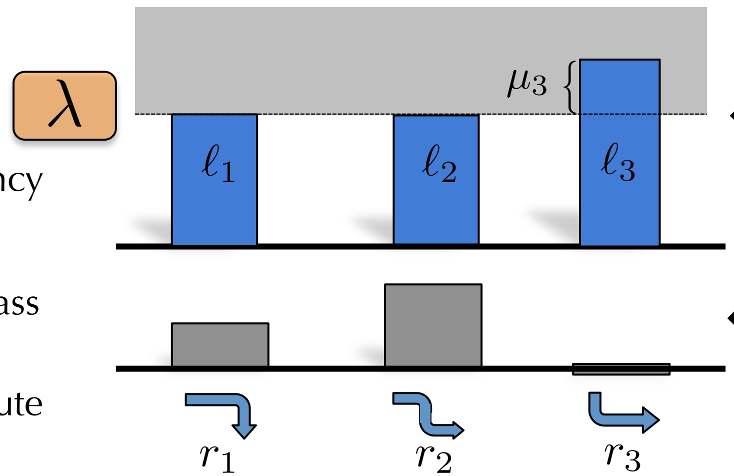

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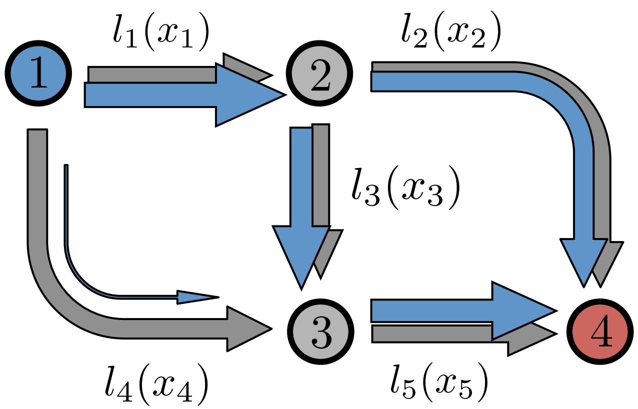

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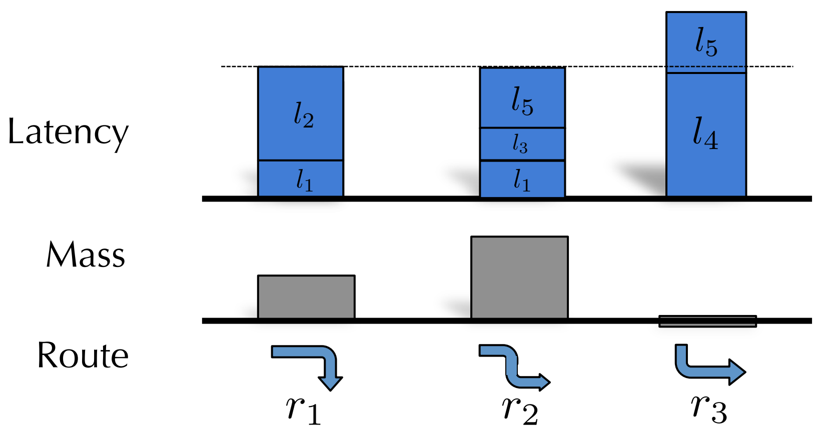

Edge Formulation

$$\min_x \quad F(x)$$
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$$\text{nodes} \quad \underbrace{\overset{\text{edges}}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix}}}_{G} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{S} m$$

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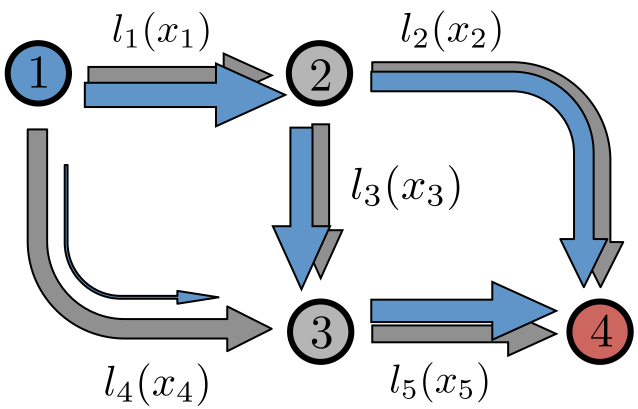

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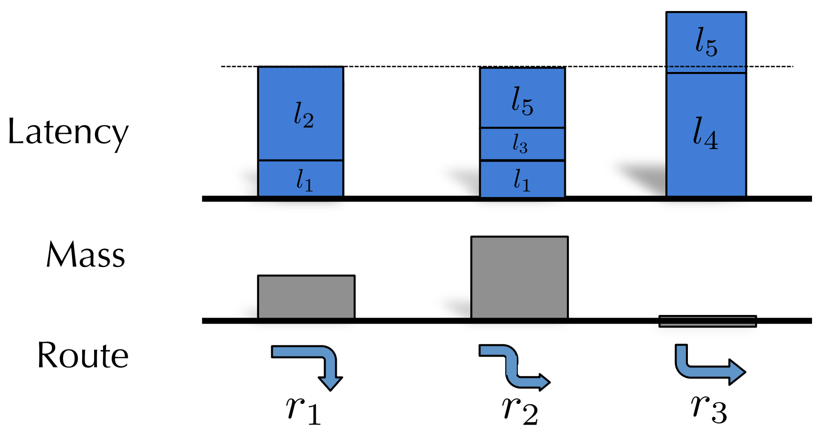

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$$G = I_o - I_i$$

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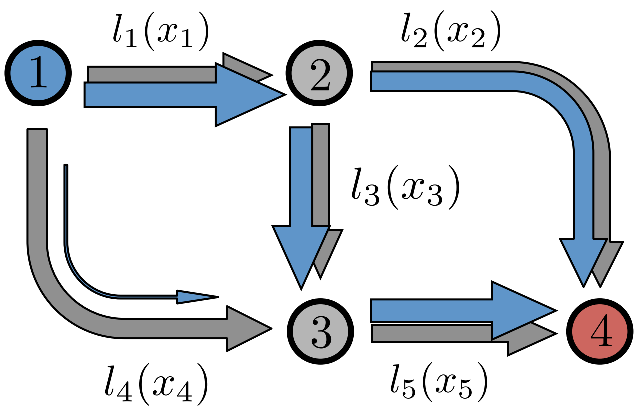

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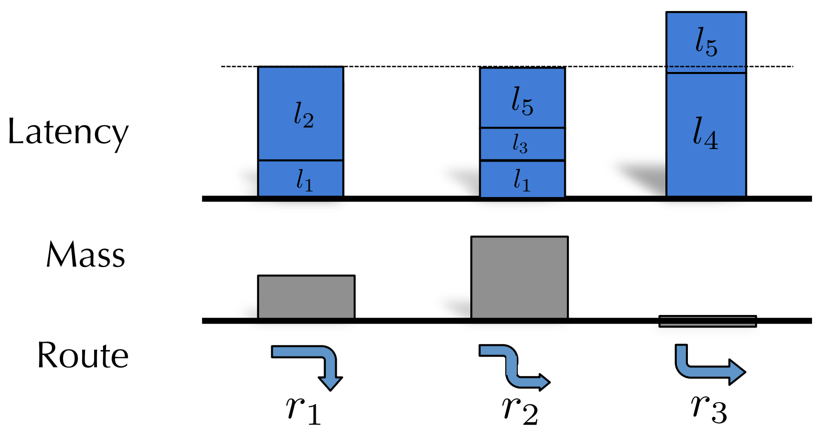

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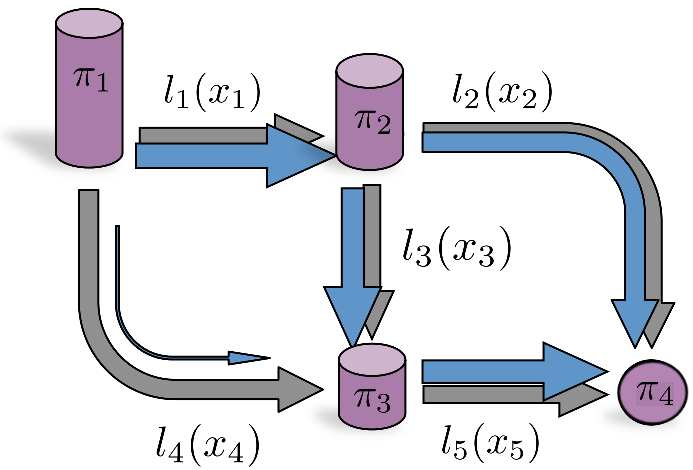
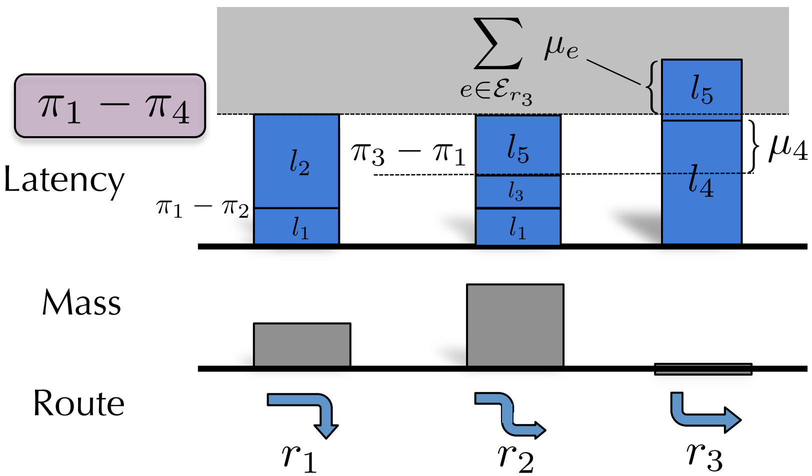


First order conditions… $l_e(x_e) = \pi_j - \pi_i + \mu_e$



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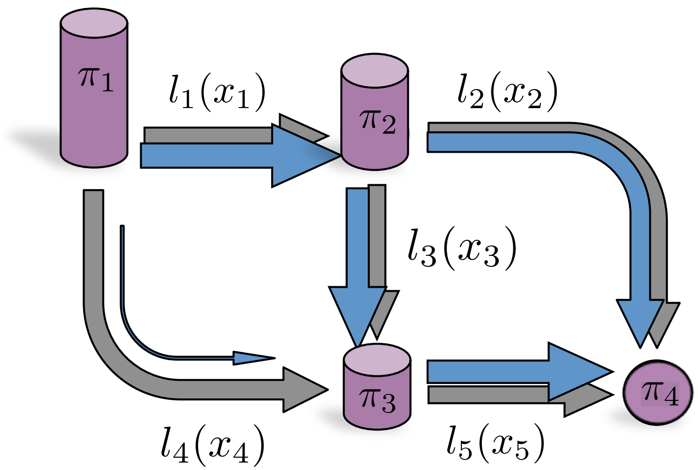
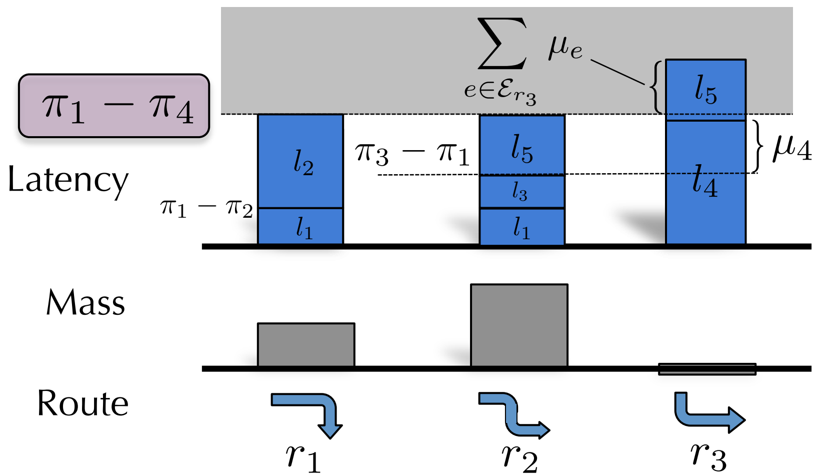
$$j \xrightarrow{e} i$$

Classic Routing Game



First order conditions… $l_e(x_e) = \pi_j - \pi_i + \mu_e$

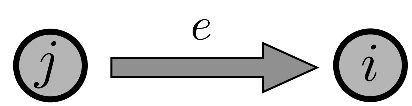
Sum up over edges in route r … $\sum_{e\in\mathcal{E}_r} l_e(x_e) = \pi_o - \pi_d + \sum_{e\in\mathcal{E}_r} \mu_e$



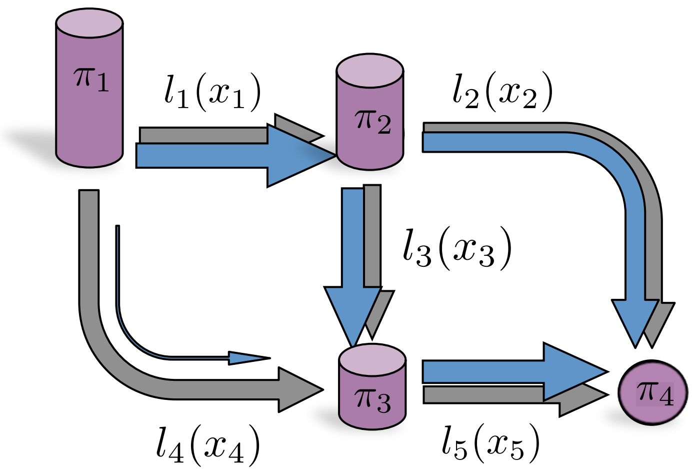
Edge Formulation

$$\min_x \quad F(x)$$
$$\text{s.t.} \quad \underbrace{Gx = Sm,}_{\pi} \quad \underbrace{x \geq 0}_{\mu}$$

$$\begin{array}{c c}
 & \text{edges} \\
\text{nodes} & \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix}}_{G}
\end{array}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{S} m$$



Classic Routing Game



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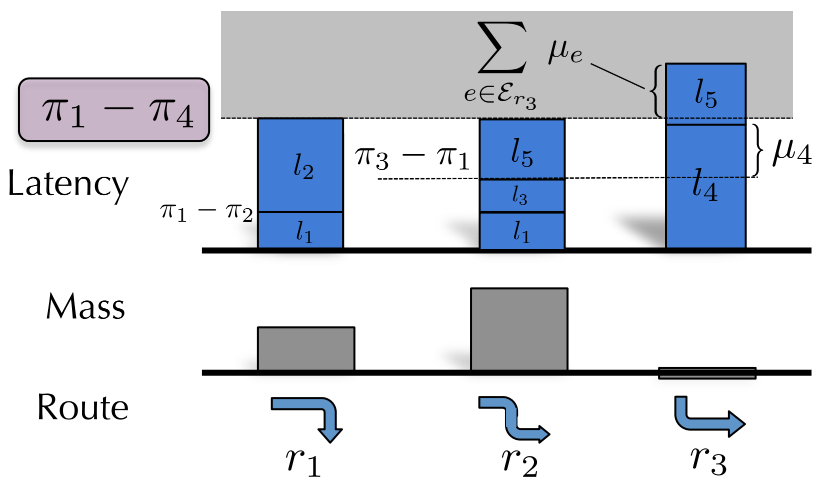

Edge Formulation

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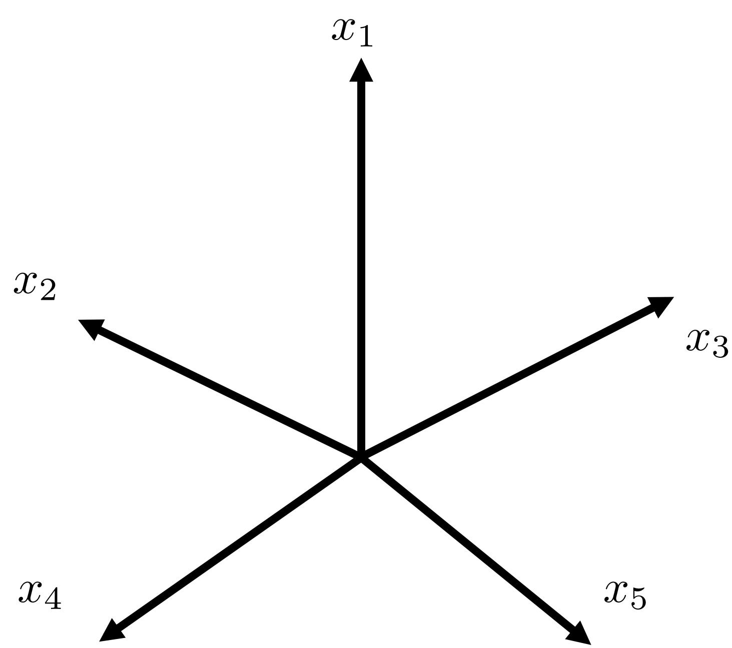
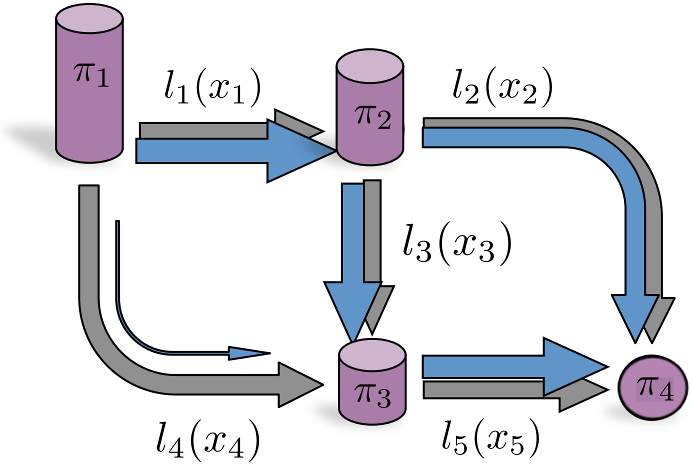


Classic Routing Game



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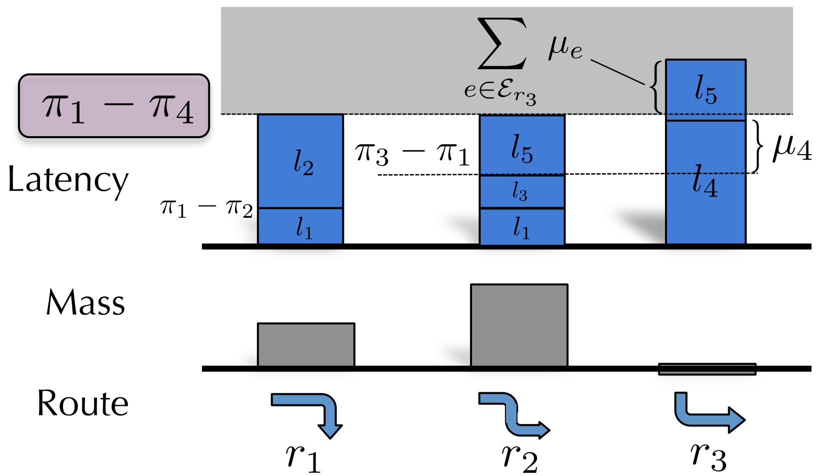

Edge Formulation

$$\min_{x} \quad F(x)$$

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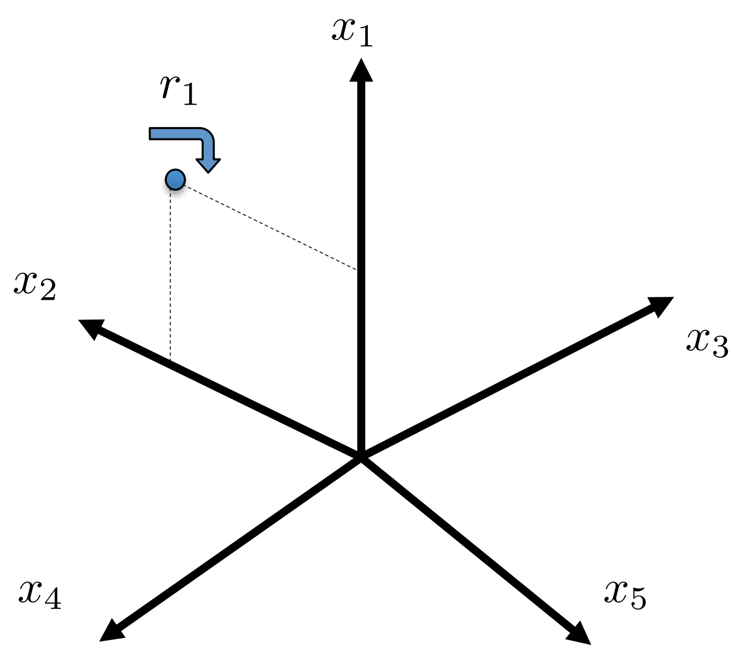
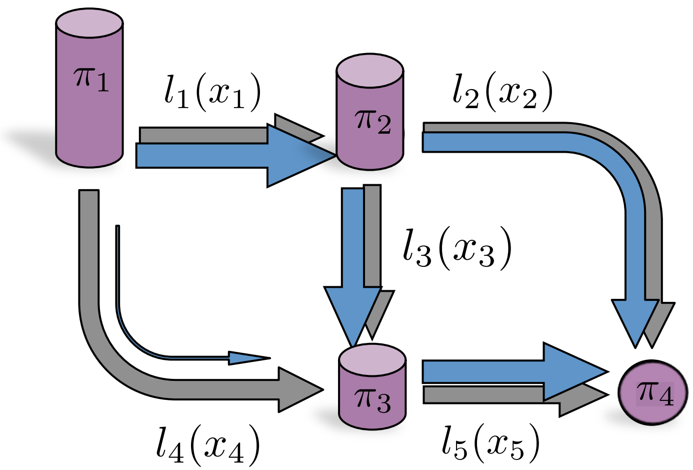


Classic Routing Game



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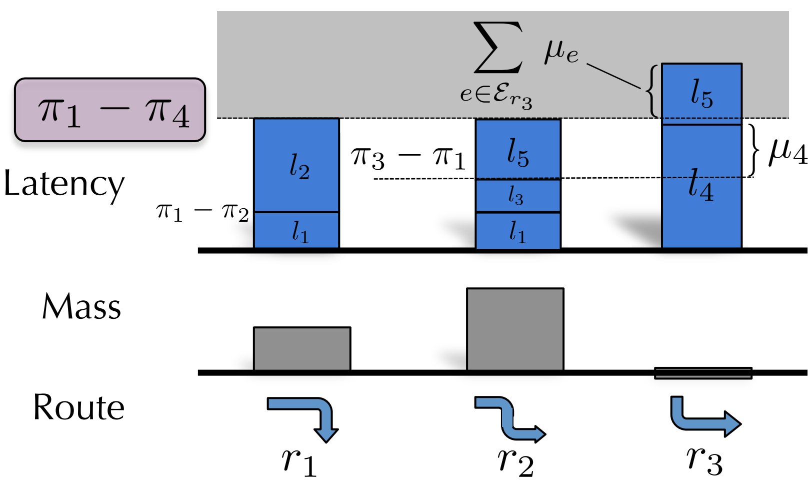

Edge Formulation

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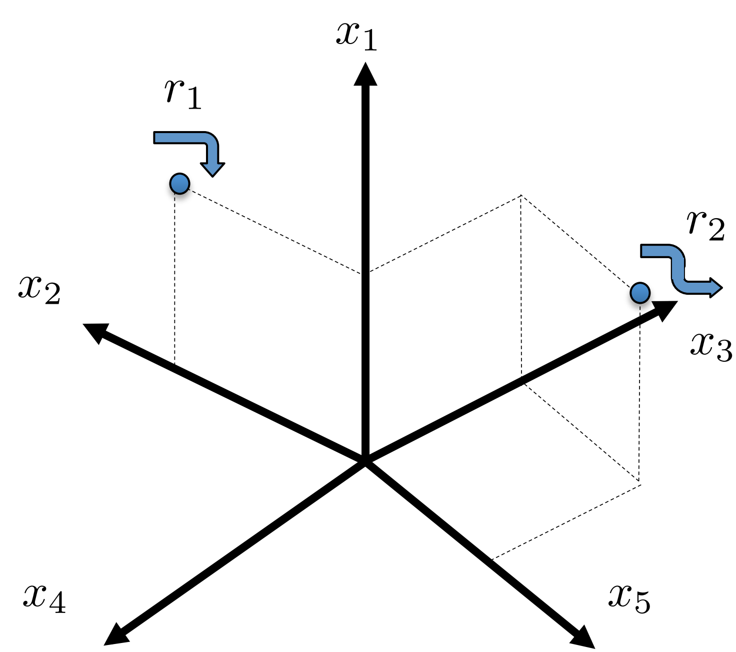
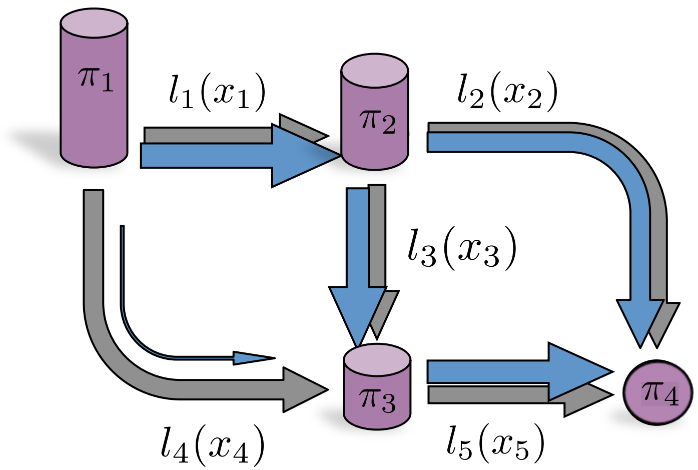


Classic Routing Game



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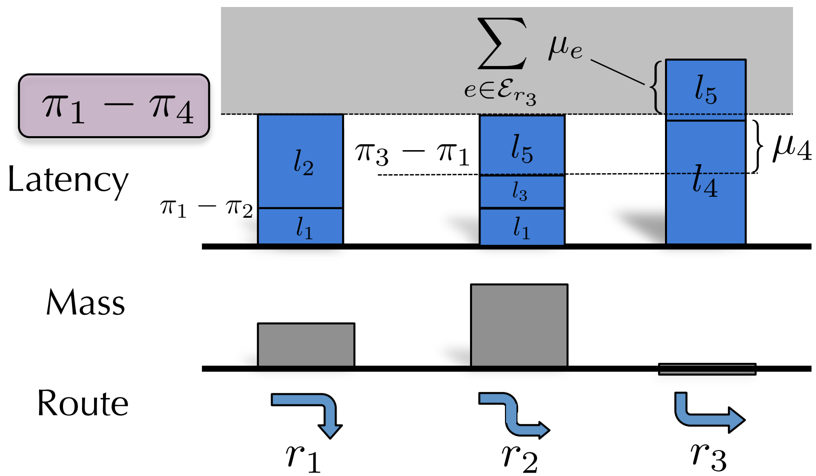

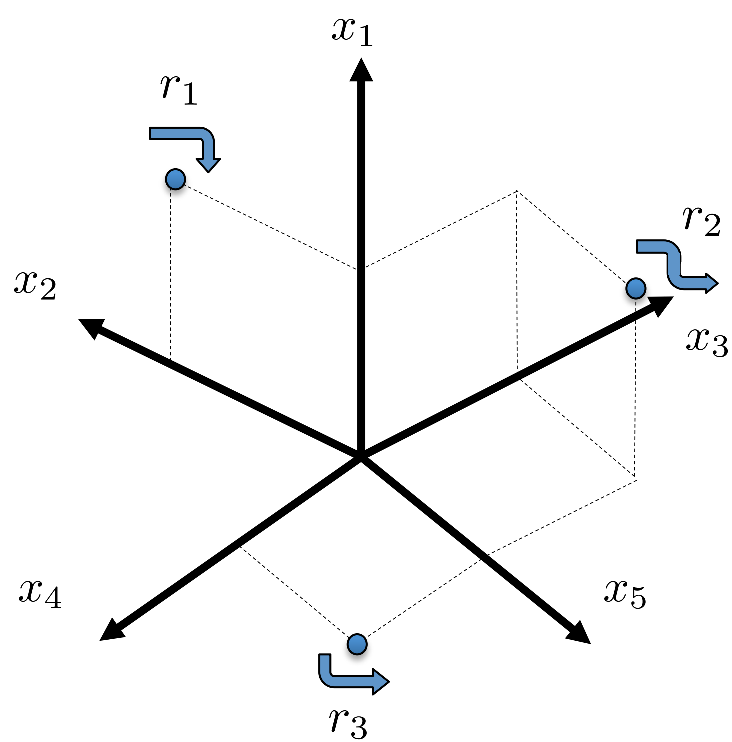

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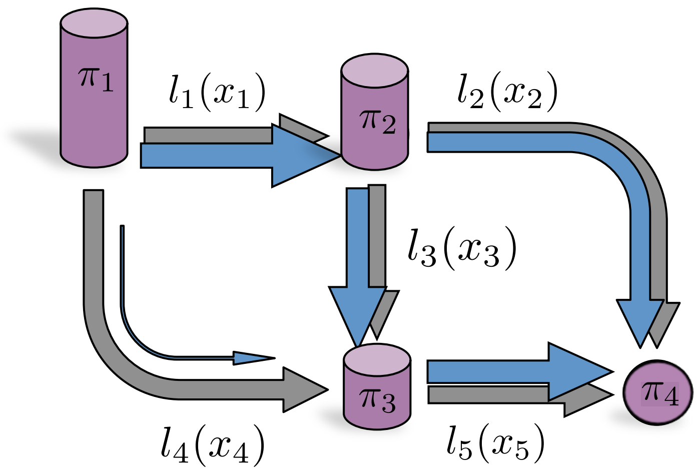

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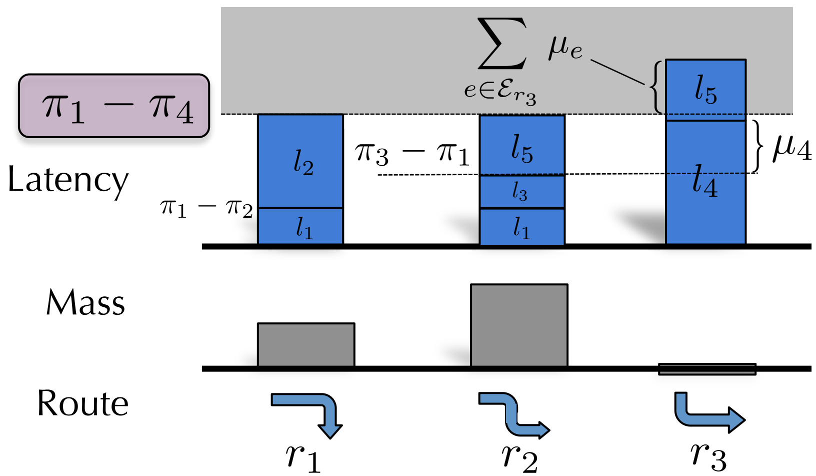

Edge Formulation

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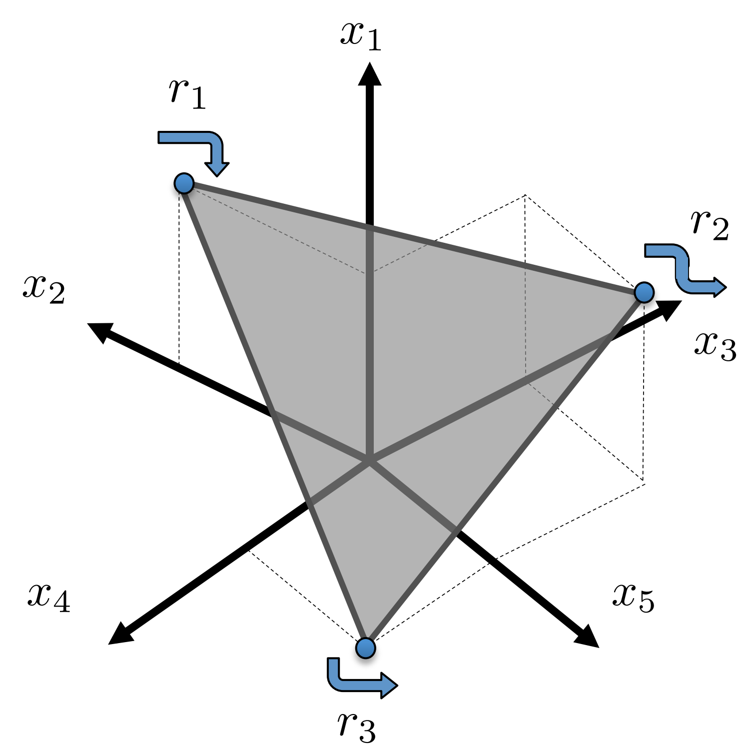
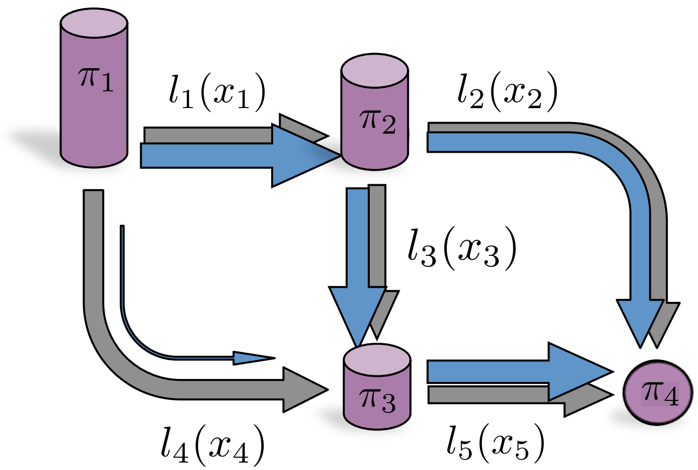
$$\text{s.t.} \quad \boxed{Gx = Sm, \;\; \pi} \quad \boxed{x \geq 0 \;\; \mu}$$



Classic Routing Game



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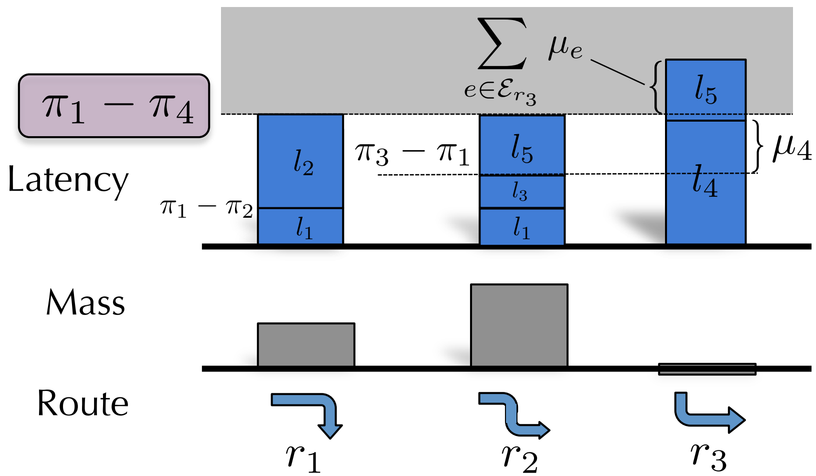

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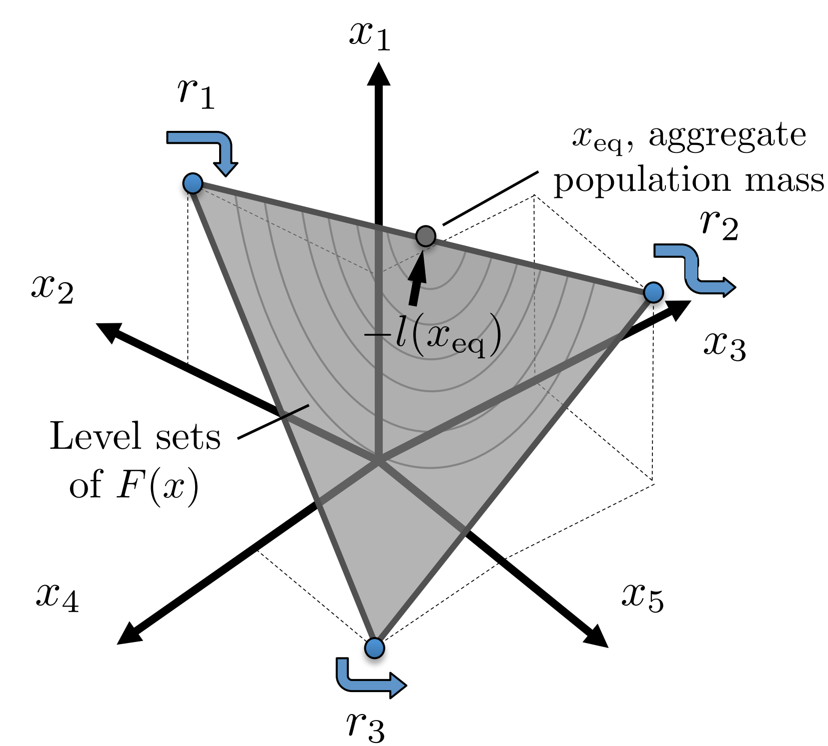
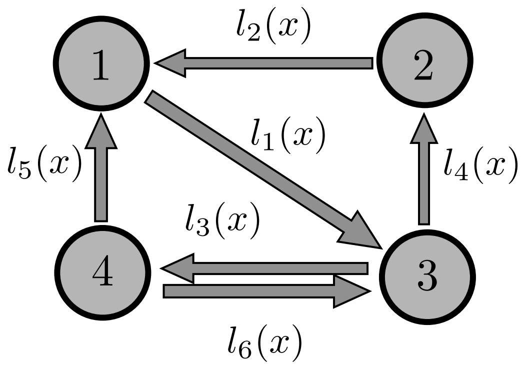


Cyclic Routing Game



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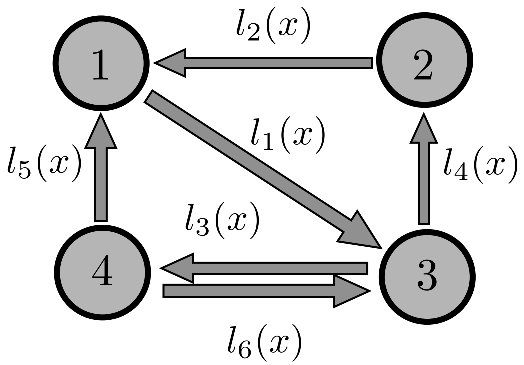
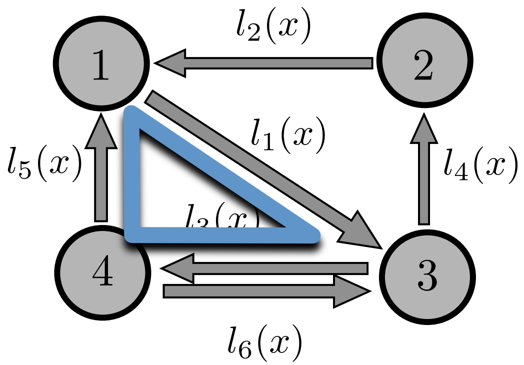


Cyclic Routing Game (Edge Formulation)

$$\min_{x} \quad F(x)$$
$$\text{s.t.} \quad \underbrace{Gx = 0,}_{\pi} \quad \underbrace{\mathbf{1}^T x = m,}_{\lambda} \quad \underbrace{x \geq 0}_{\mu}$$

Cyclic Routing Game



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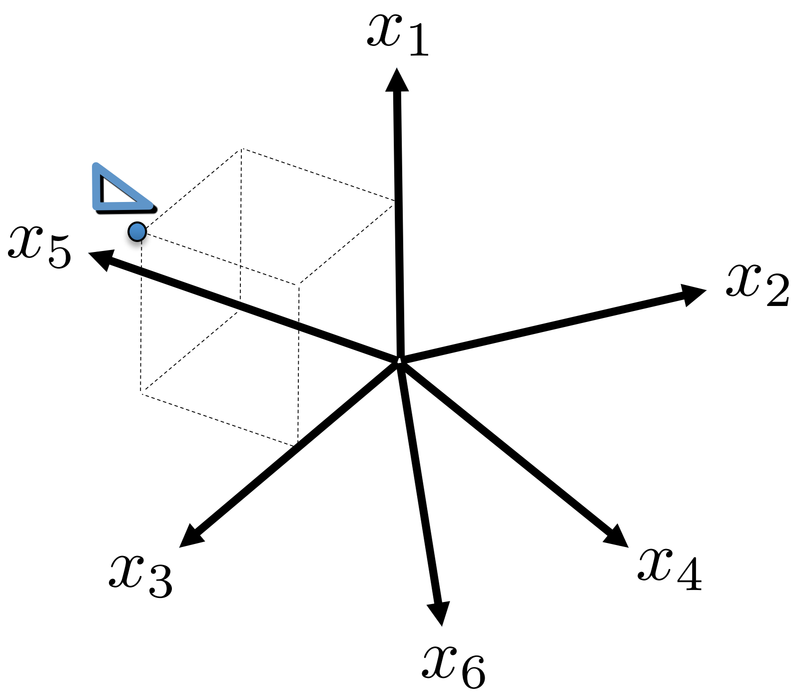
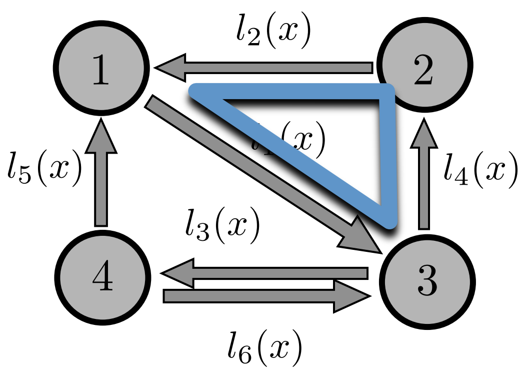


Cyclic Routing Game



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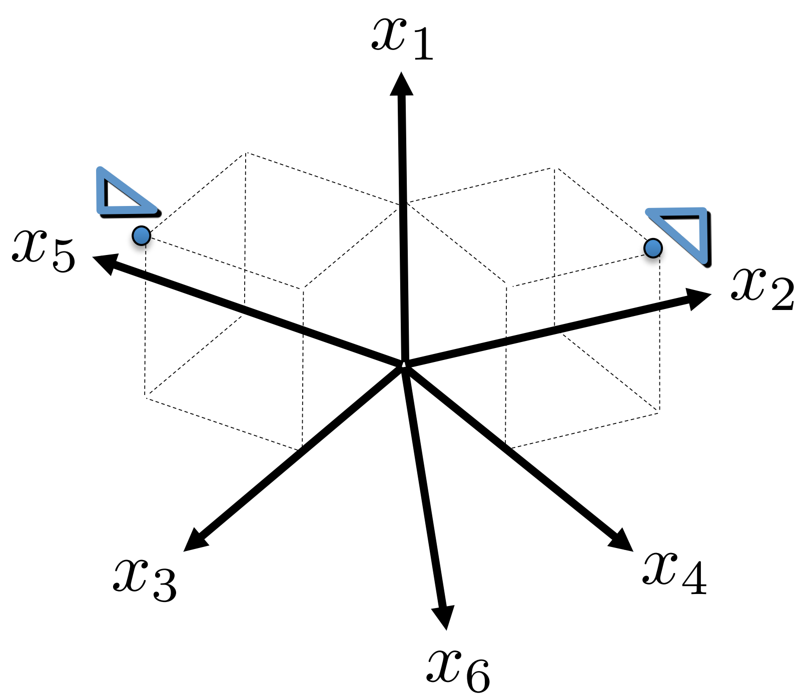
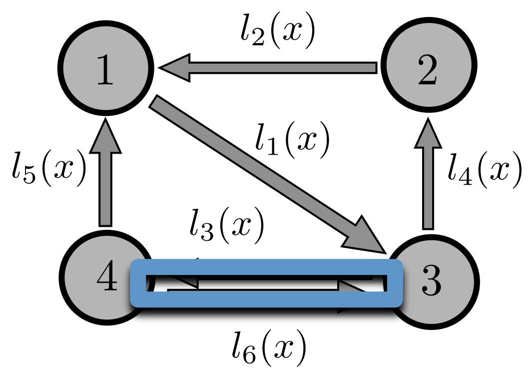


Cyclic Routing Game



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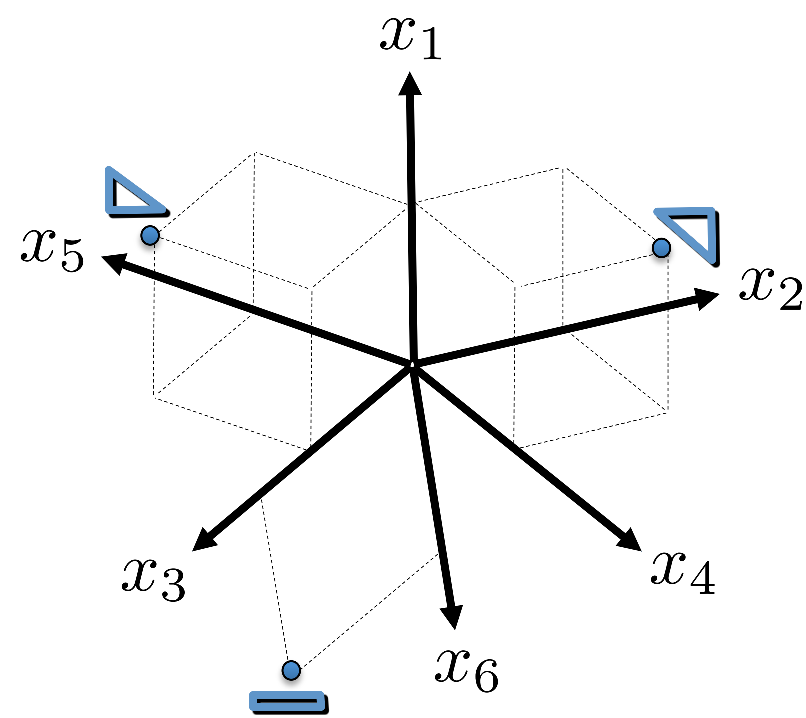

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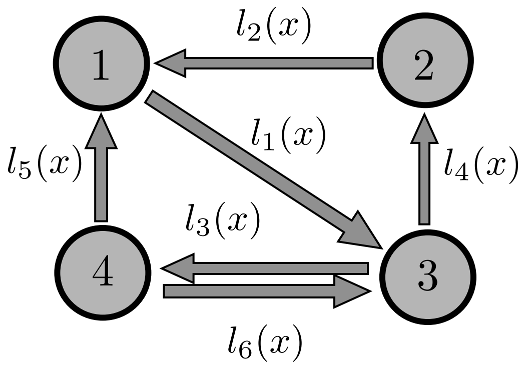

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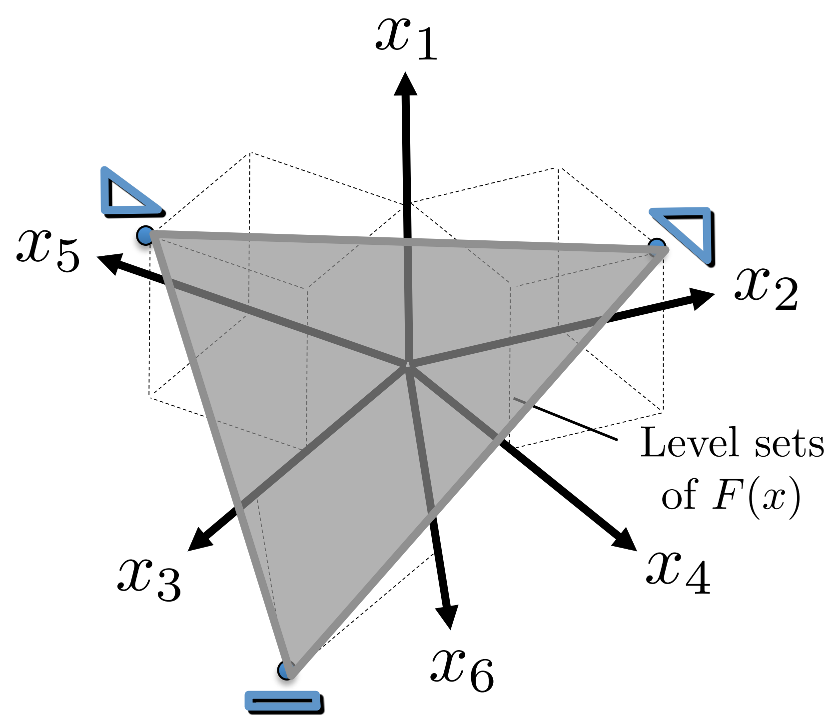
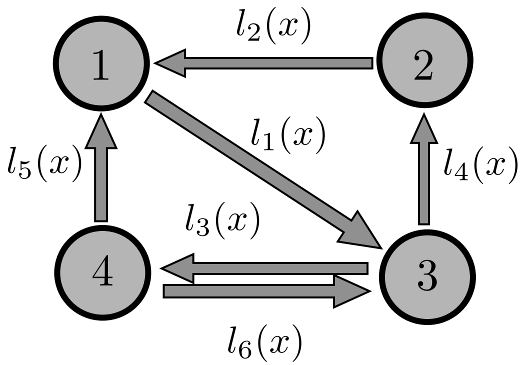


Cyclic Routing Game



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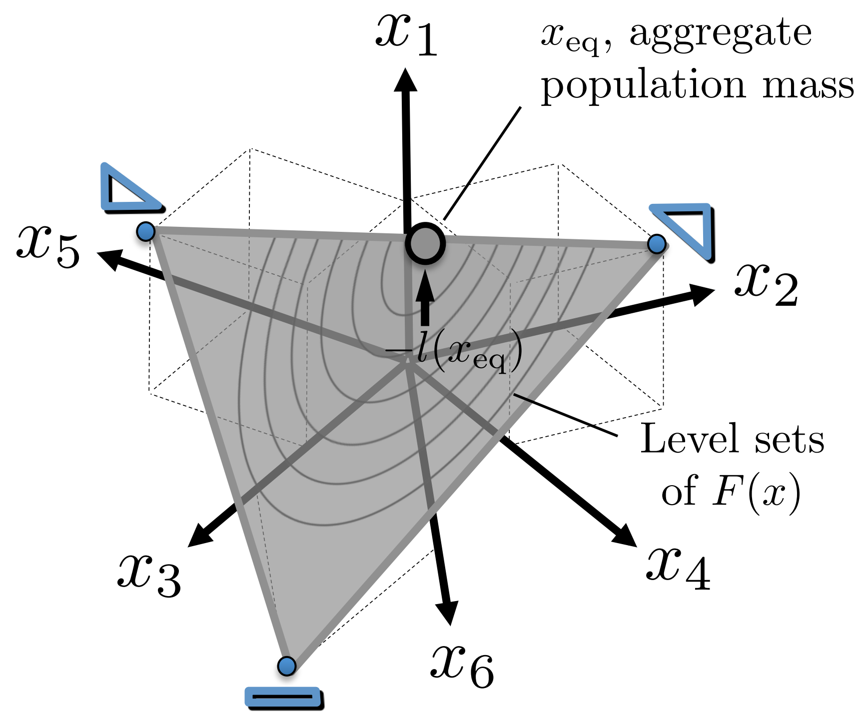

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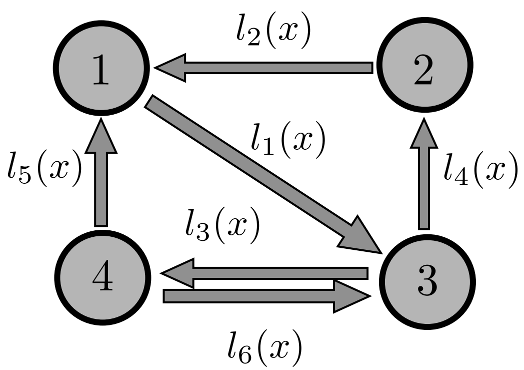

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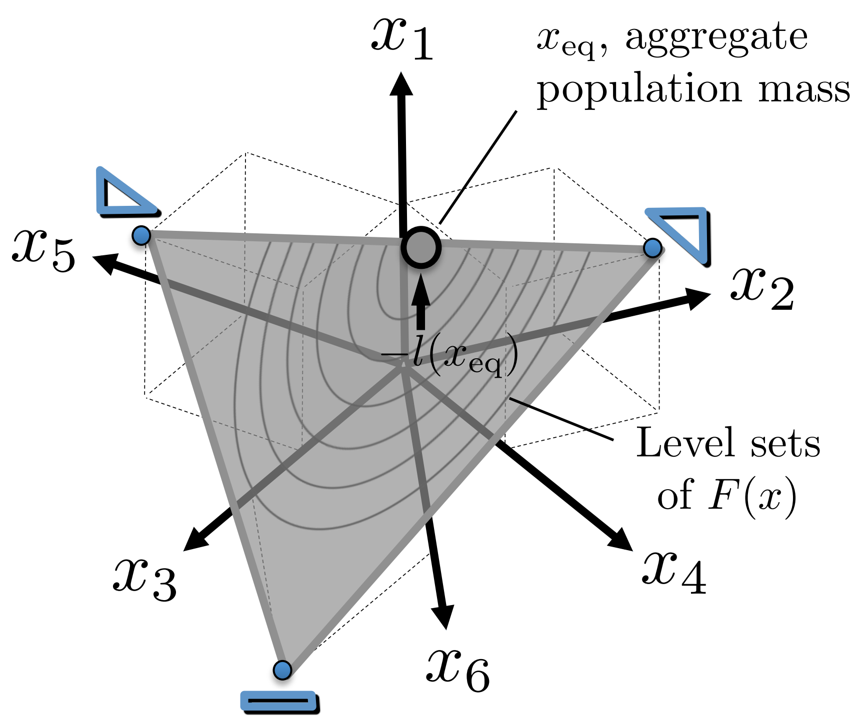

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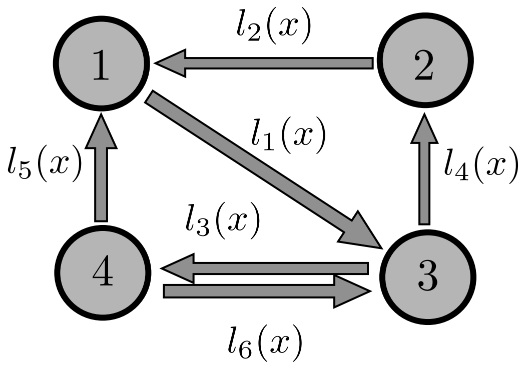

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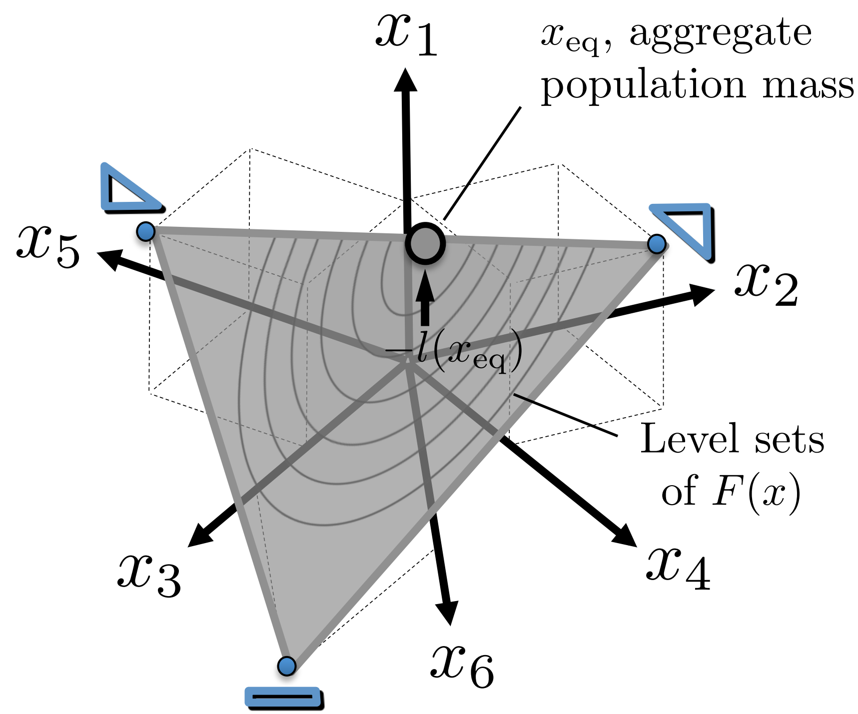
Sum up over edges in cycle c ... $\displaystyle\sum_{e \in \mathcal{E}_c} l_e(x_e) = |\mathcal{E}_c|\lambda + \sum_{e \in \mathcal{E}_c} \mu_e$



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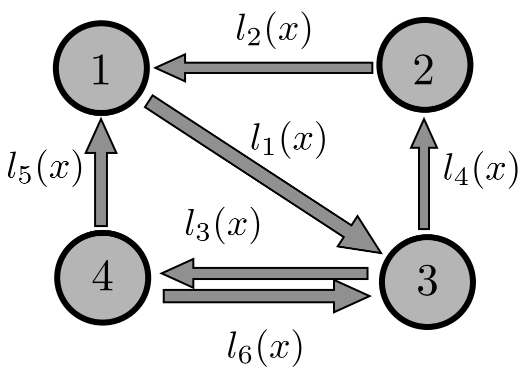

Cyclic Routing Game (Edge Formulation)

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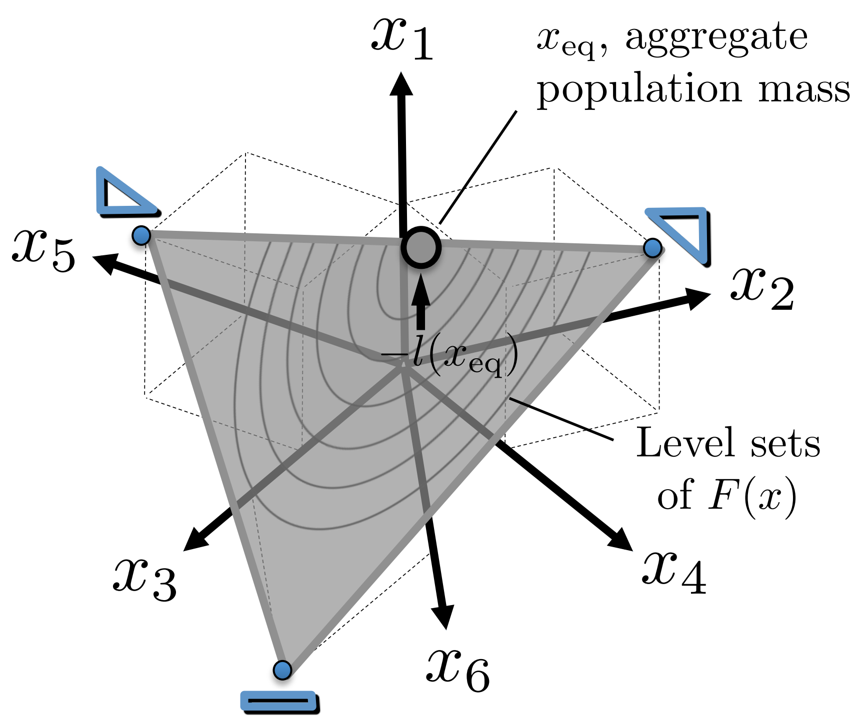
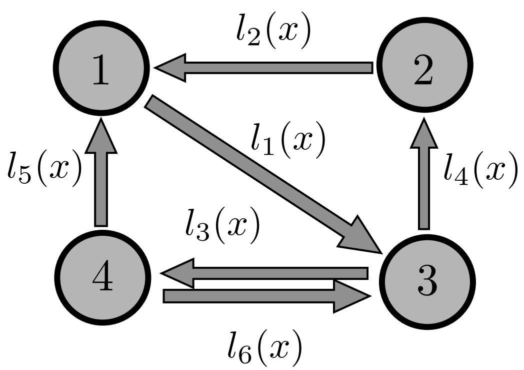
$$\text{s.t.} \quad \boxed{Gx = 0,}_{\ \pi} \quad \boxed{\mathbf{1}^T x = m,}_{\ \lambda} \quad \boxed{x \geq 0}_{\ \mu}$$

First order conditions... $l_e(x_e) = \pi_j - \pi_i + \lambda + \mu_e$

Sum up over edges in cycle c ... $\frac{1}{|\mathcal{E}_c|} \sum_{e \in \mathcal{E}_c} l_e(x_e) = \lambda + \frac{1}{|\mathcal{E}_c|} \sum_{e \in \mathcal{E}_c} \mu_e$



Cyclic Routing Game



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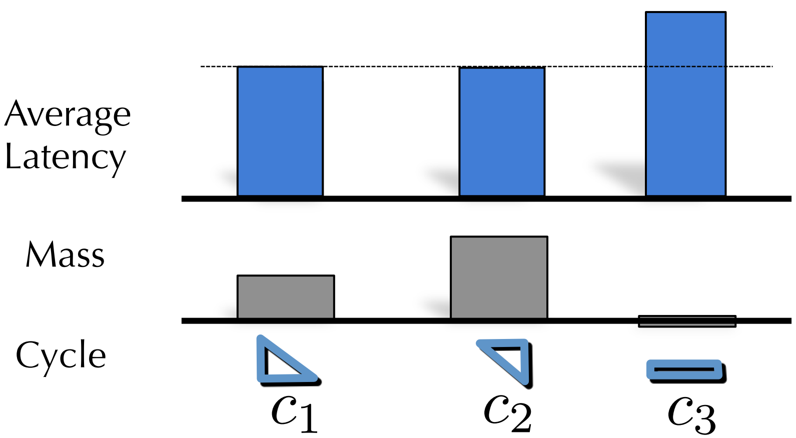

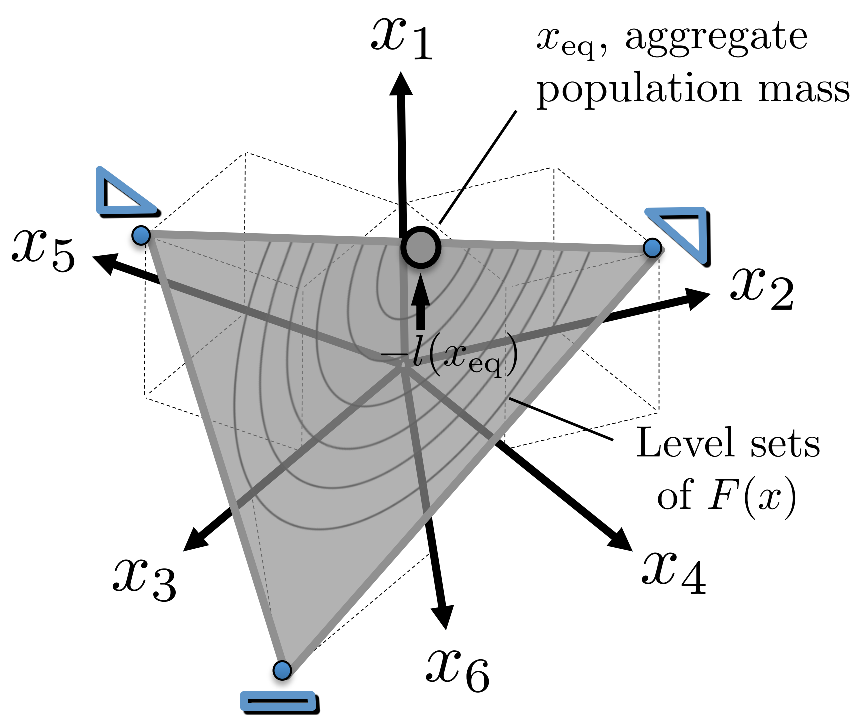

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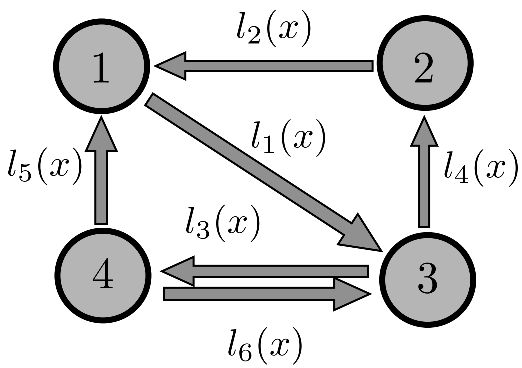

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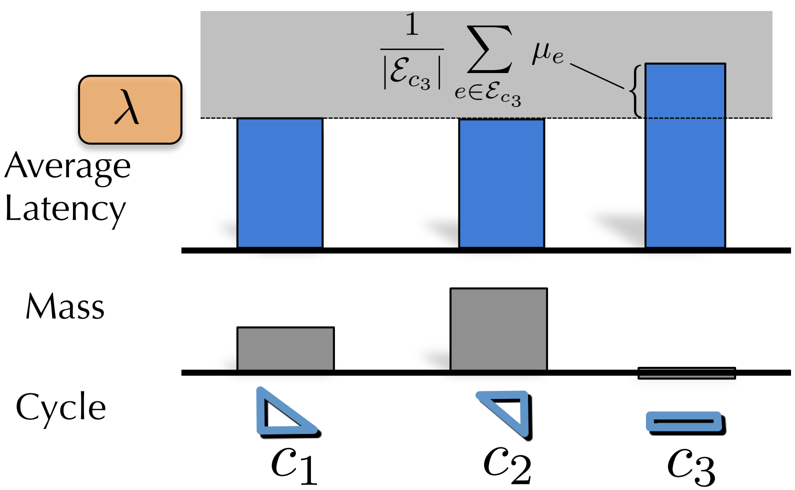

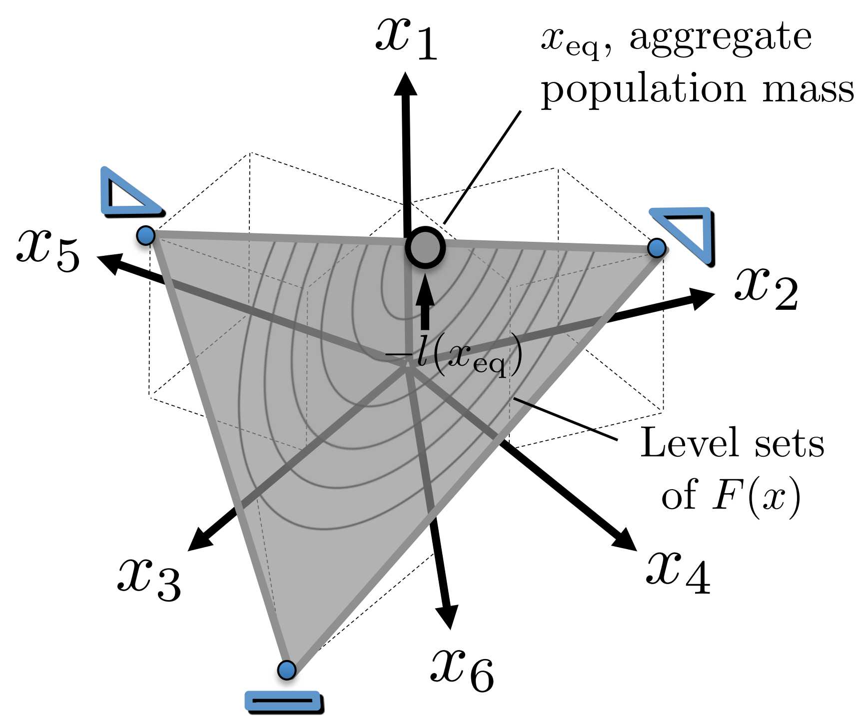

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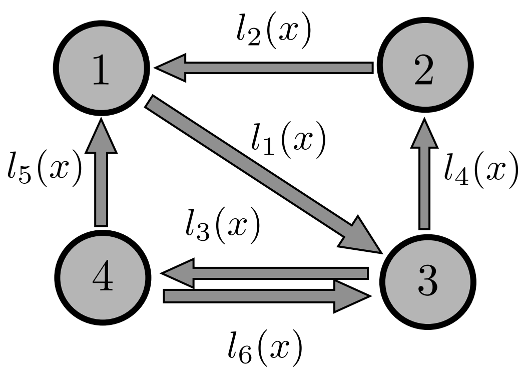


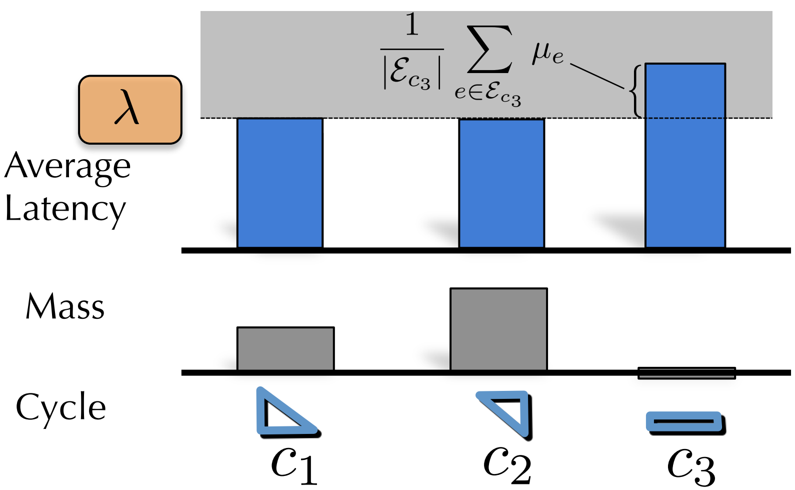
Cyclic Routing Game (Route Formulation)

$$\min_{z} \quad F(x) = F(\mathbf{C}z)$$

s.t. $\quad \mathbf{1}^T z = m \;\; (\lambda) \qquad z \geq 0 \;\; (\mu)$

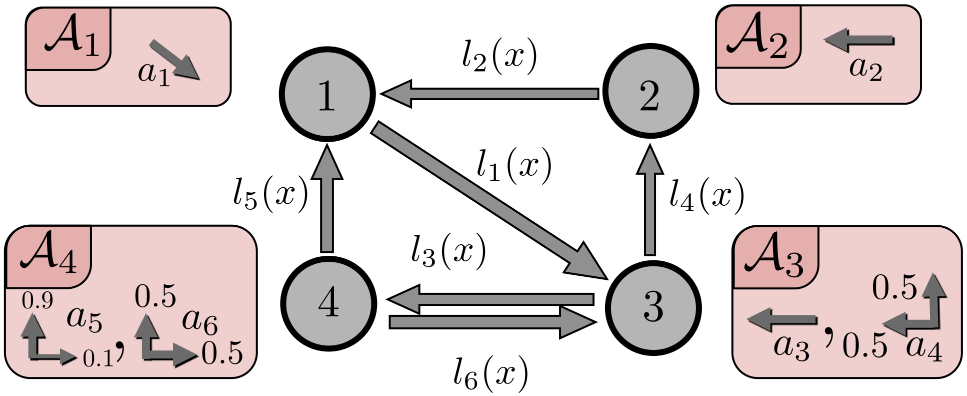
First order conditions… $\quad l_e(x_e) = \pi_j - \pi_i + \lambda + \mu_e$

Sum up over edges in cycle c … $\quad \frac{1}{|\mathcal{E}_c|}\sum_{e\in\mathcal{E}_c} l_e(x_e) = \lambda + \frac{1}{|\mathcal{E}_c|}\sum_{e\in\mathcal{E}_c} \mu_e$



$$\begin{array}{cc} & \text{cycles} \\ \mathbf{C} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 1/3 & 0 & 1/2 \\ 0 & 1/3 & 0 \\ 1/3 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} & \text{edges} \end{array}$$

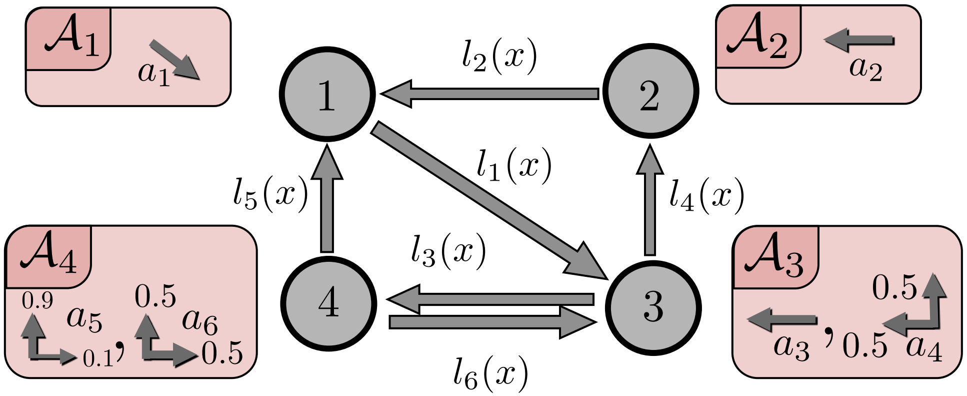
MDP Routing Game



Cyclic Routing Game (Edge Formulation)

$$\min_{x} \quad F(x)$$
$$\text{s.t.} \quad Gx = 0, \; (\pi) \quad \mathbf{1}^T x = m, \; (\lambda) \quad x \geq 0 \; (\mu)$$

MDP Routing Game

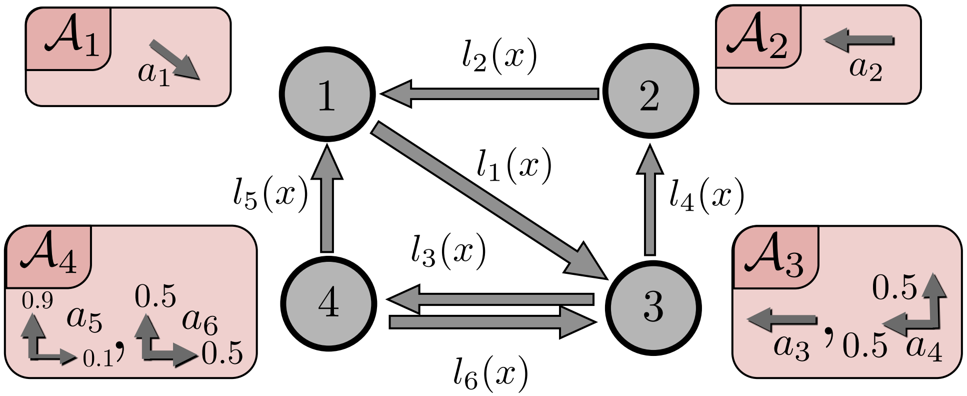


All actions: $\mathcal{A} = \cup_{j \in \mathcal{N}} \mathcal{A}_j$

Cyclic Routing Game (Edge Formulation)

$$\min_{x} \quad F(x)$$
$$\text{s.t.} \quad Gx = 0, \; (\pi) \quad \mathbf{1}^T x = m, \; (\lambda) \quad x \geq 0 \; (\mu)$$

MDP Routing Game



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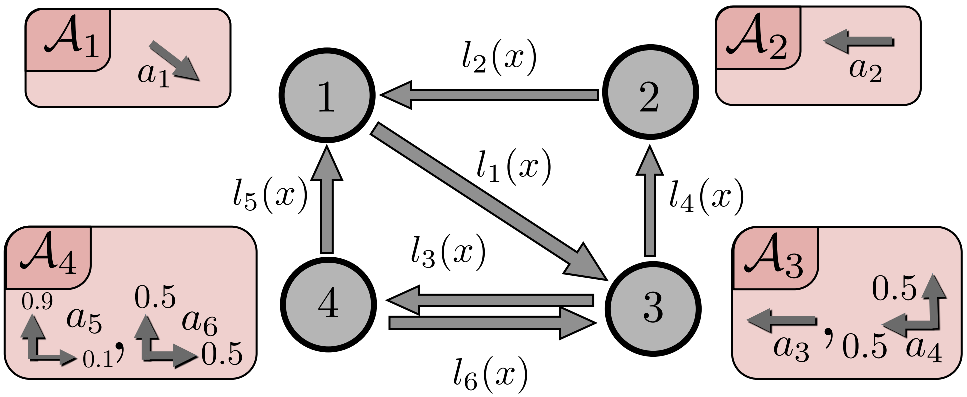

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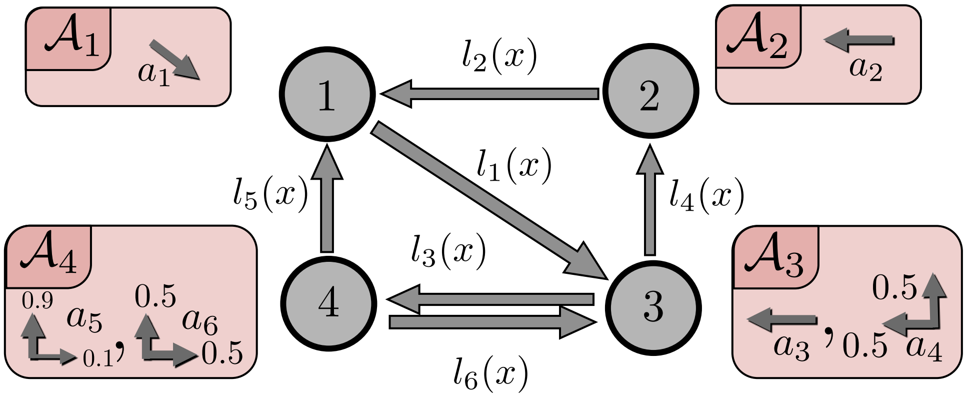

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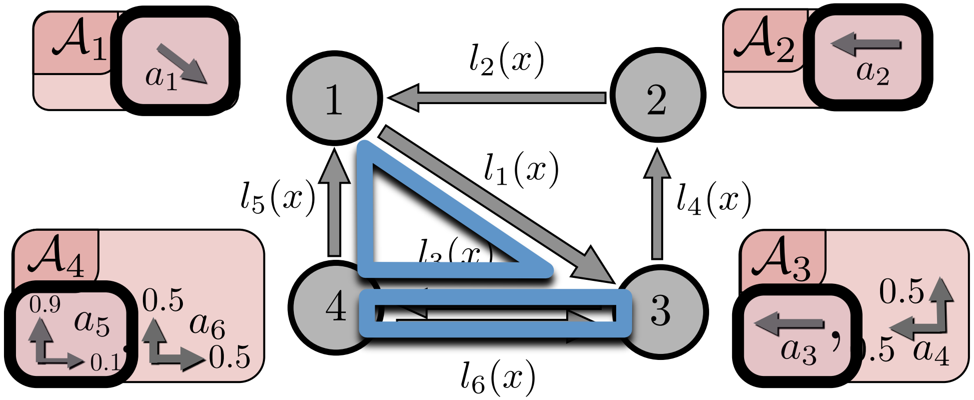

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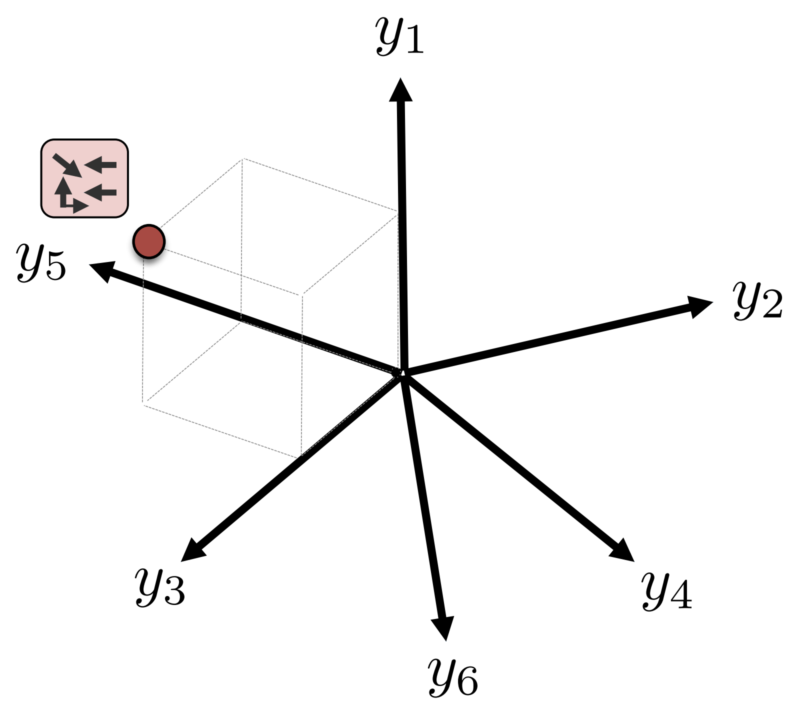

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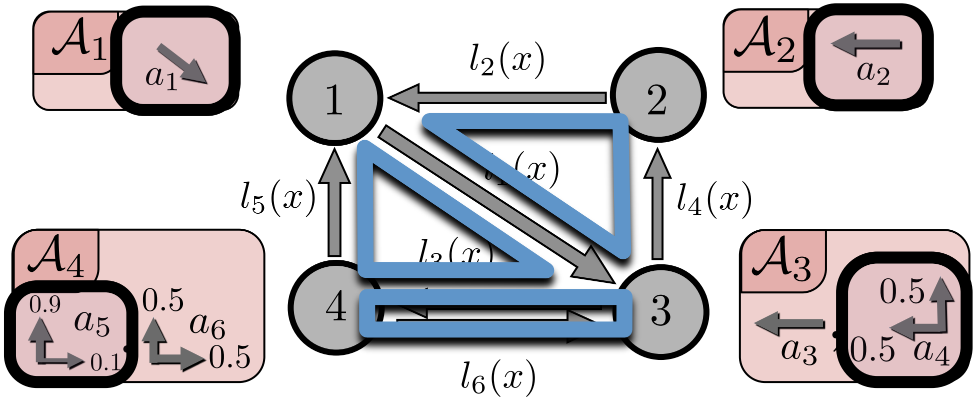

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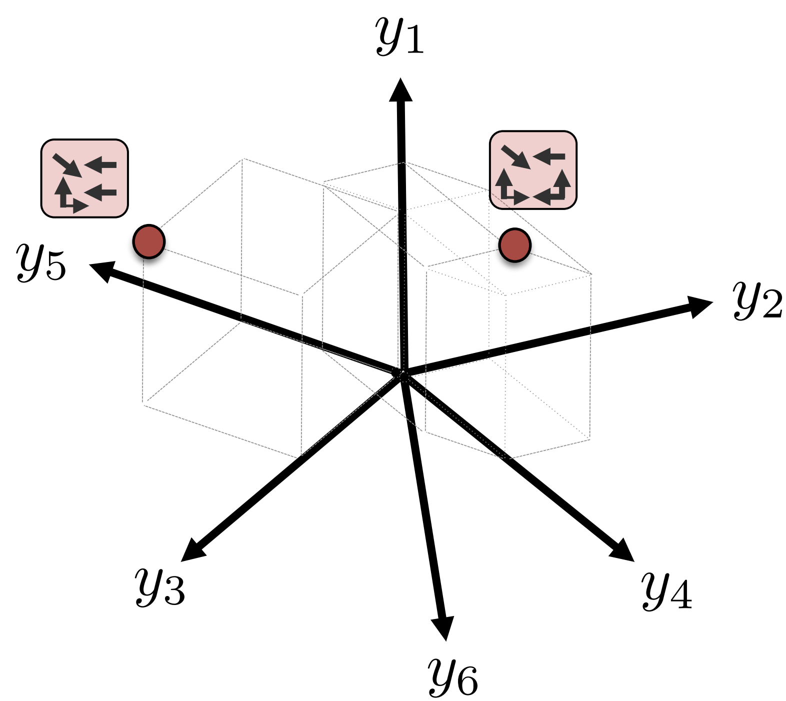

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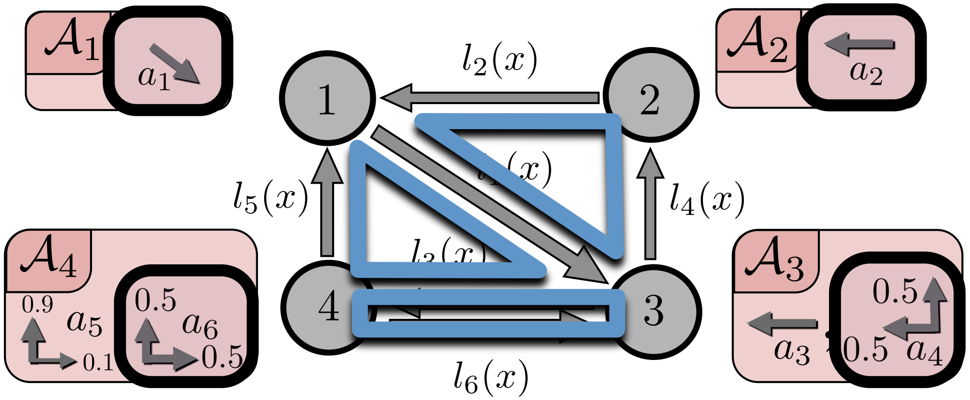

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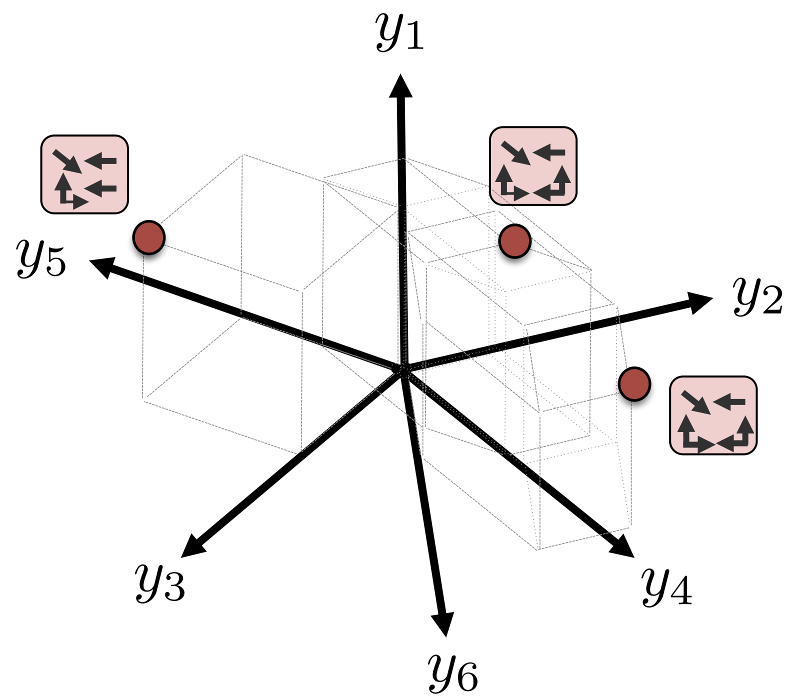

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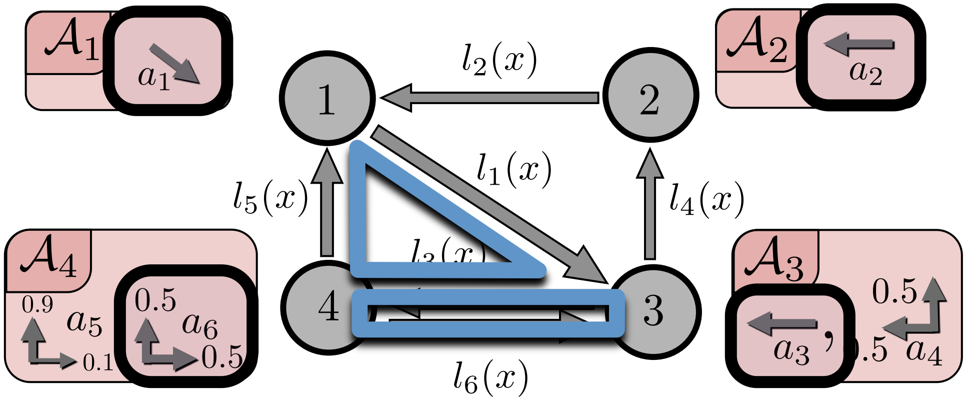

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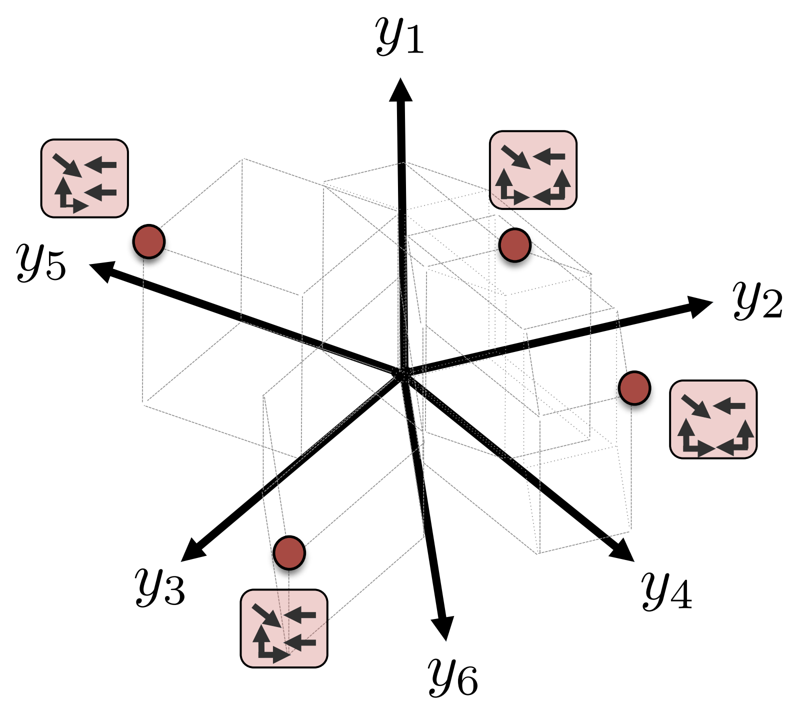
s.t. $GTy = 0$ (π) $\quad \mathbf{1}^T y = m,$ (λ) $\quad y \geq 0$ (μ)



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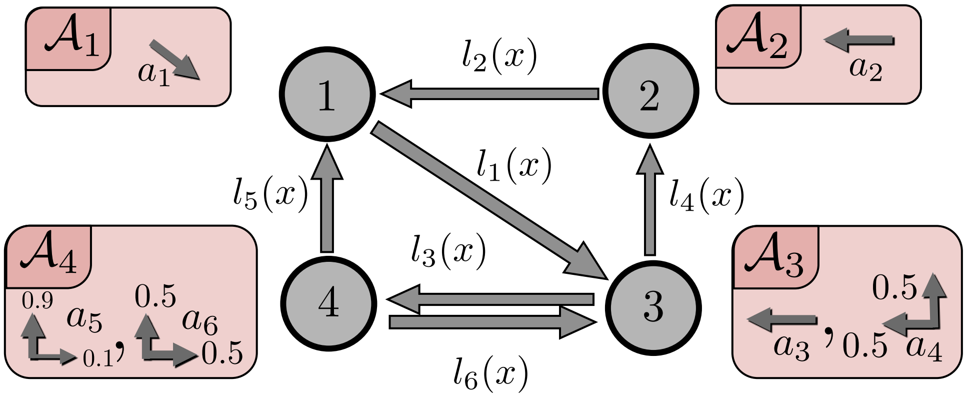

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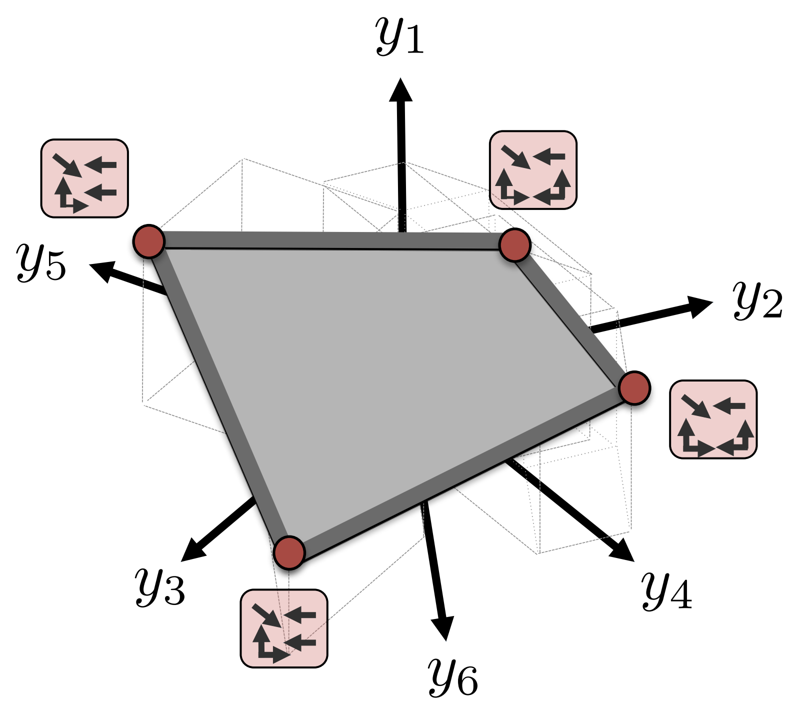

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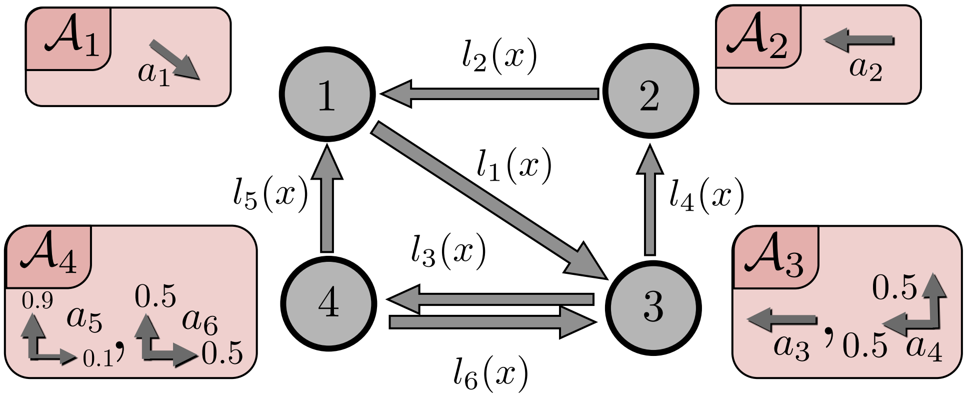

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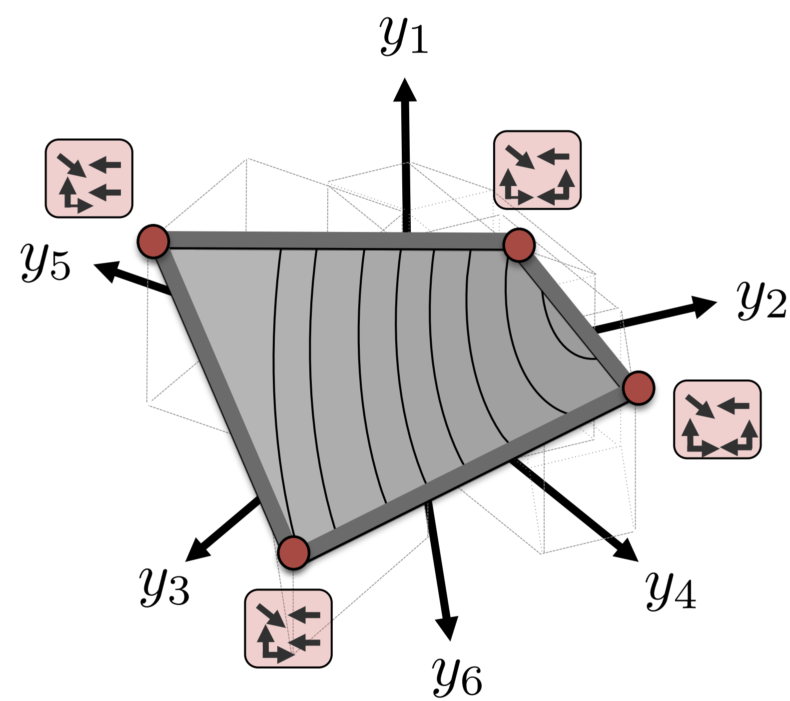

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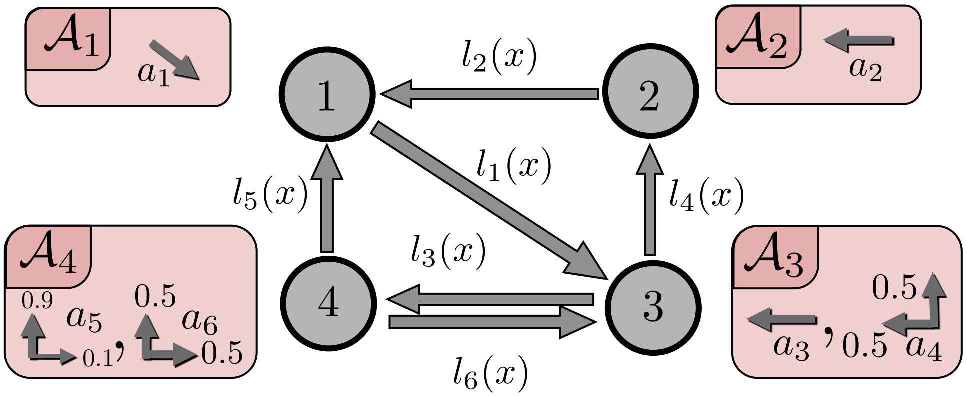

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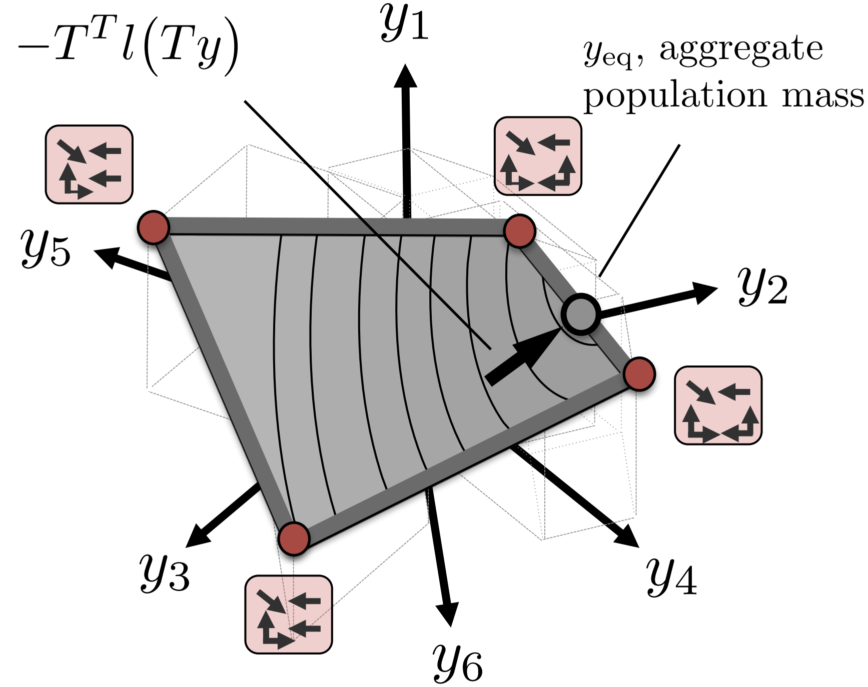

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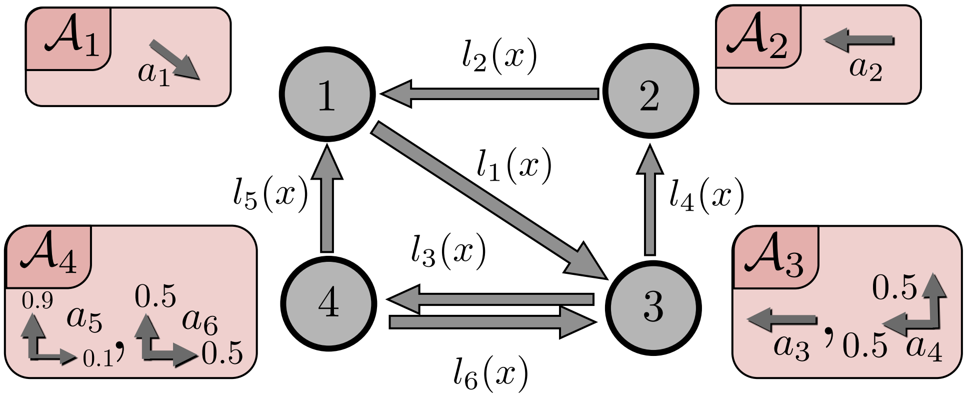

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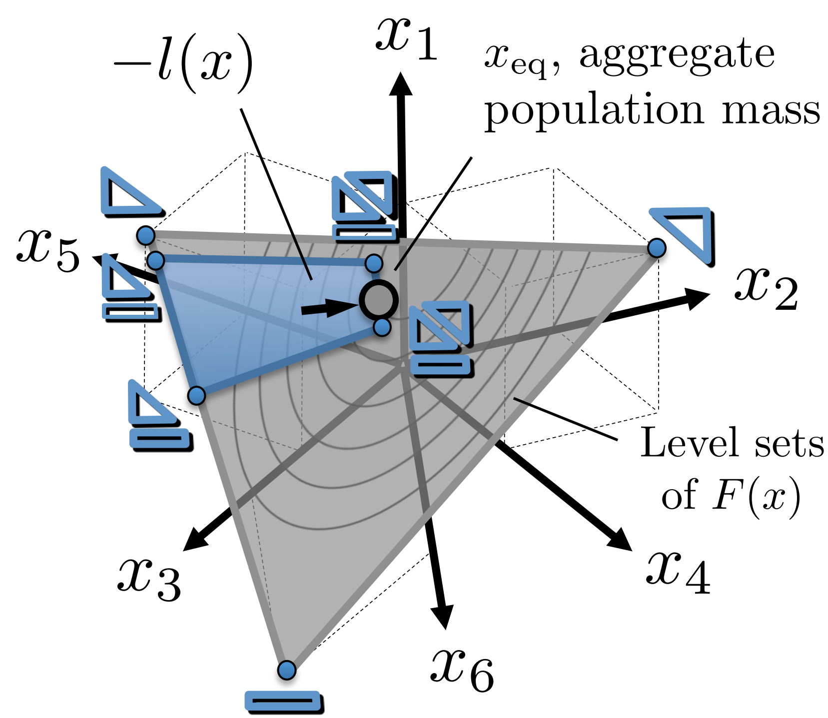

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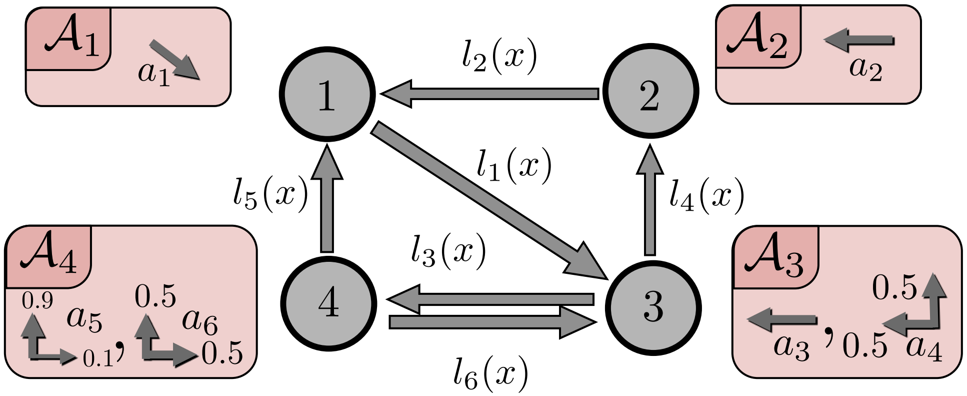

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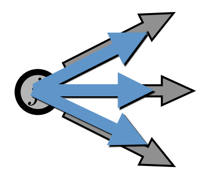
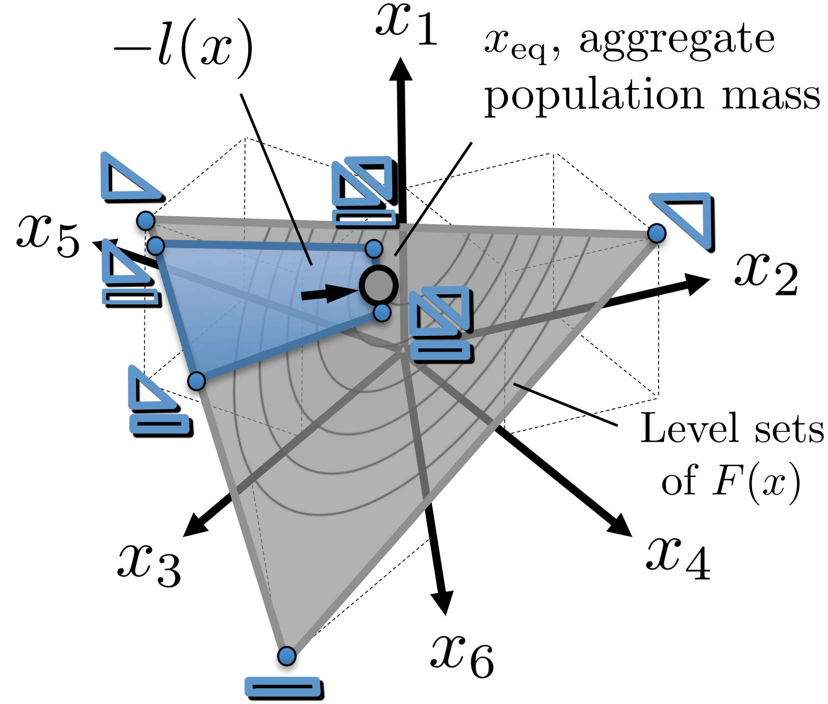
First order conditions…

$$\left(l(x) + I_i \pi\right)^T T_{:a} - \pi_j - \lambda - \mu_a = 0$$





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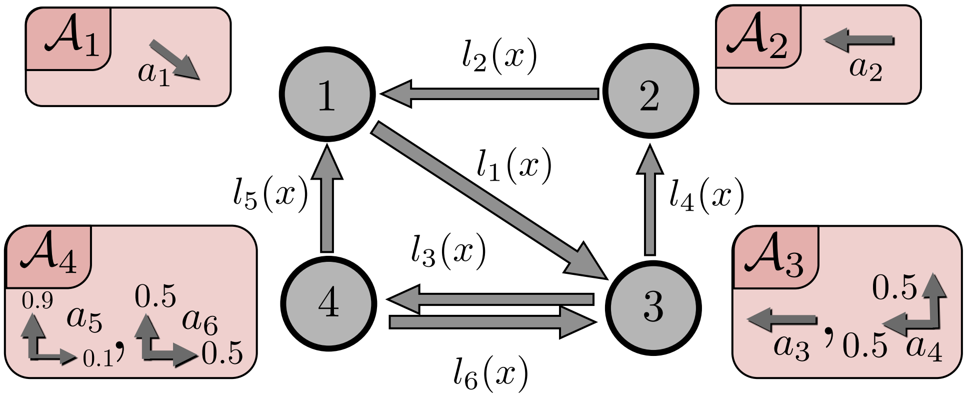

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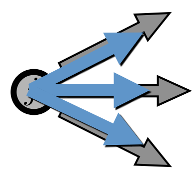
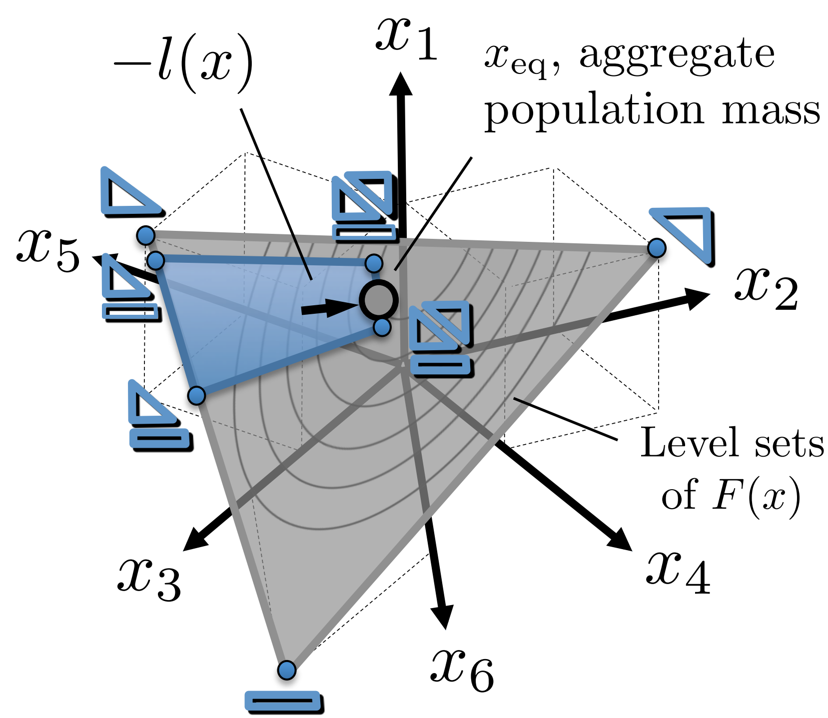


Summing over a policy η and the resulting stationary distribution $p(\eta)$…

$$\sum_{j,a} \left[\left(l(x)^T + \pi^T I_i\right) T_{:a} - \pi_j - \lambda - \mu_a\right] \eta_a^j p_j(\eta) = 0$$



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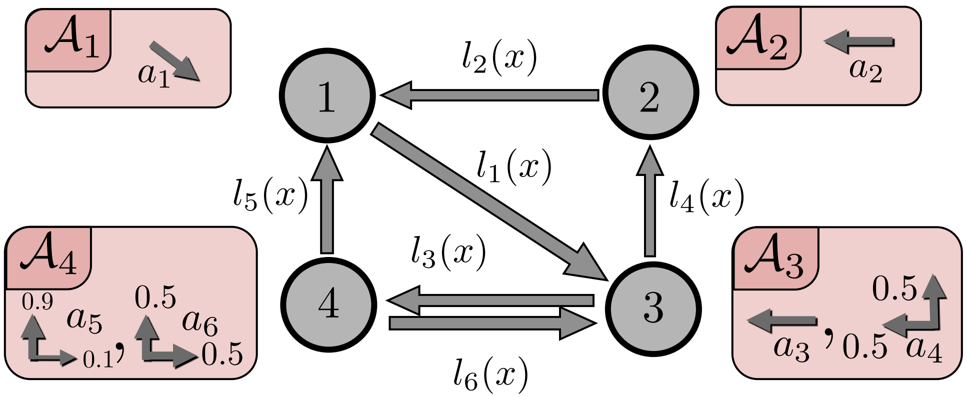

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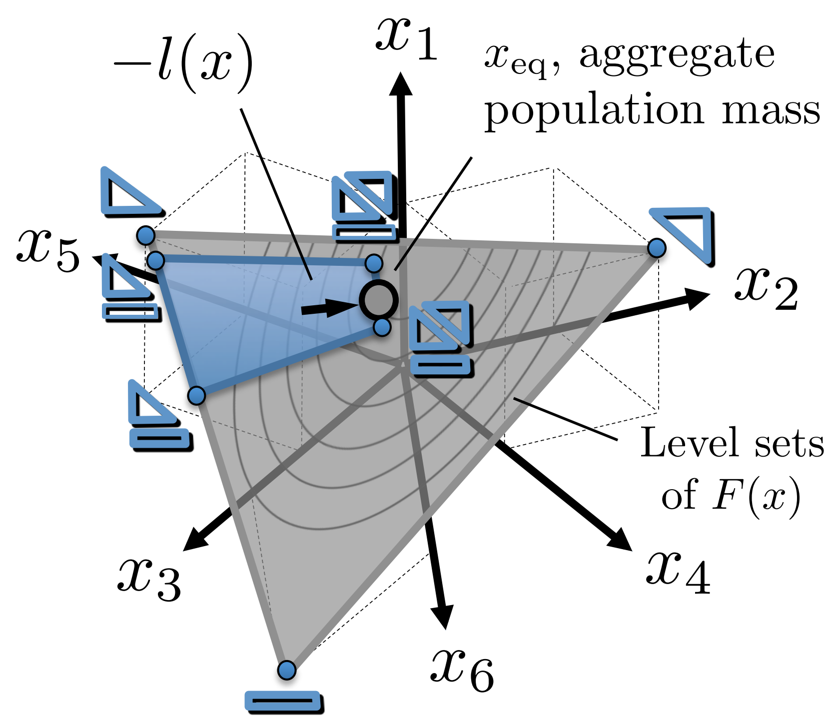

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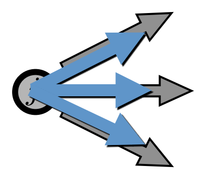

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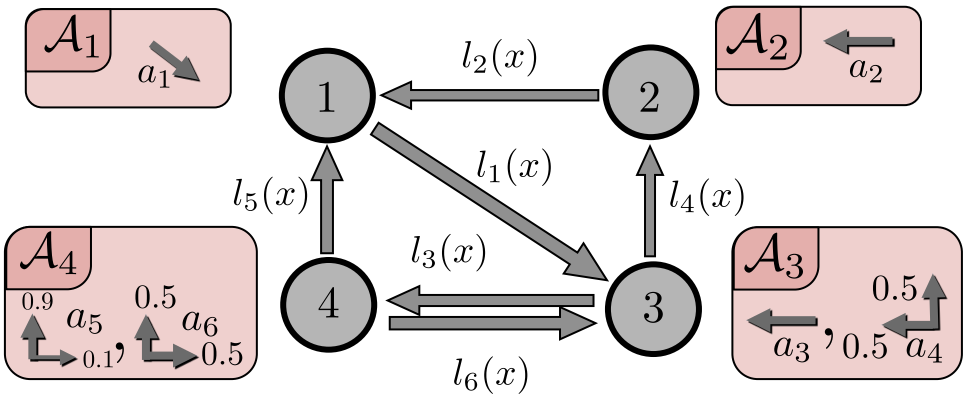
$$\sum_{j,a} \left[l(x)^T T_{:a} \right] \eta_a^j p_j(\eta) = \lambda + \sum_{j,a} \mu_a \eta_a^j p_j(\eta)$$

MDP Routing Game



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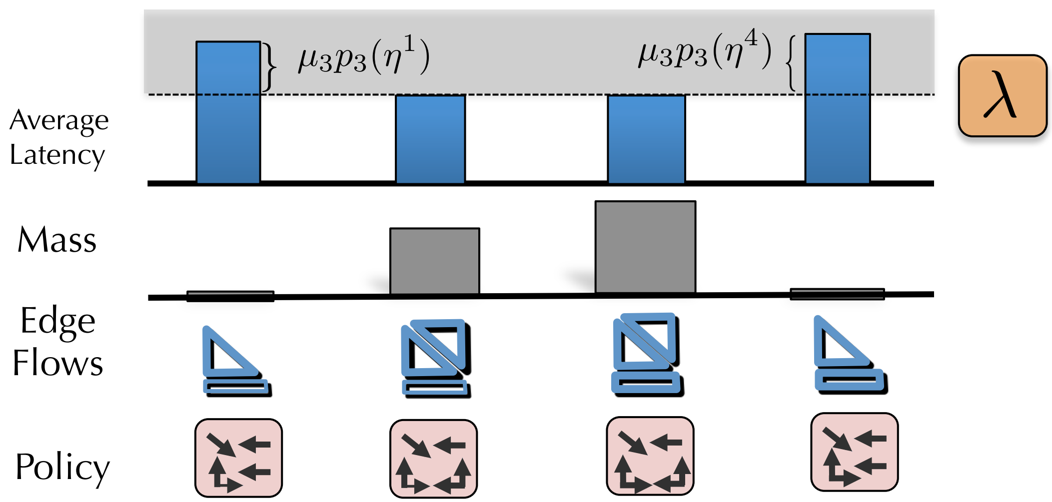
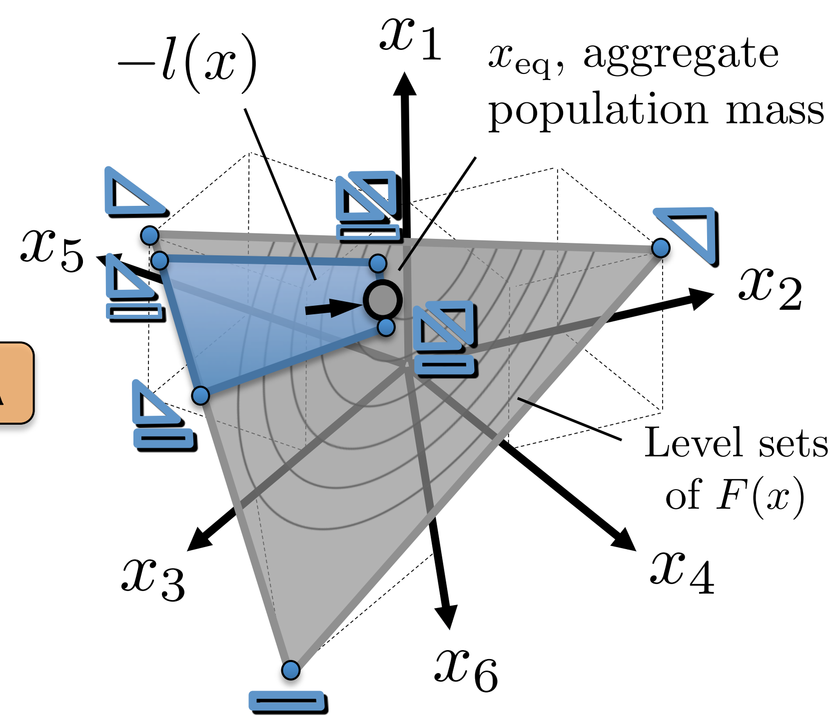
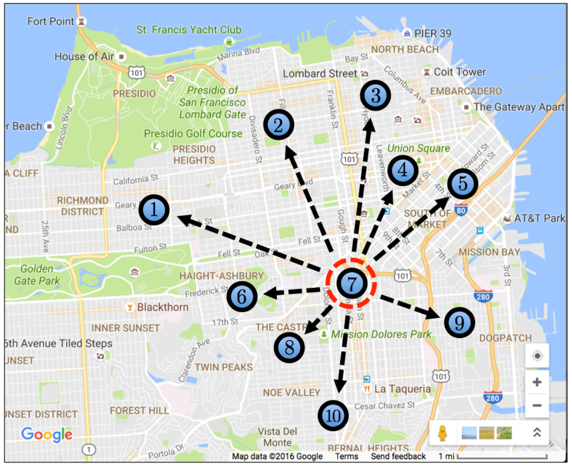


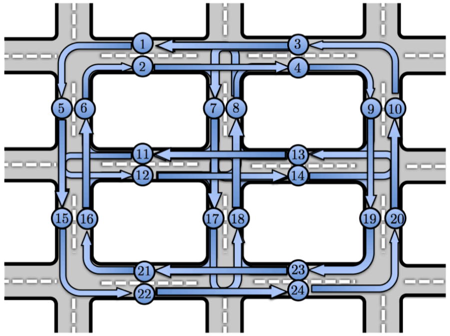


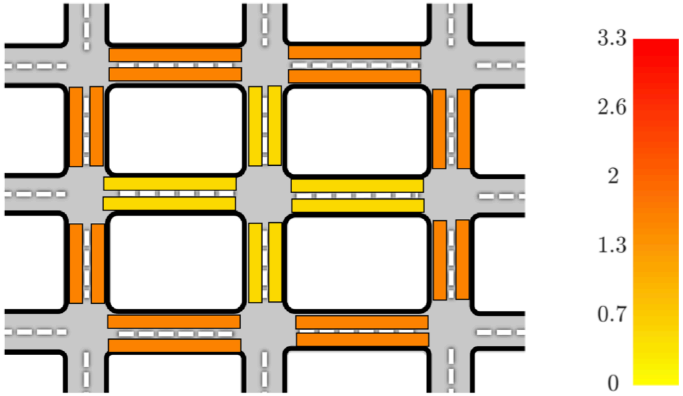
Applications

Ride Sharing



Parking on urban streets





Conclusion – Future Work

Extensions

- Braess's Paradox
- Price of anarchy

Future work

- Discounted infinite horizon case

Thanks!