Queue-Routing Game

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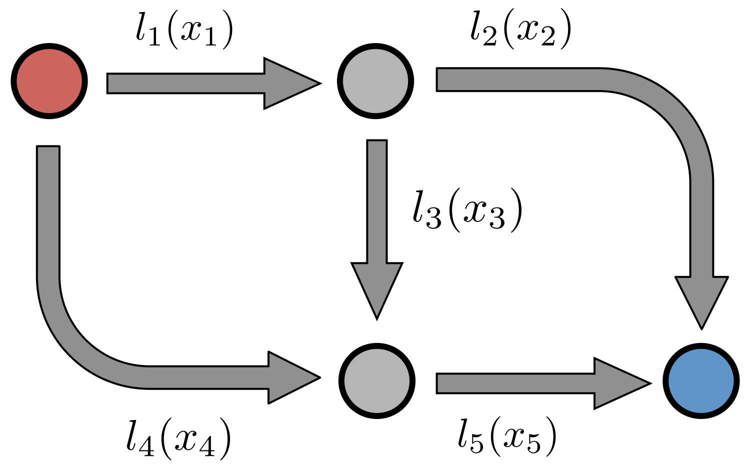
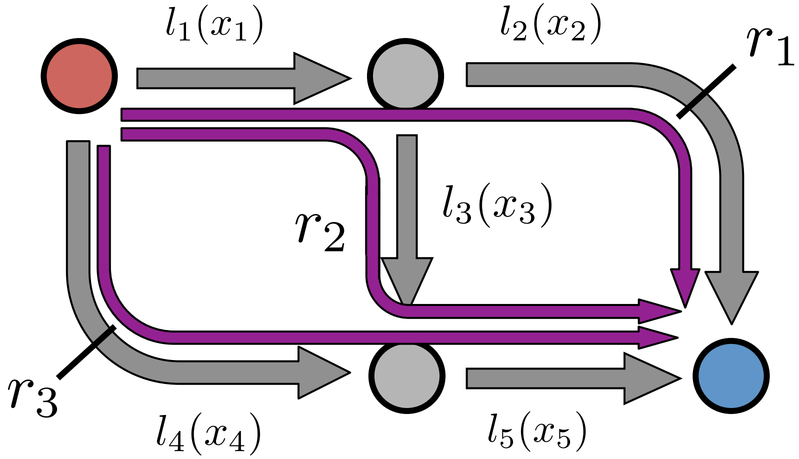
UC Berkeley
April 8^{th}, 2016

Routing Game Tutorial



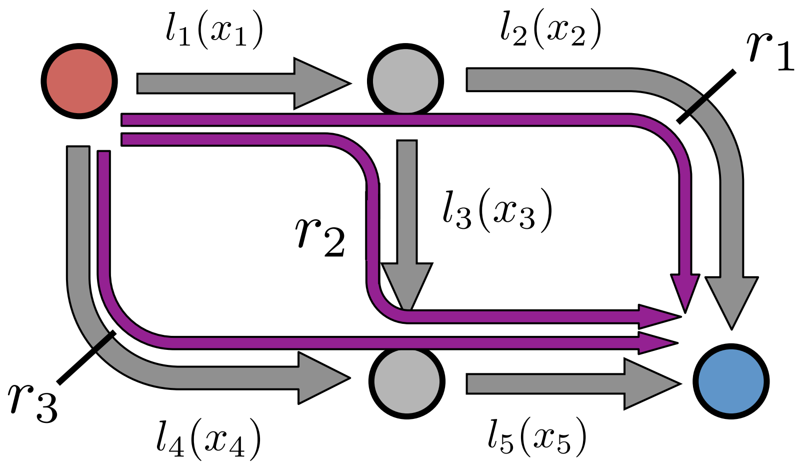
Edge & Route Flows



$$\begin{array}{cc} & \text{Paths} \\ \text{Edges} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Routing Matrix $\mathbb{R}$

Edge & Route Flows



Routes

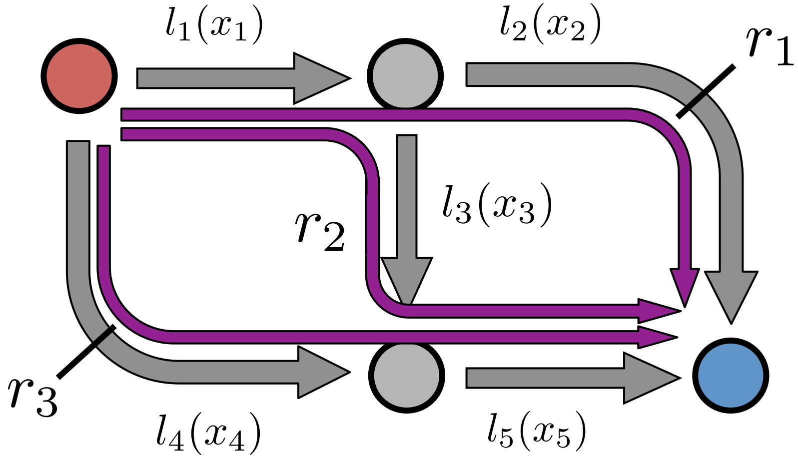
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{array}{c} \\ \text{Edges} \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x_1^R \\ x_2^R \\ x_3^R \end{bmatrix}$$

Edge Flows

Routing Matrix $\mathbb{R}$

Route Flows

Edge & Route Flows



$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{array}{cc} & \text{Routes} \\ \text{Edges} & \begin{bmatrix}1&1&0\\1&0&0\\0&1&0\\0&0&1\\0&1&1\end{bmatrix}\end{array} \times \begin{bmatrix}x_1^R\\x_2^R\\x_3^R\end{bmatrix}$$

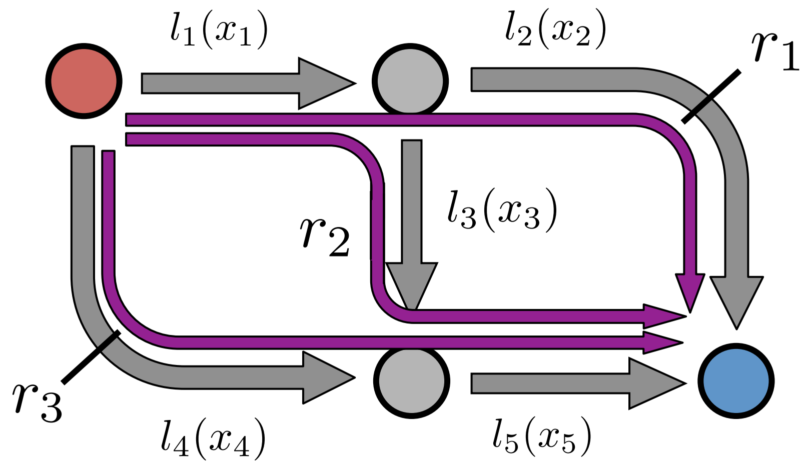
Edge Flows

Routing Matrix $\mathbb{R}$

Route Flows

$$x = \mathbb{R} \times x^R$$

Edge & Route Latencies



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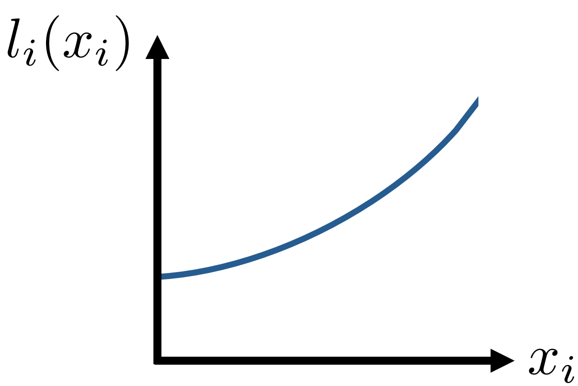
Edge Flows | **Routing Matrix** $\mathbb{R}$ | Route Flows

$$x = \mathbb{R} \times x^R$$

<u>Latencies</u>

Edge $\quad l(x) = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{bmatrix}$

Path $\quad l^R(x) = \begin{bmatrix} l_1^R & l_2^R & l_3^R \end{bmatrix}$



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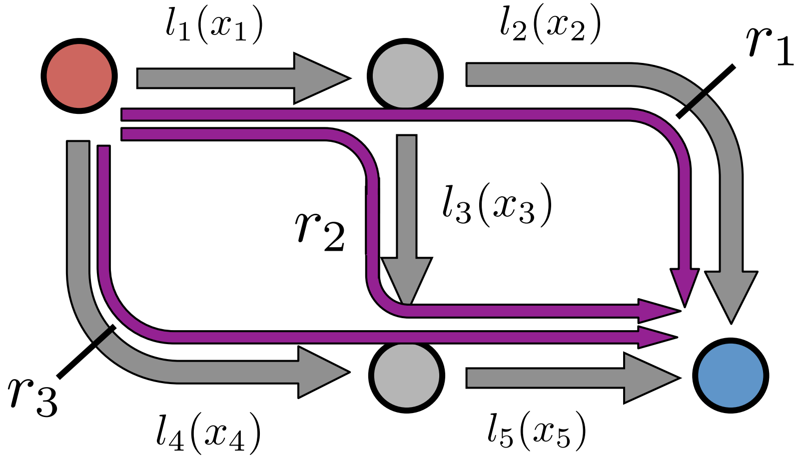

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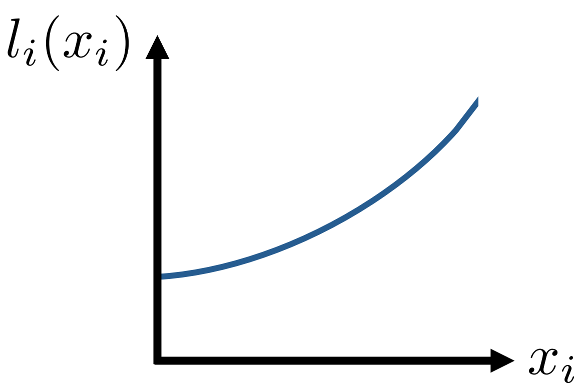
Edge Flows | **Routing Matrix** $\mathbb{R}$ | Route Flows

$$\underbrace{\quad}_{x = \mathbb{R} \times x^R}$$



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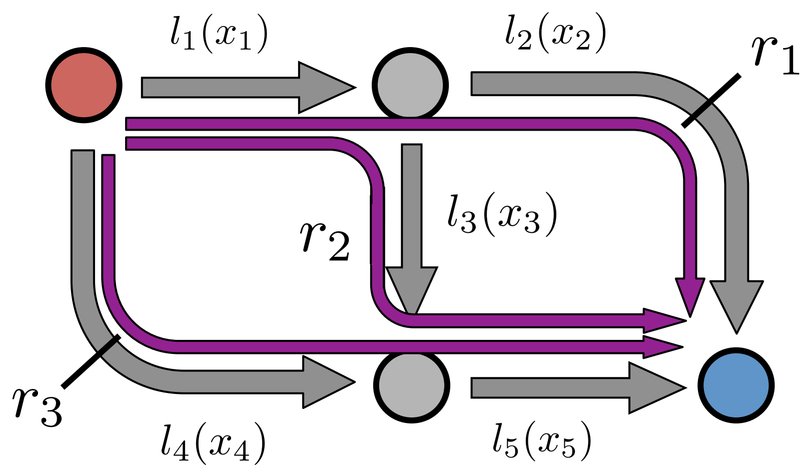
$$l \times \mathbb{R} = l^R$$

Potential Function



Routes

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\text{Edge Flows}} = \text{Edges}\ \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{\textbf{Routing Matrix}\ \mathbb{R}} \times \underbrace{\begin{bmatrix} x_1^R \\ x_2^R \\ x_3^R \end{bmatrix}}_{\text{Route Flows}}$$

$$x = \mathbb{R} \times x^R$$

Potential Function $\quad P(x) = \sum_e \int_0^{x_e} l_e(u)\ du$

<u>Latencies</u>

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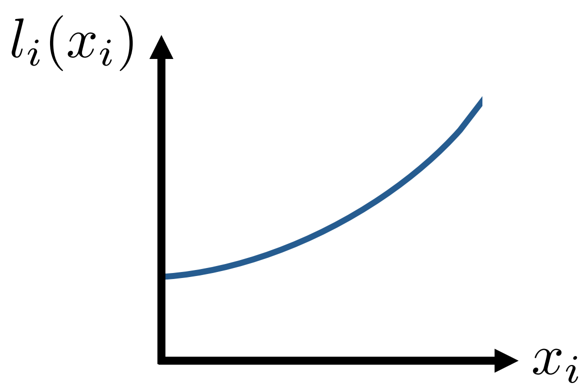
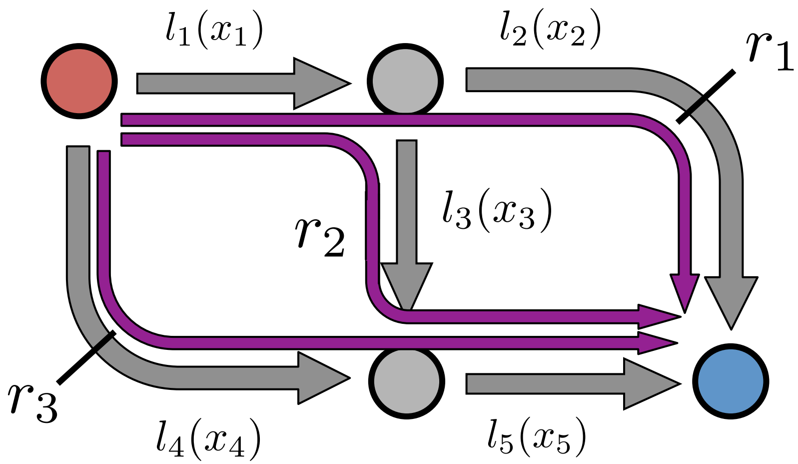


Potential Function



Potential Function $$P(x) = \sum_e \int_0^{x_e} l_e(u)\ du$$

$$\nabla_x P(x) = l(x)$$

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Edge Flows

Routing Matrix $\mathbb{R}$

Route Flows

$$x = \mathbb{R} \times x^R$$

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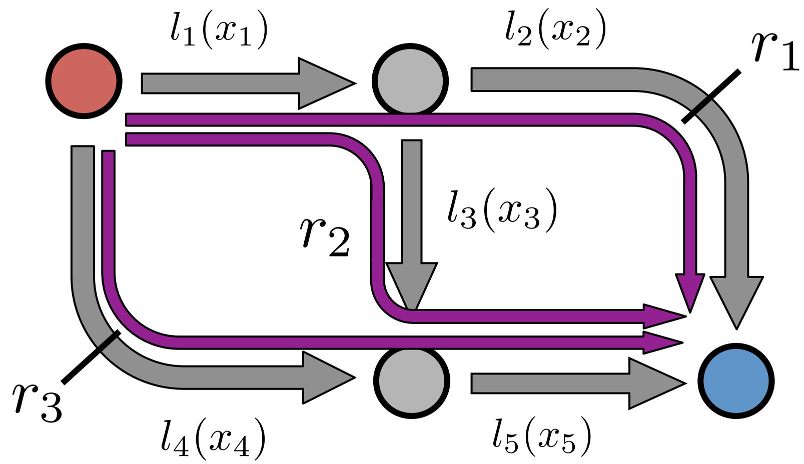
$$l \times \mathbb{R} = l^R$$

Routing Game Formulations



Routing Matrix
$$\mathbb{R} = \begin{array}{cc} & \text{Routes} \\ \text{Edges} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Flows $\quad x = \mathbb{R} \times x^R$

Latencies $\quad l \times \mathbb{R} = l^R$

Potential Function $\quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du$

$$\min_{x^R} \quad P(x)$$
$$\text{s.t.} \quad x^R \geq 0, \quad x = \mathbb{R}x^R, \quad \sum_r x_r^R = s$$

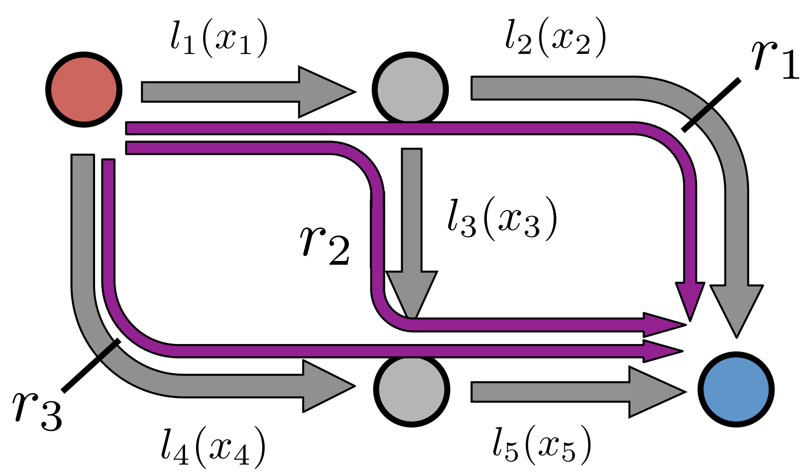
Path Formulation

OR

$$\min_{x} \quad P(x)$$
$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

Edge Formulation

Routing Game Formulations



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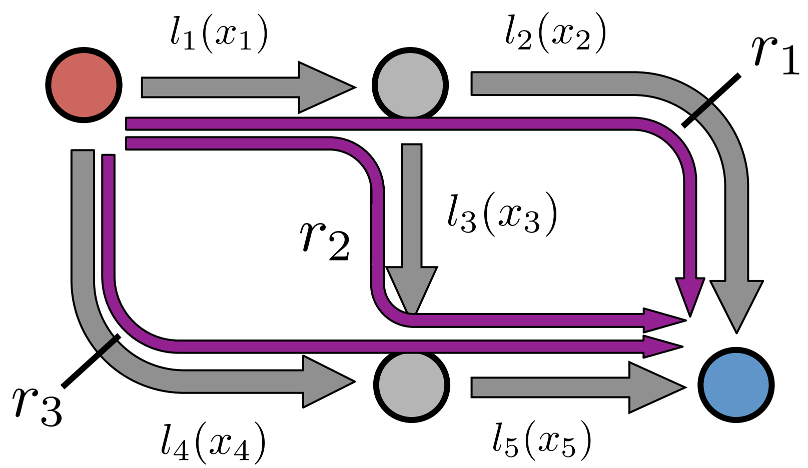
Edge Formulation

KEY ARGUMENTS

Routing Game Formulations



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Routing Matrix $\mathbb{R} = $

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Flows $x = \mathbb{R} \times x^R$

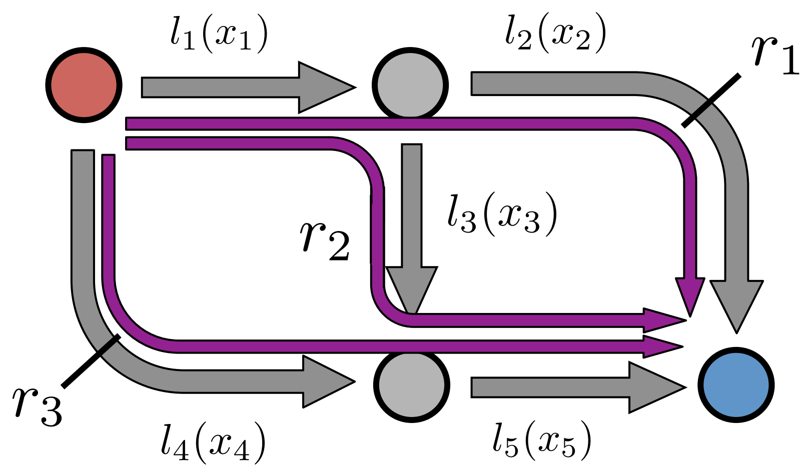
Latencies $l \times \mathbb{R} = l^R$

KEY ARGUMENTS

First Order Optimality

$$\underbrace{\nabla P(x)}_{l(x)} + \lambda \cdot \text{CONST} - \underbrace{\mu}_{\geq 0} = 0$$

Routing Game Formulations



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Edge Formulation

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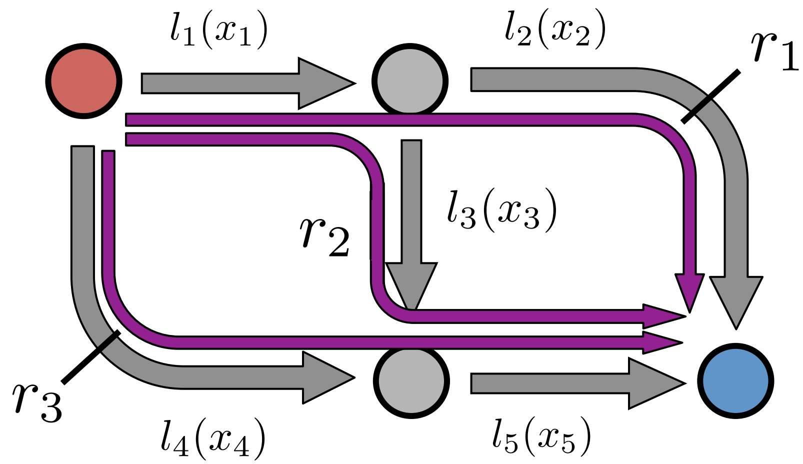
$$\Downarrow$$

Wardrop Equilibrium

Wardrop Equilibrium



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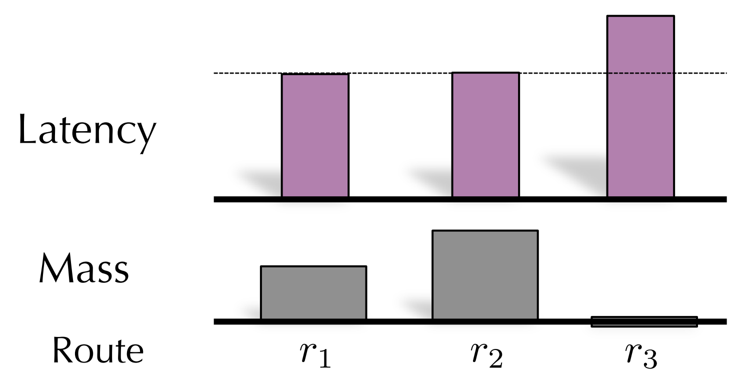
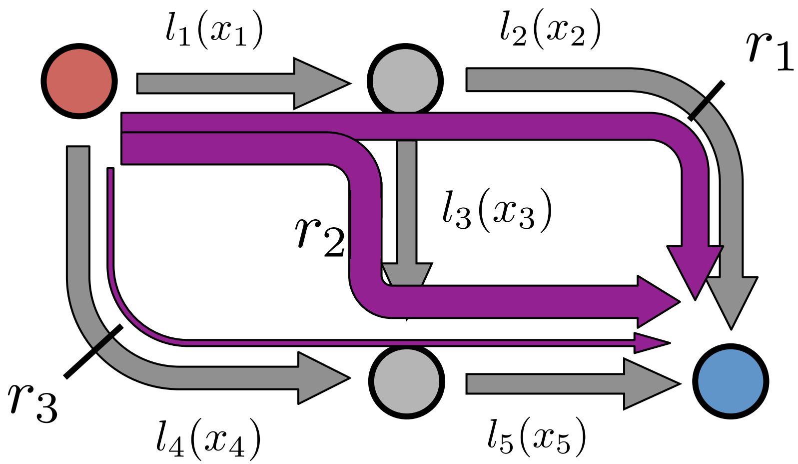
Wardrop Equilibrium



Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\ du$$

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Flows $\quad x = \mathbb{R} \times x^R$

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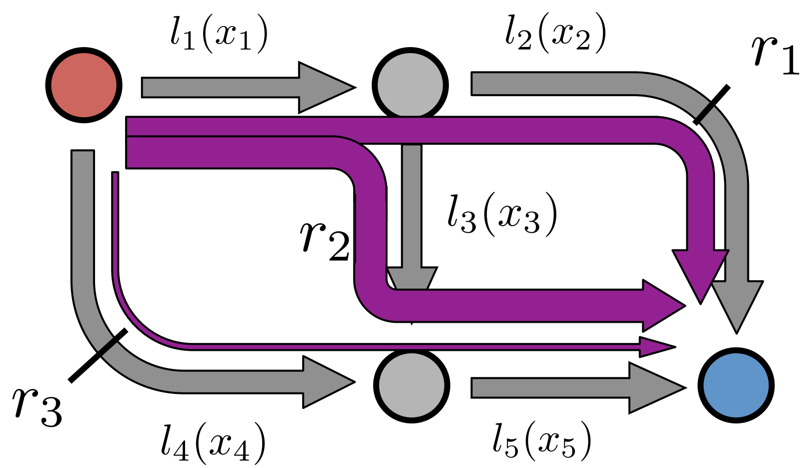
First Order Optimality…

Path Formulation



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Gradient… $\nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

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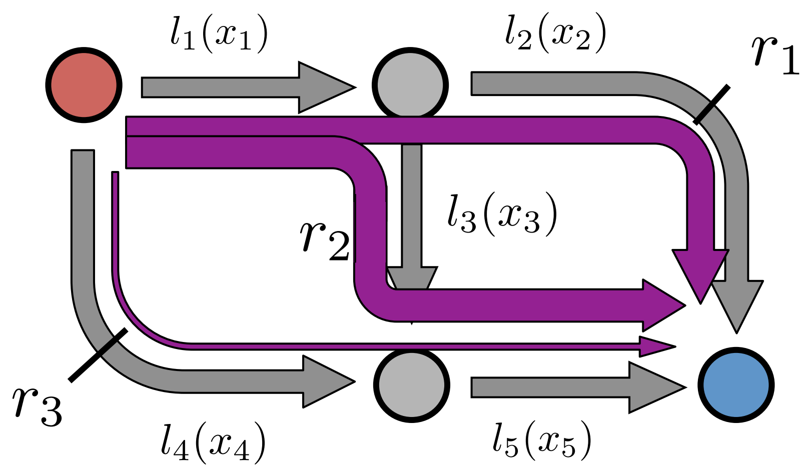

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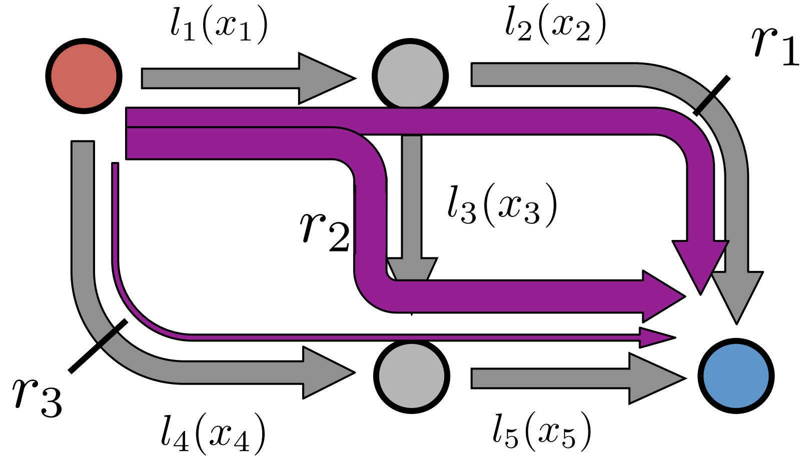
First Order Optimality…

Gradient… $\nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

Path Formulation



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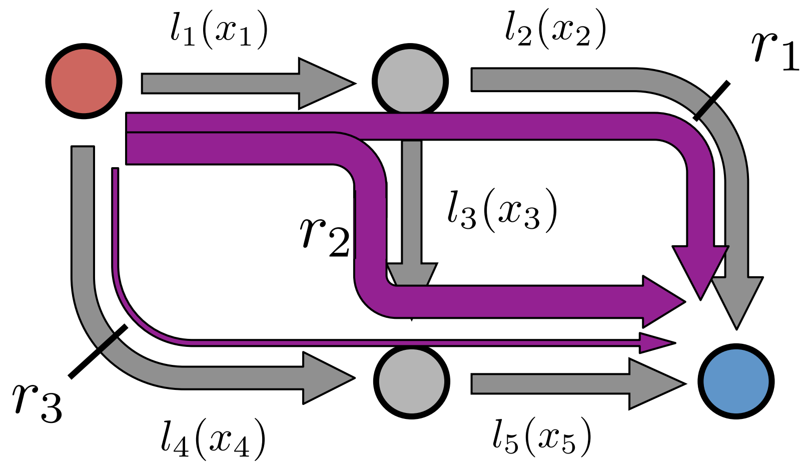
Path Formulation



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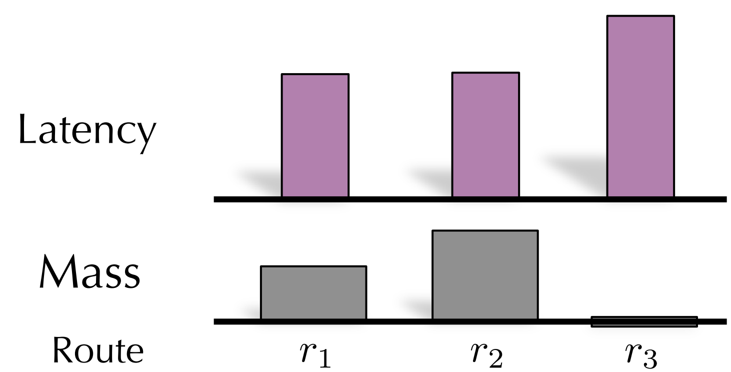

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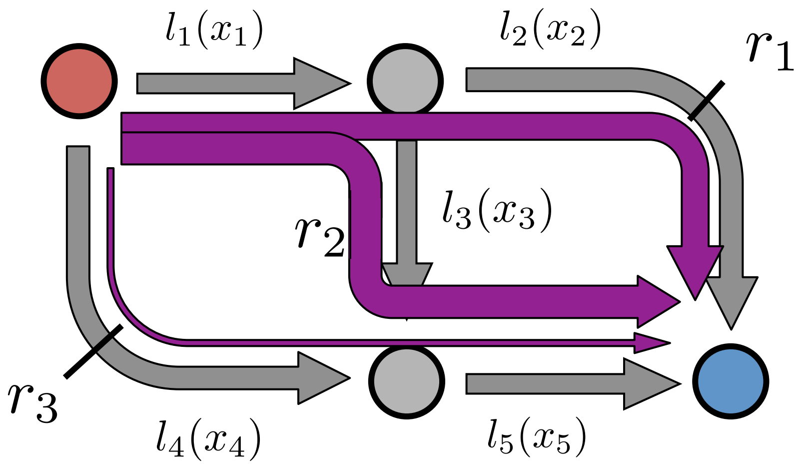

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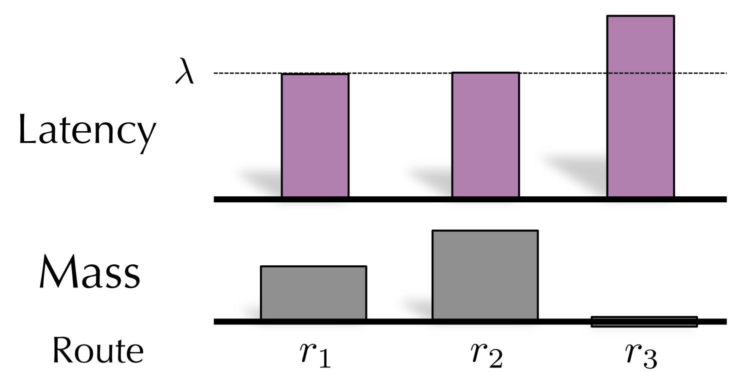
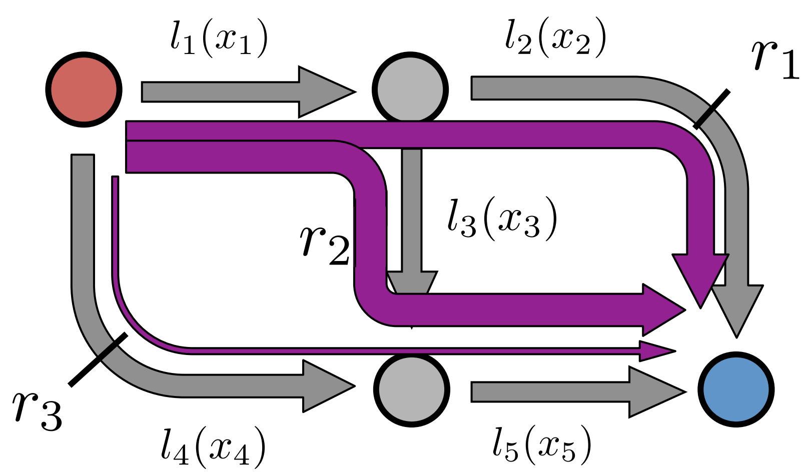


Path Formulation



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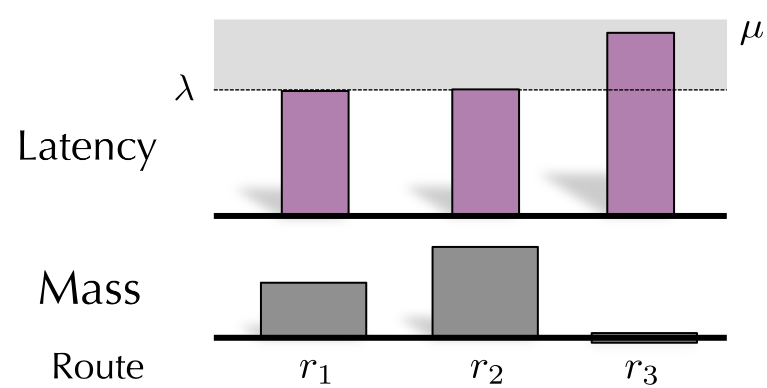

Path Formulation

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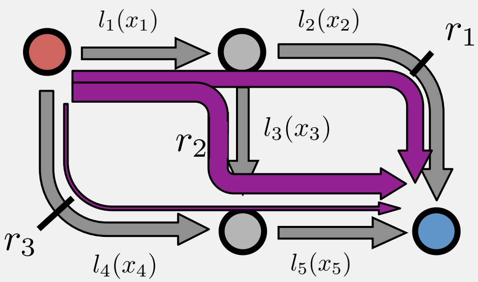

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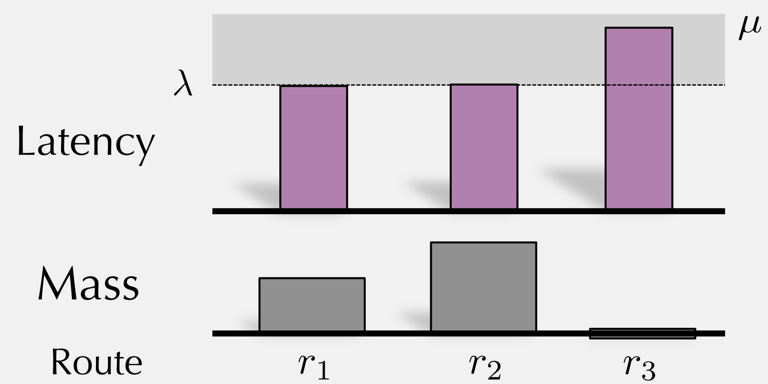
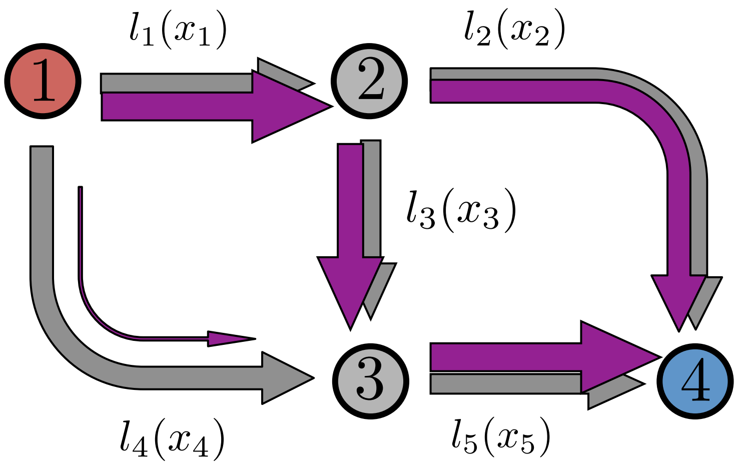


Path Formulation



Edge Formulation

$$\min_{x} \quad P(x) = \sum_{e} \int_0^{x_e} l_e(u)\ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

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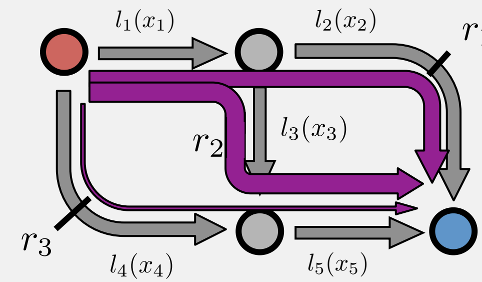

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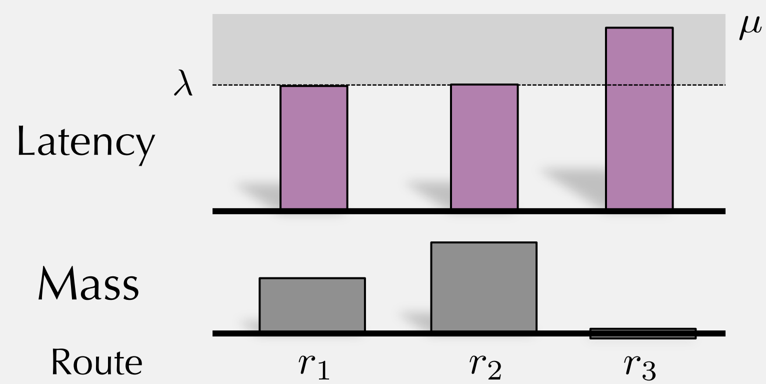

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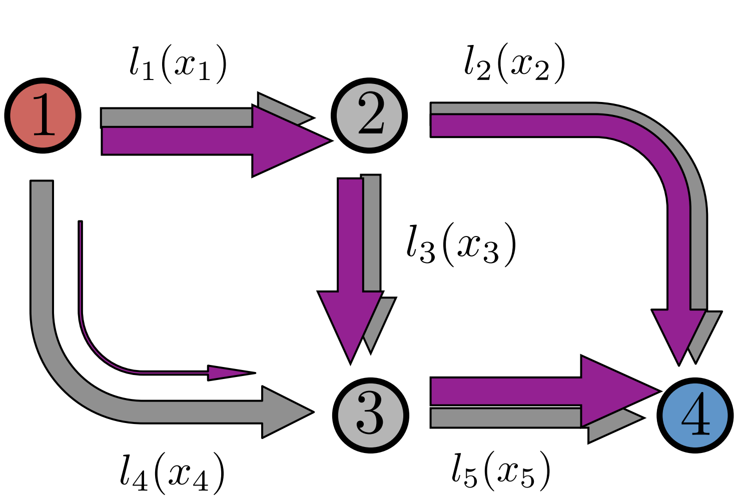

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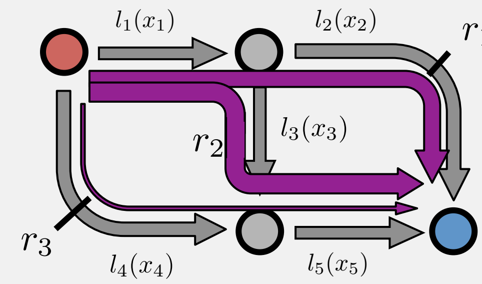

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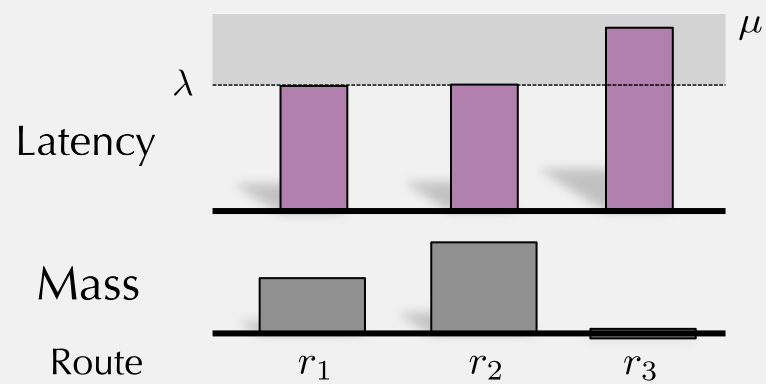

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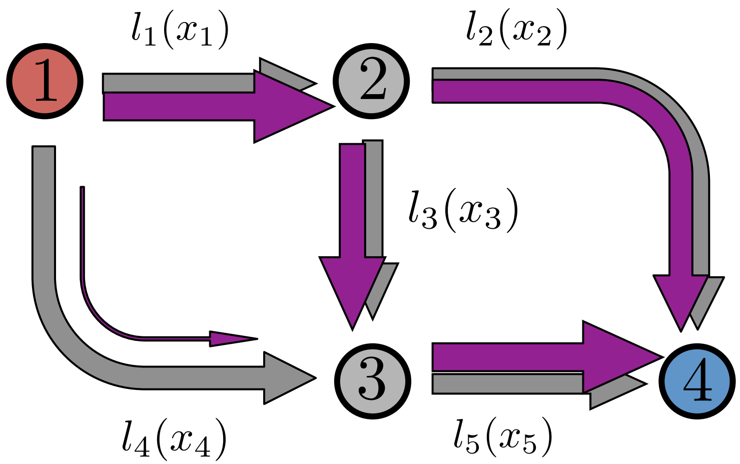

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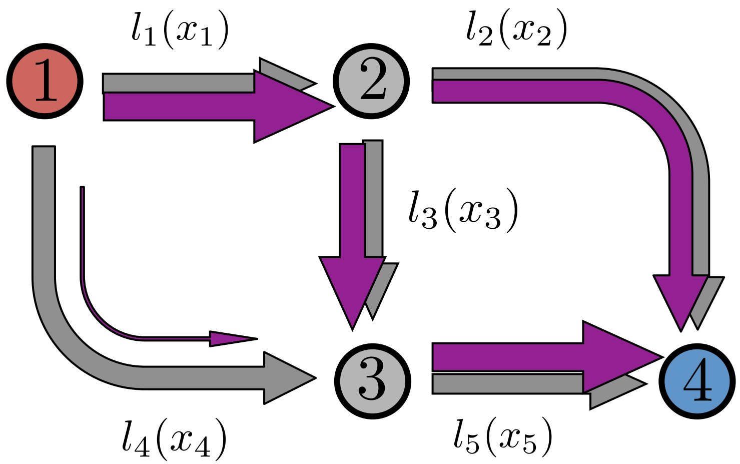

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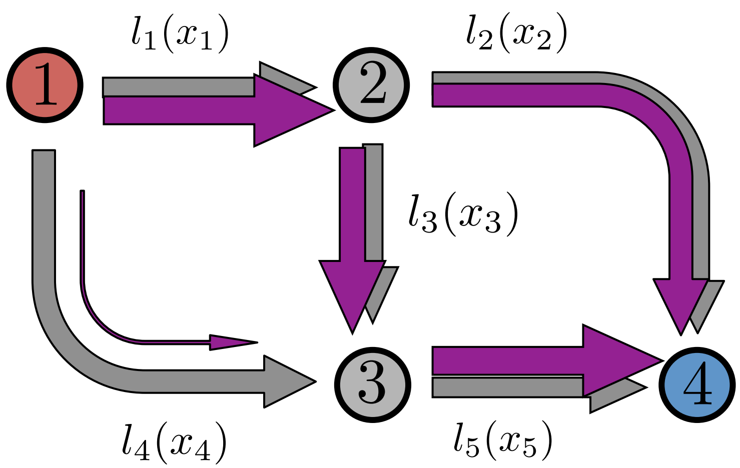

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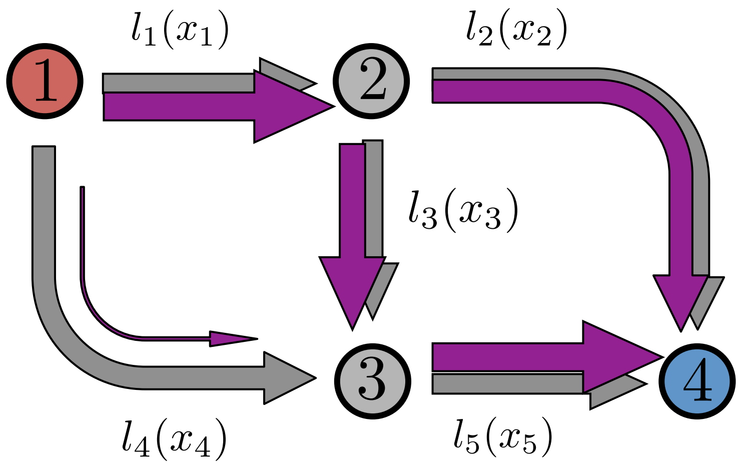

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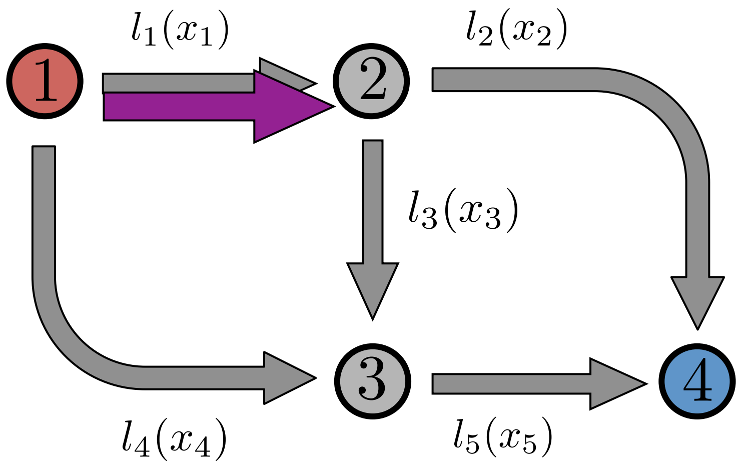

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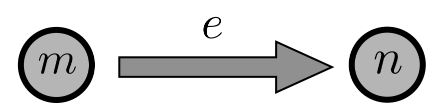
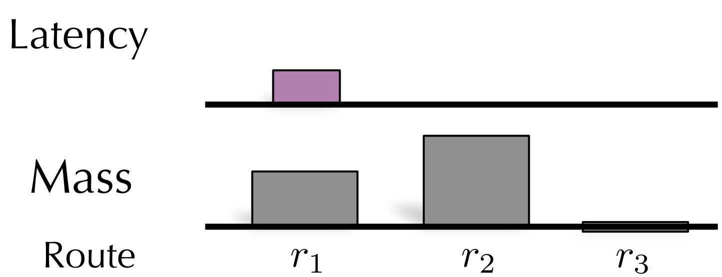


$$r_1 : \qquad l_1 = \pi_2 - \pi_1 + \nu_1$$



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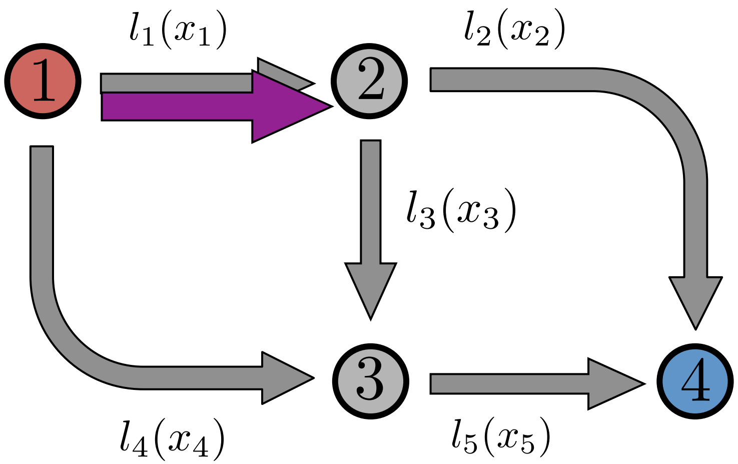

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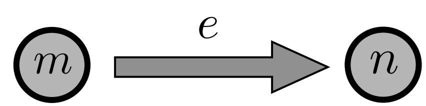

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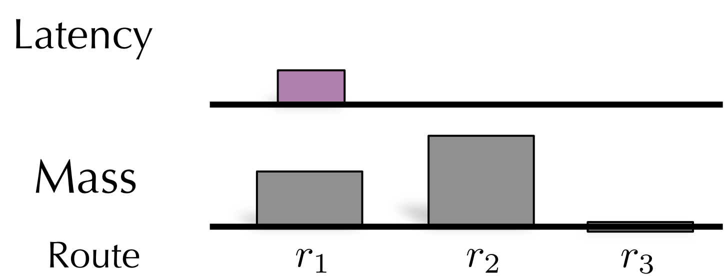

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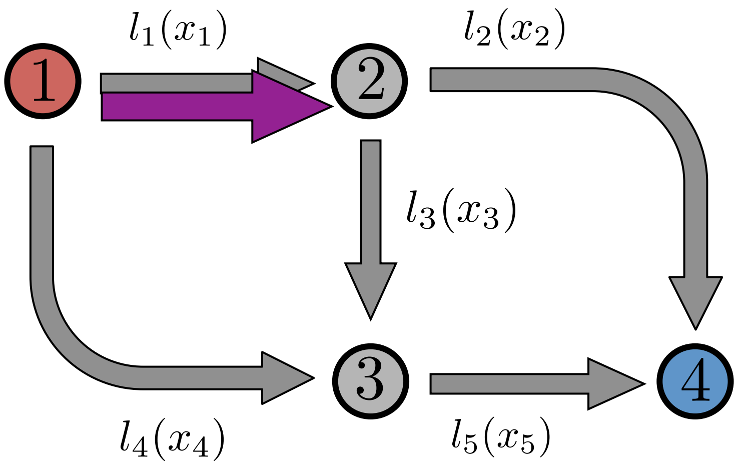

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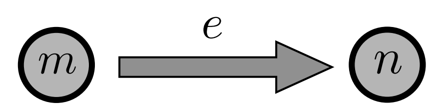

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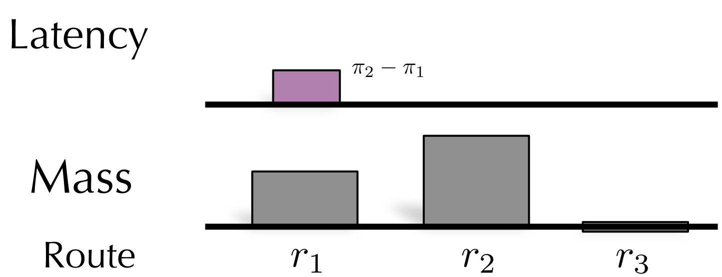

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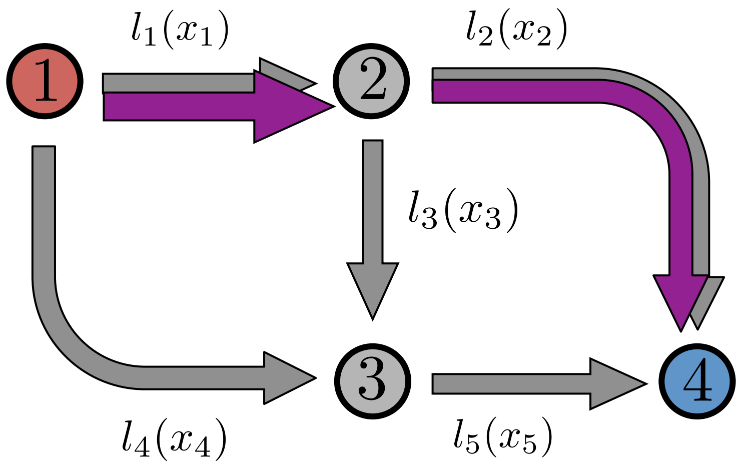

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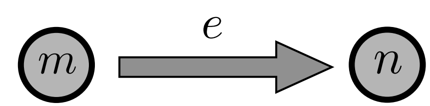

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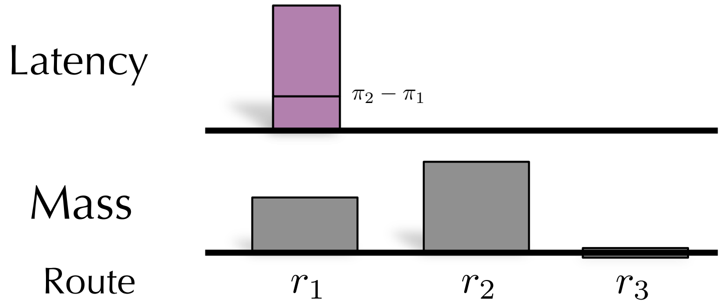

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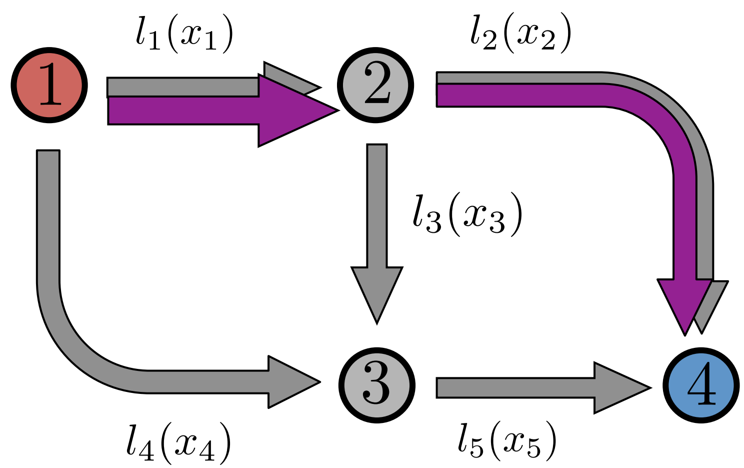

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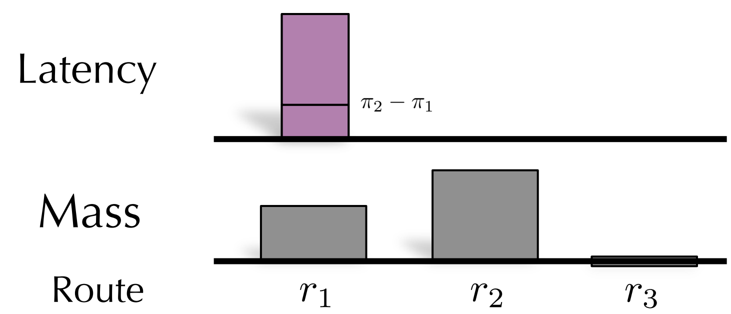

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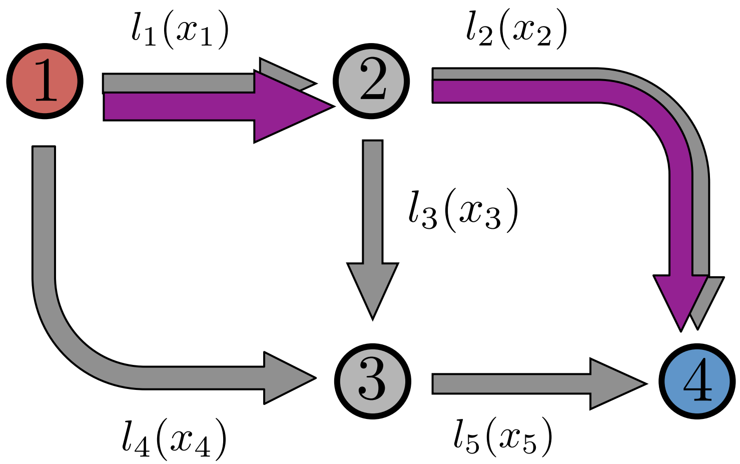

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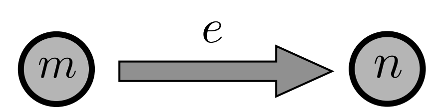

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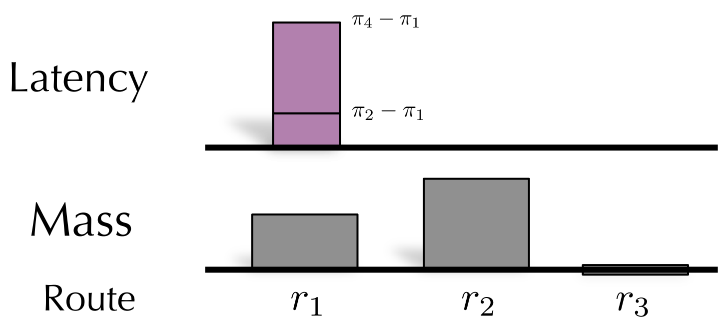

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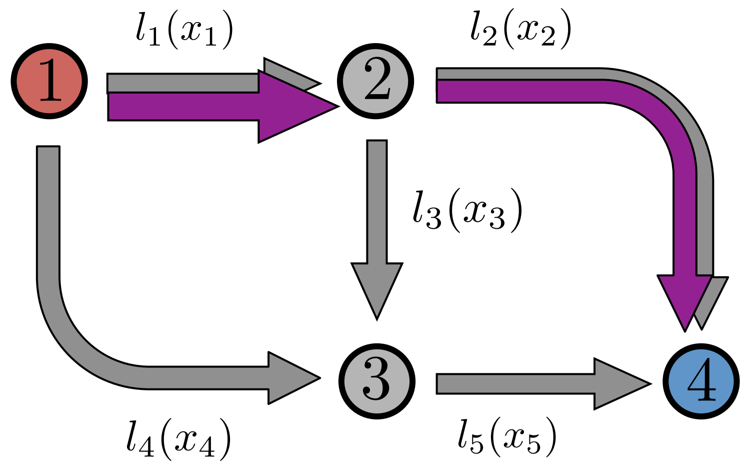

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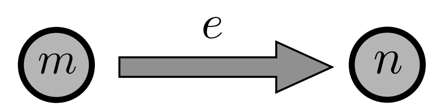

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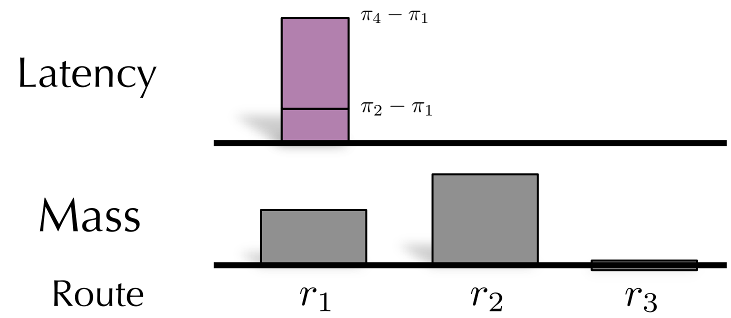

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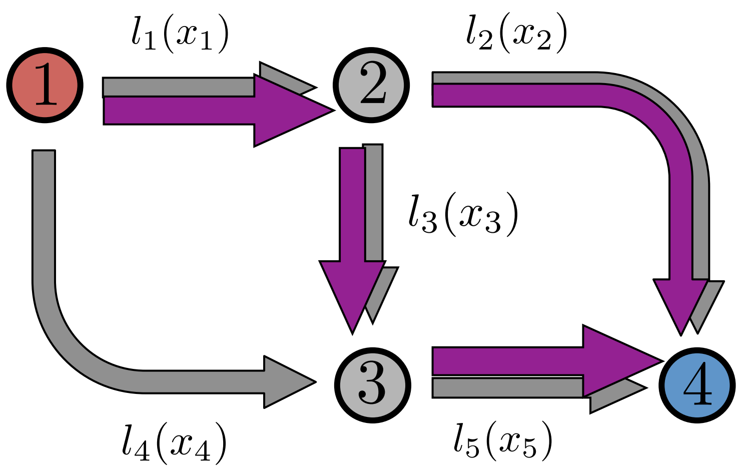

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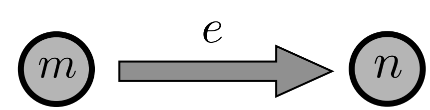

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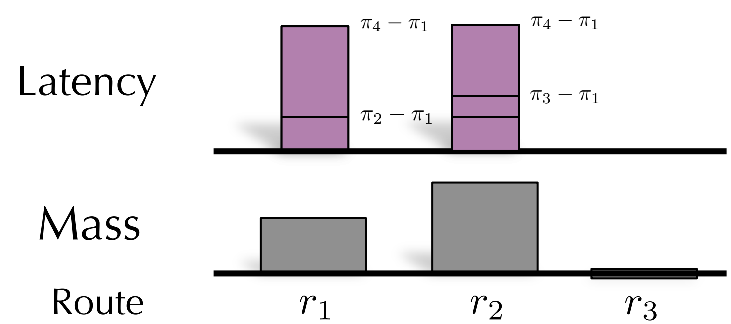
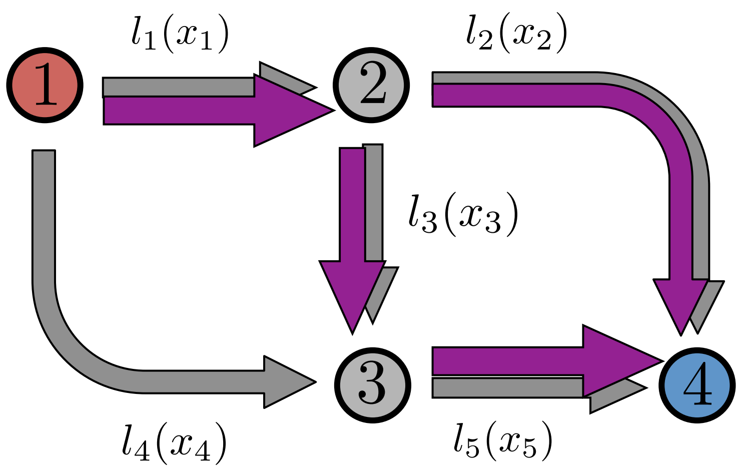


Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

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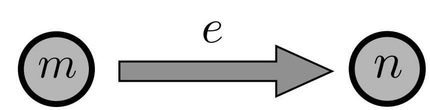
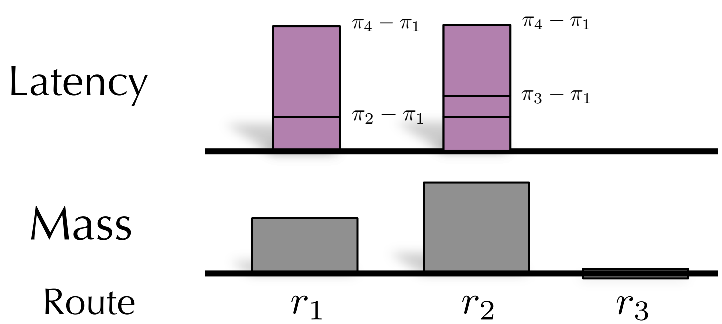


$$r_1 : \qquad l_1^R = \pi_4 - \pi_1$$

$$r_2 : l_1 + l_3 + l_5 = \pi_4 - \pi_1 + \cancelto{0}{\nu_1} + \cancelto{0}{\nu_3} + \cancelto{0}{\nu_5}$$



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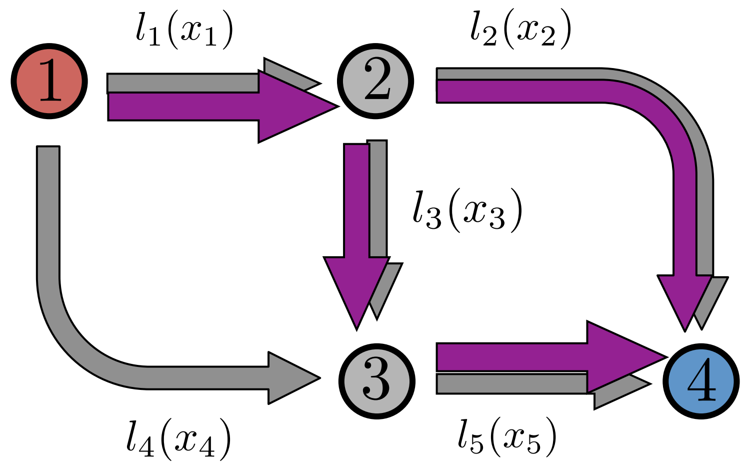

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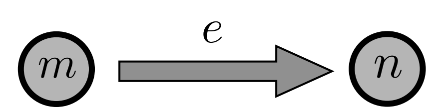

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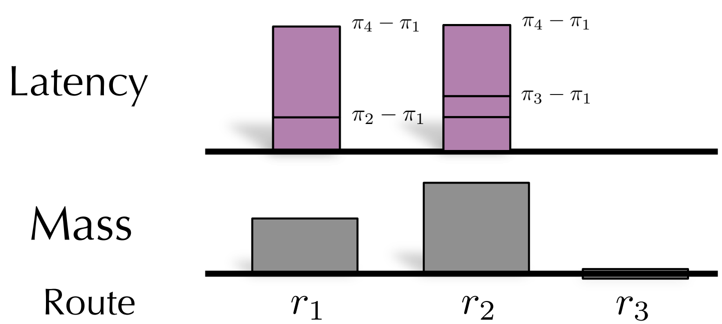

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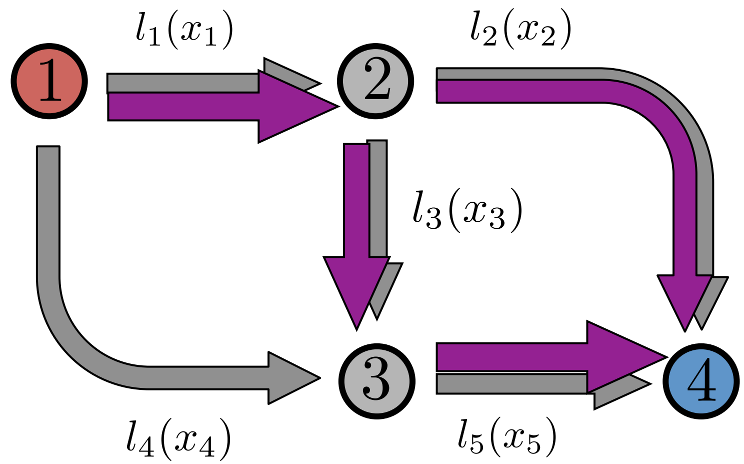

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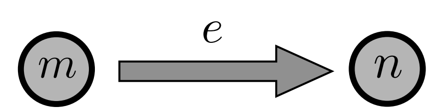

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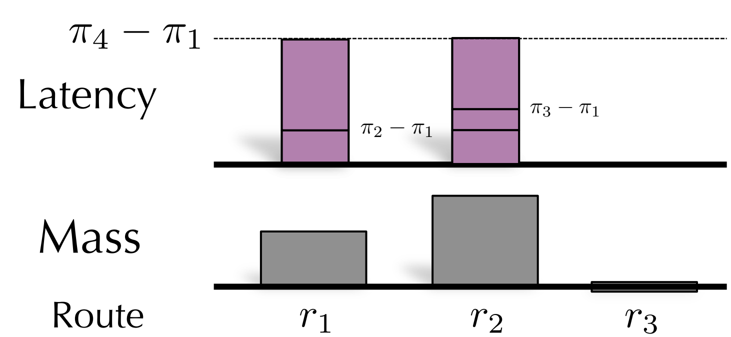

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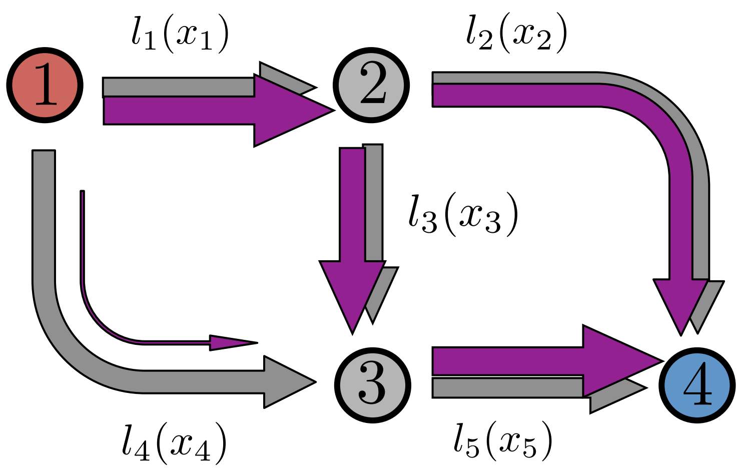

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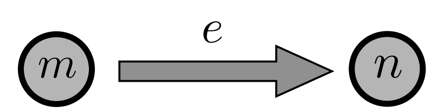

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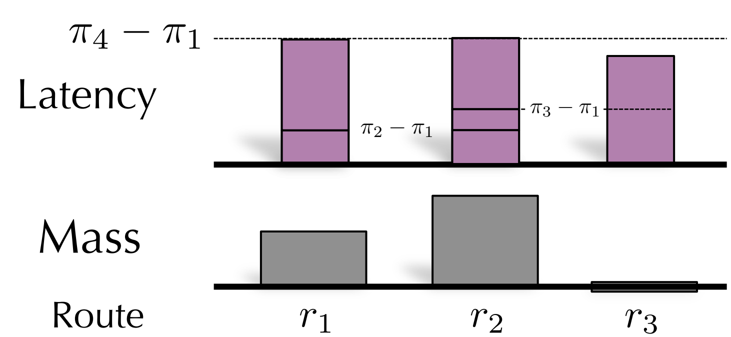
$r_3 : \quad l_4 = \pi_3 - \pi_1 + \nu_4$



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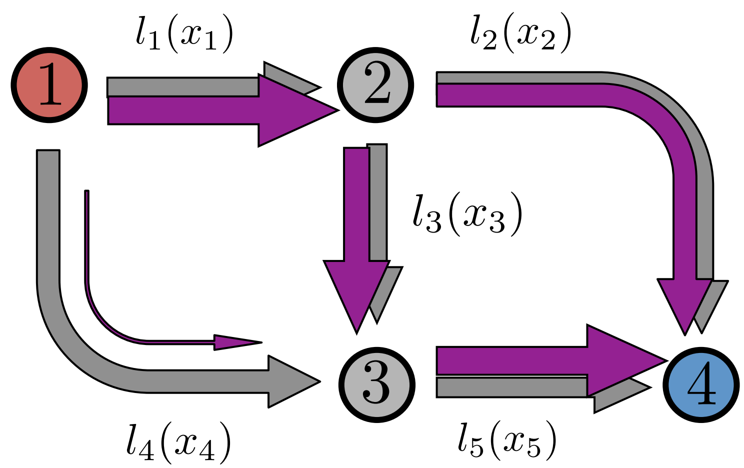

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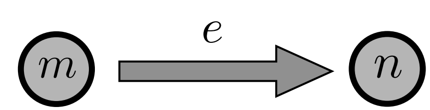

$r_1 :\quad l_1^R = \pi_4 - \pi_1$

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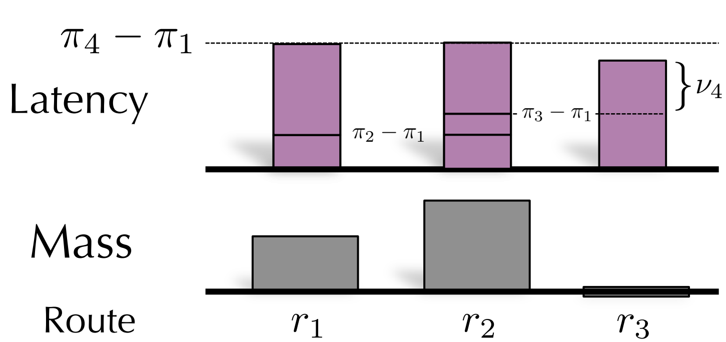
$r_3 :\quad l_4 = \pi_3 - \pi_1 + \nu_4 \neq 0$



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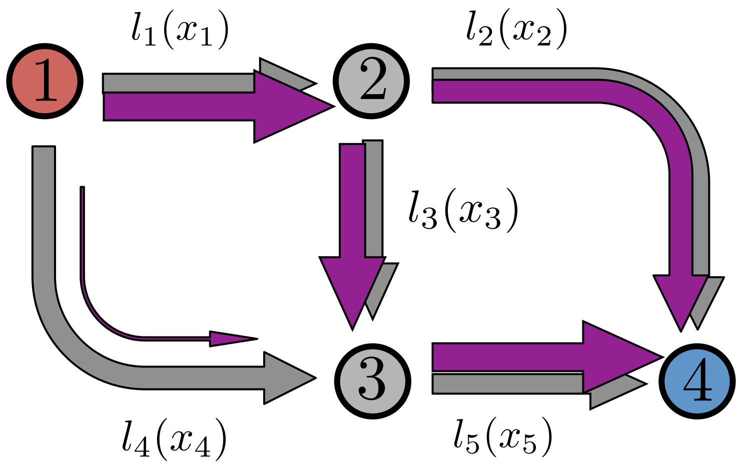

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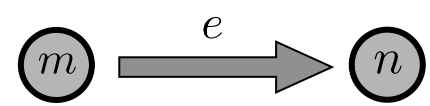

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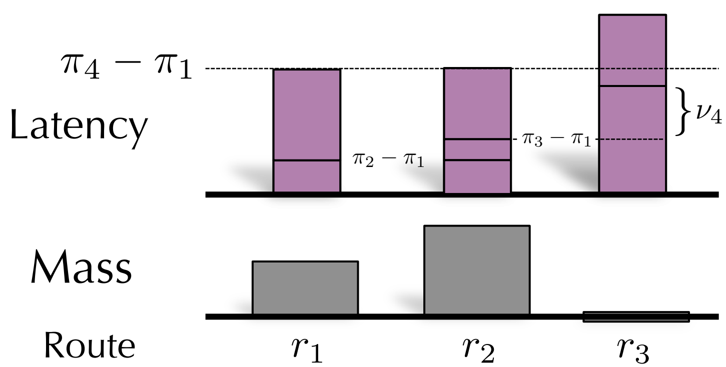

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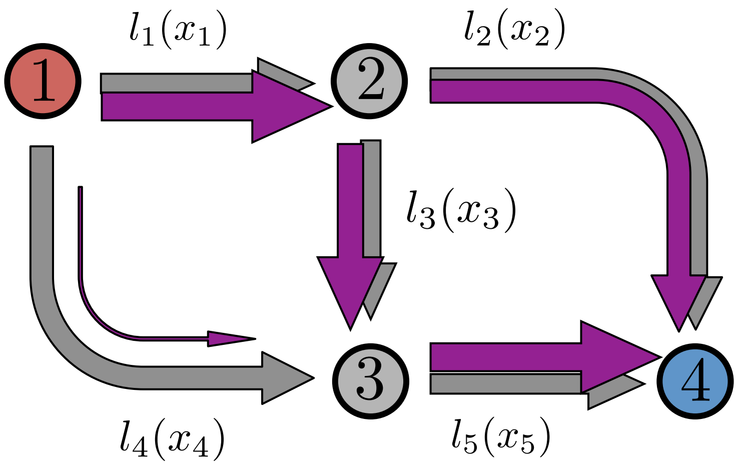

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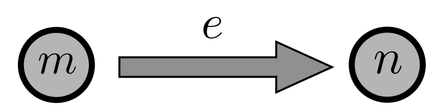

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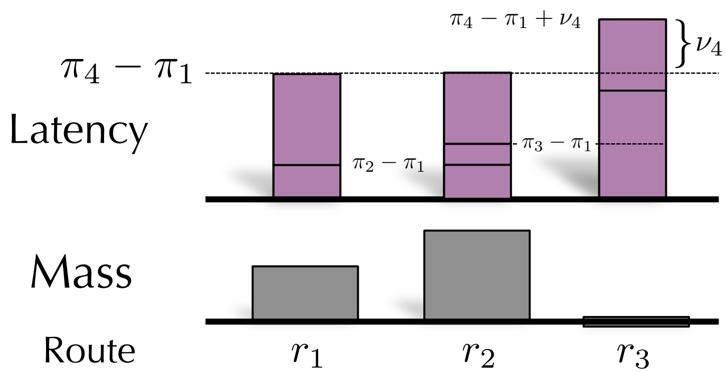

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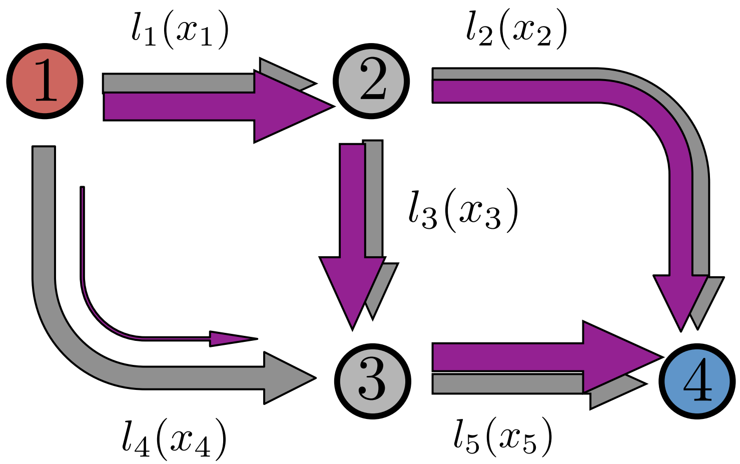

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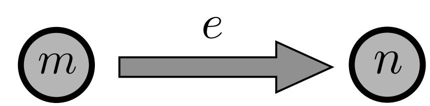

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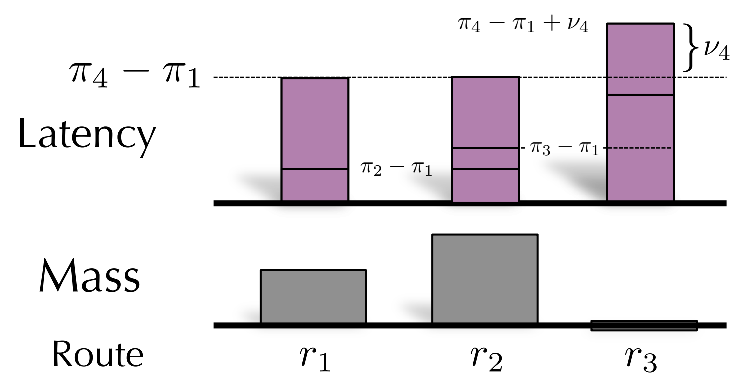

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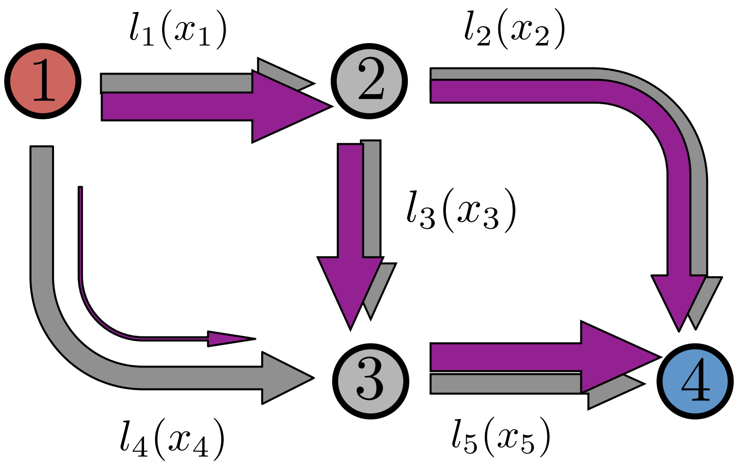

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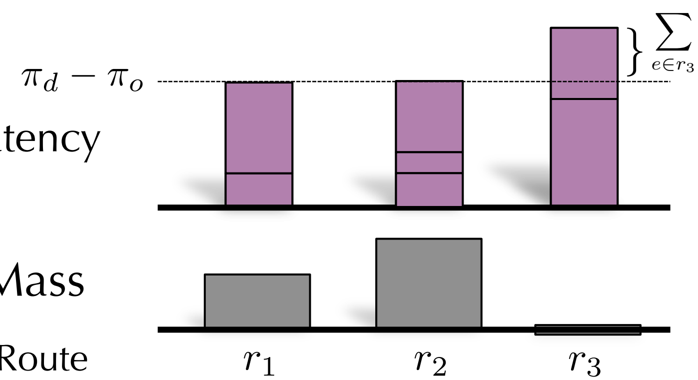
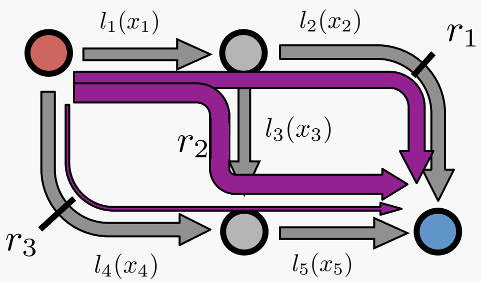
$$m \xrightarrow{e} n$$

$$l_r^R = \textstyle\sum_{e \in r} l_e(x_e)$$
$$= \pi_d - \pi_o + \textstyle\sum_{e \in r} \nu_e$$



Summary

Path Formulation



$$\min_{x^R} \; P(x) \quad \text{s.t.} \quad x^R \geq 0, \; x = \mathbb{R}x^R, \; \textstyle\sum_r x_r^R = s$$

Gradient... $\quad \nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

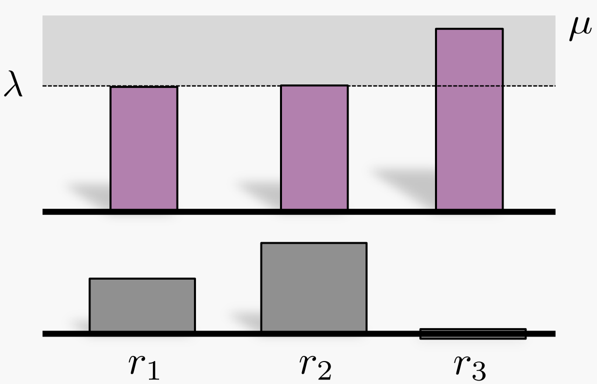
$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

$$\frac{\partial \mathcal{L}}{\partial x^R} : \qquad l^R(x) = \lambda \mathbf{1}^T + \mu^T$$

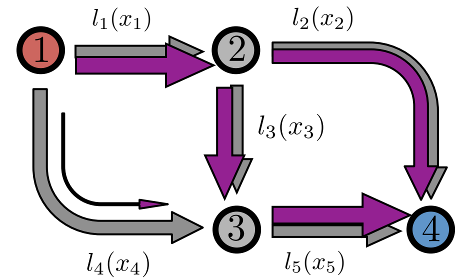
Latency

Mass

Route



Edge Formulation



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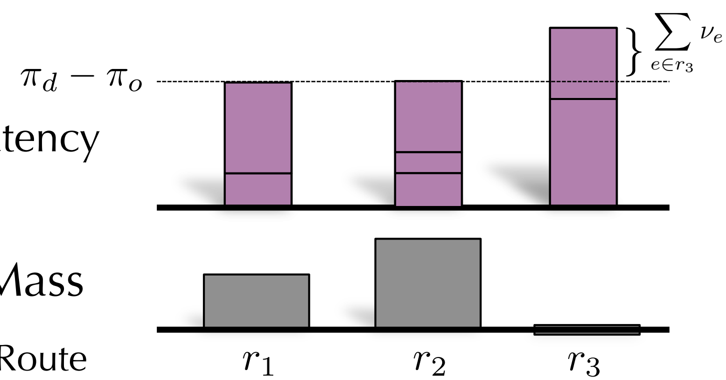
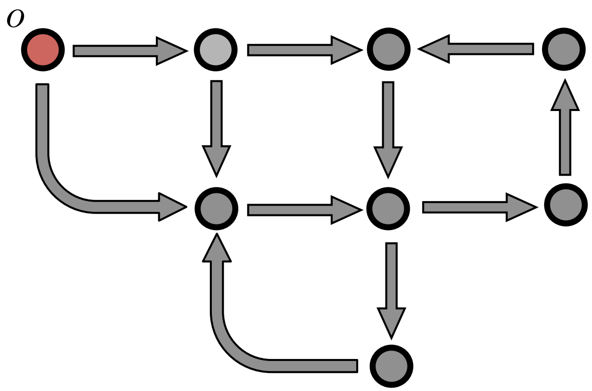
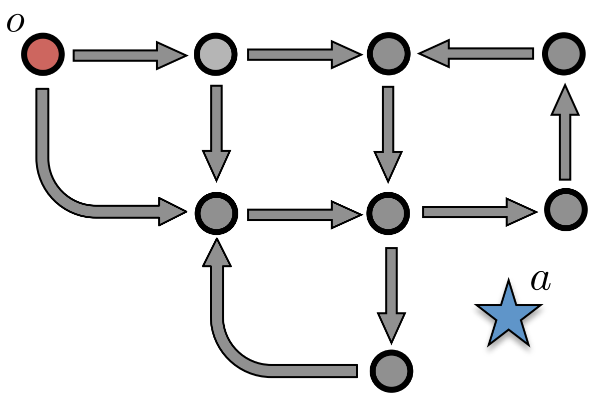


Parking Traffic Problem

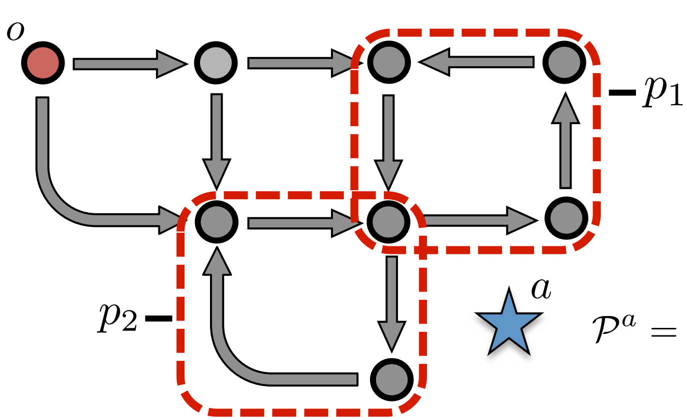
Edge Formulation



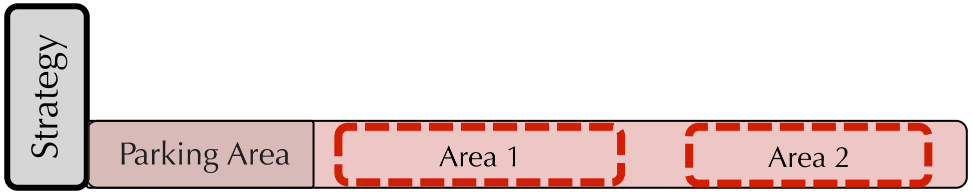
Edge Formulation



Edge Formulation



$\mathcal{P}^a = \{p_1, p_2\}$



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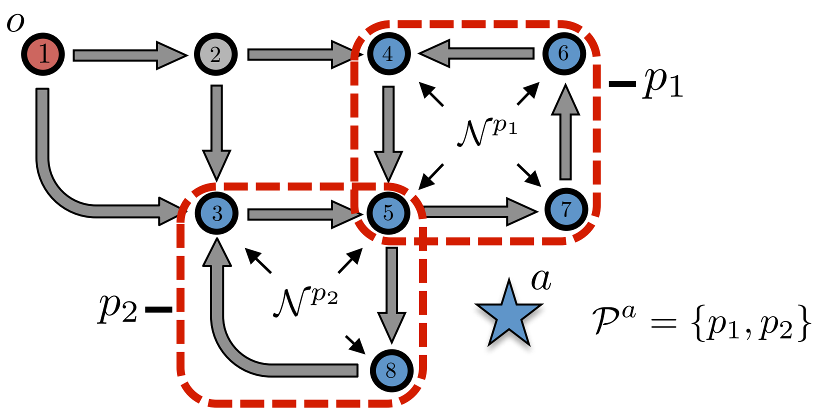

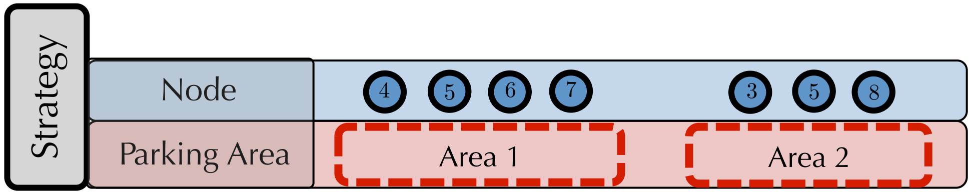

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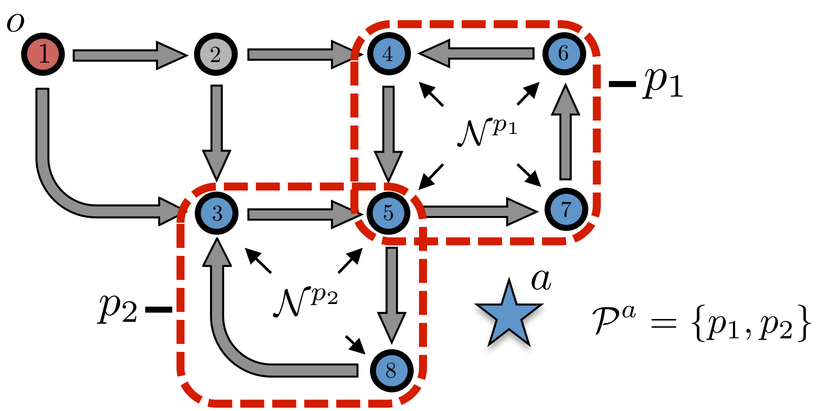

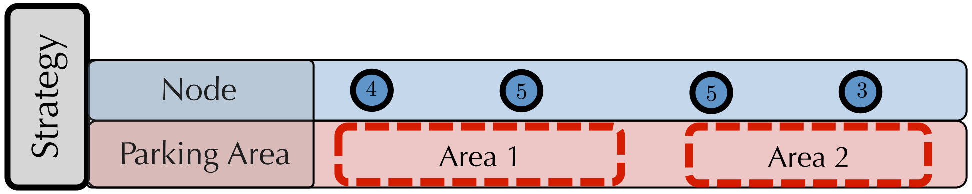

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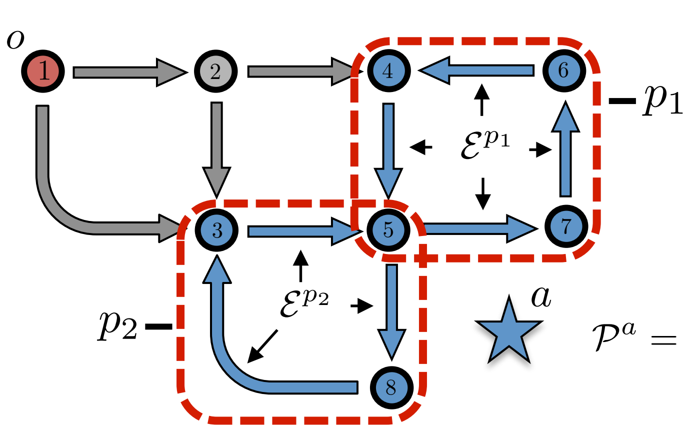

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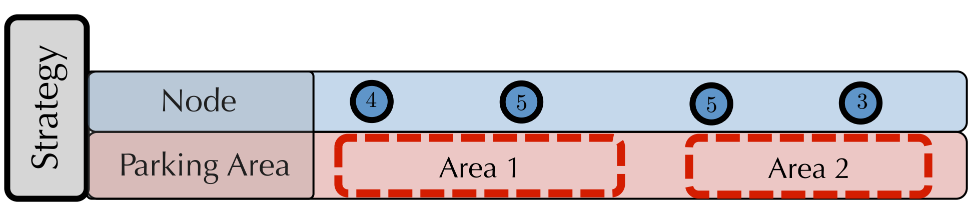
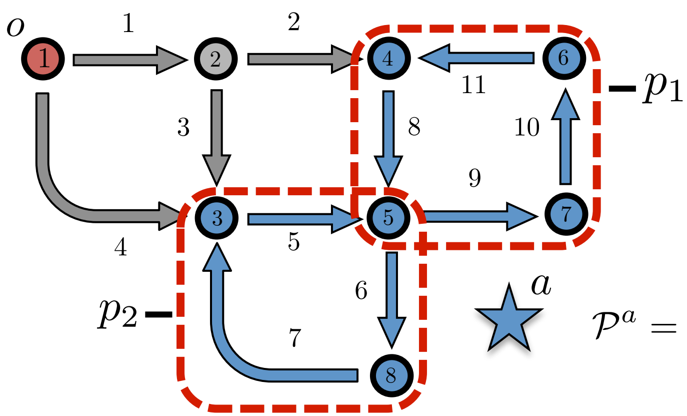
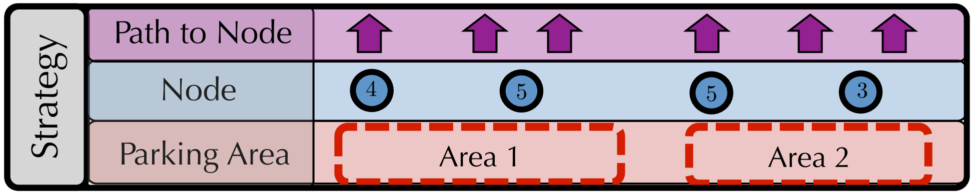


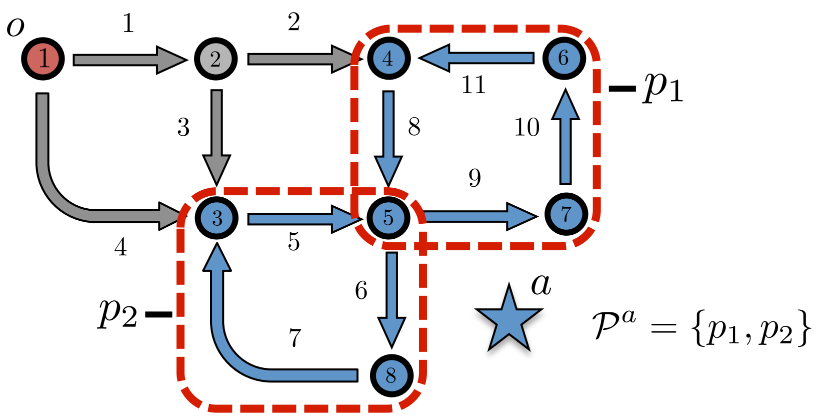
Edge Formulation

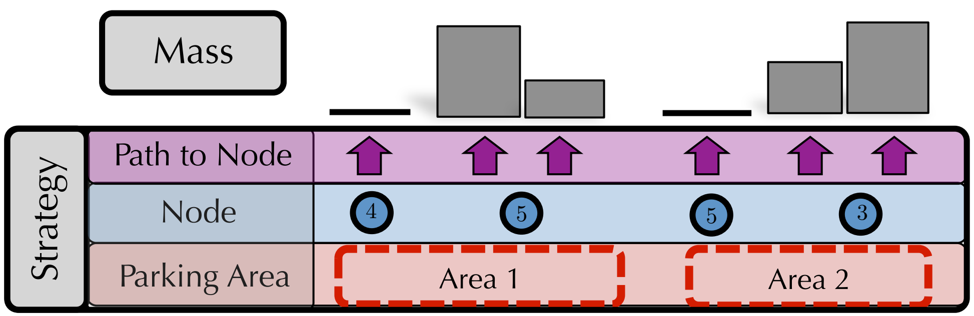


$$\mathcal{P}^a = \{p_1, p_2\}$$

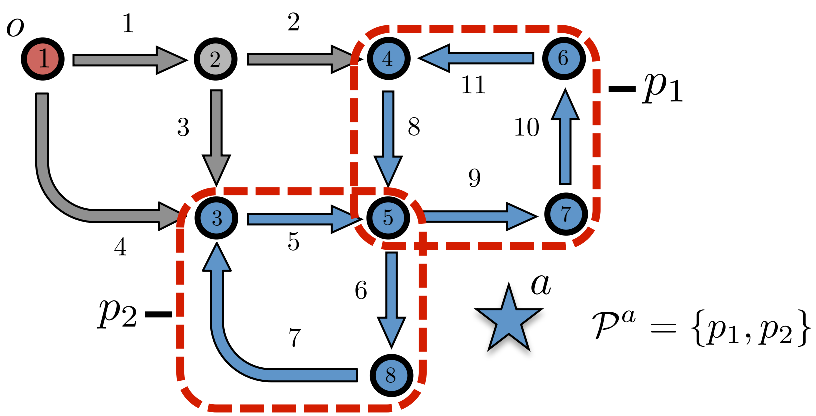


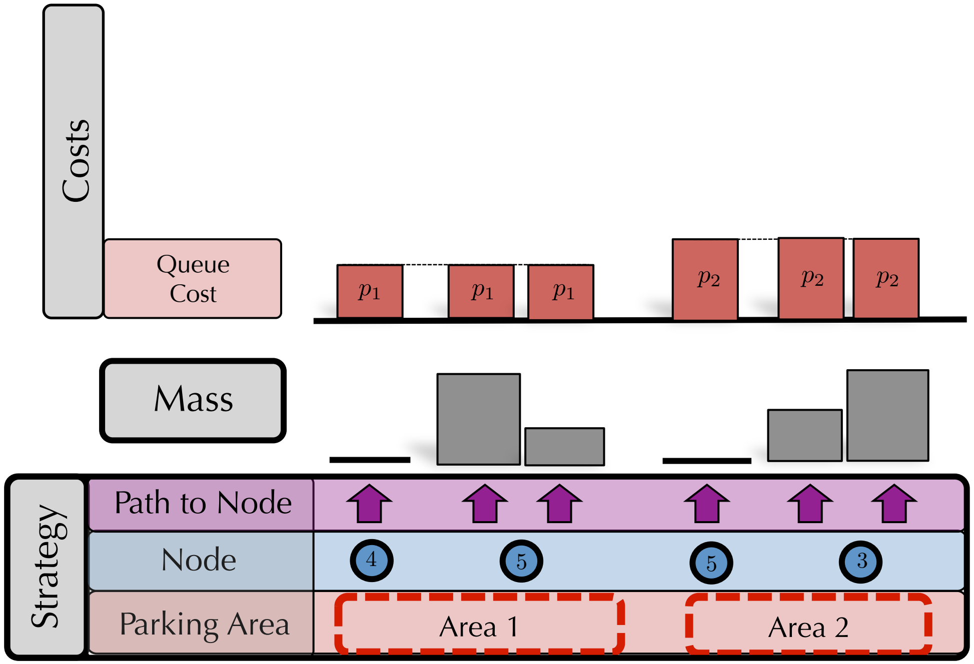
Edge Formulation



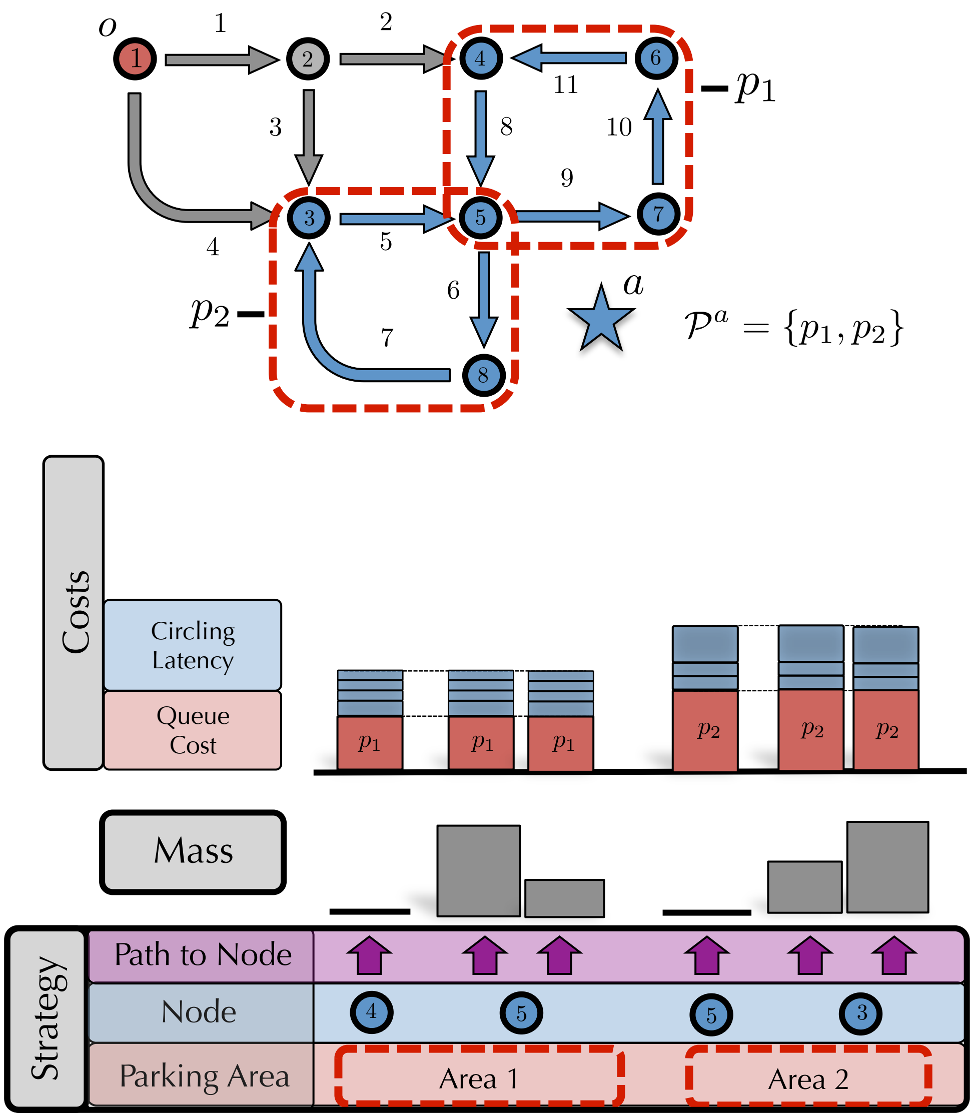


Edge Formulation

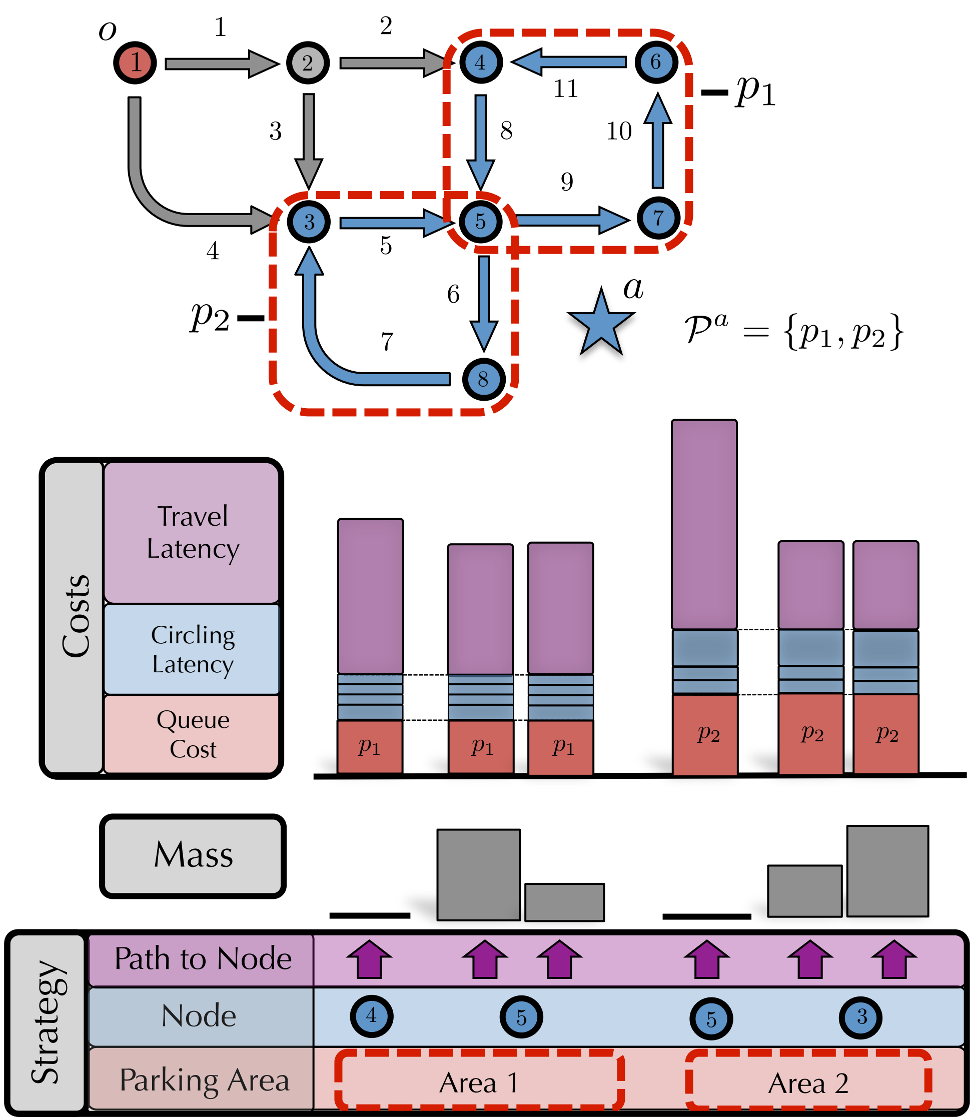




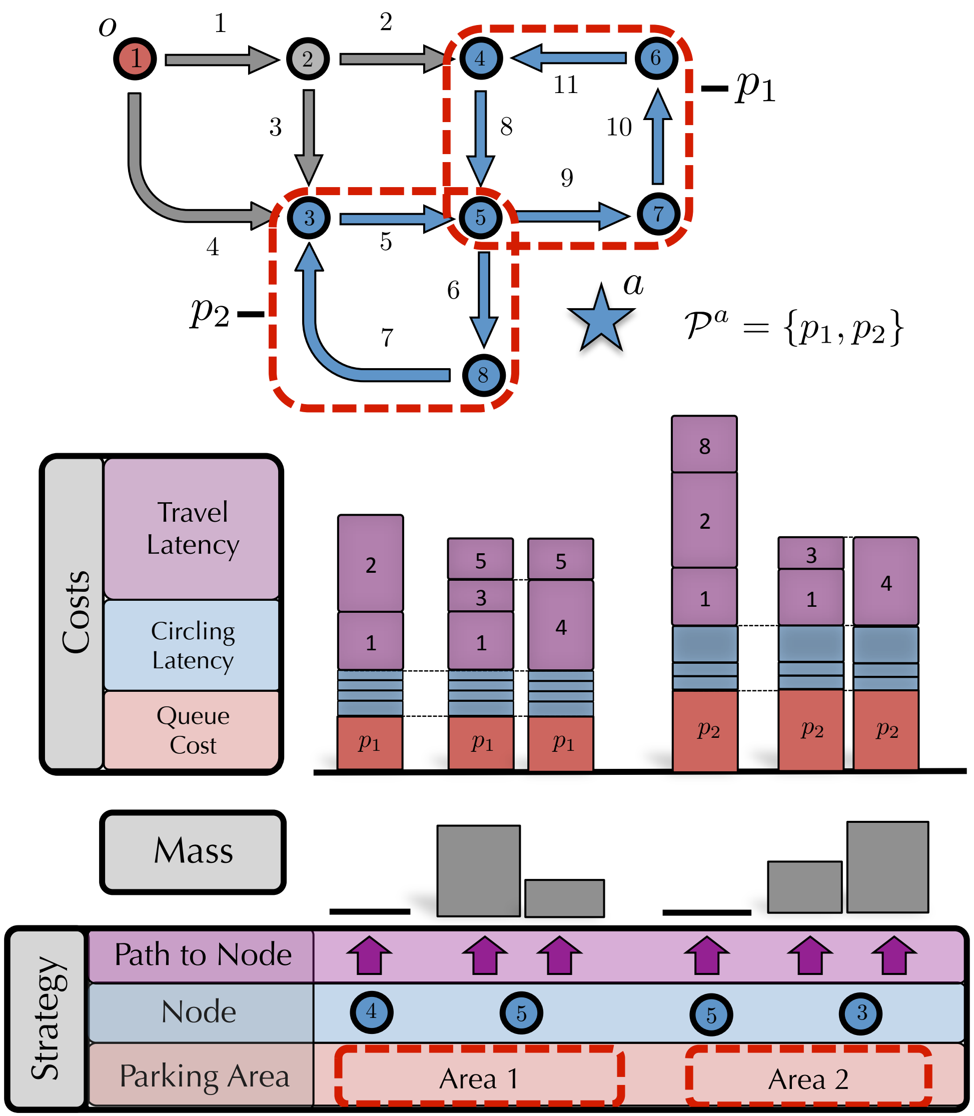
Edge Formulation



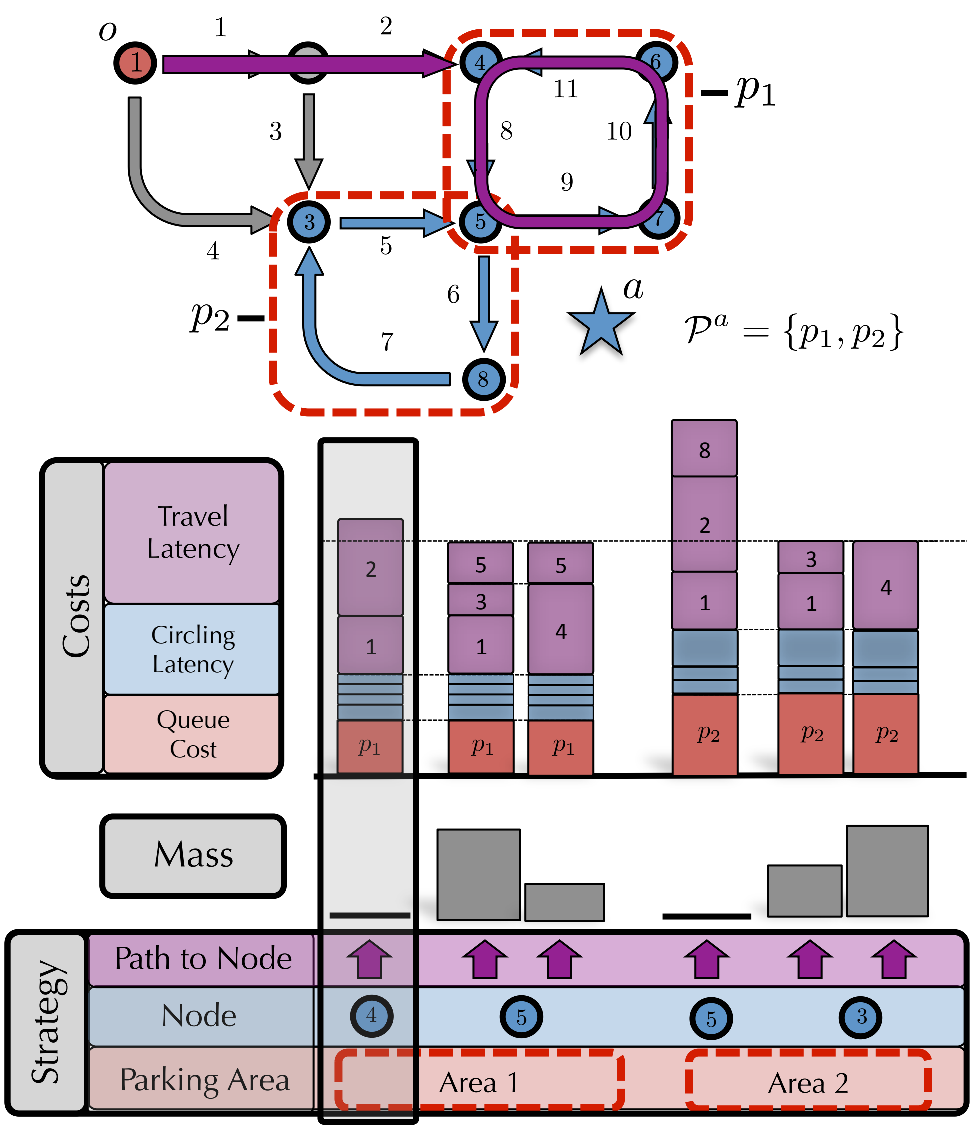
Edge Formulation



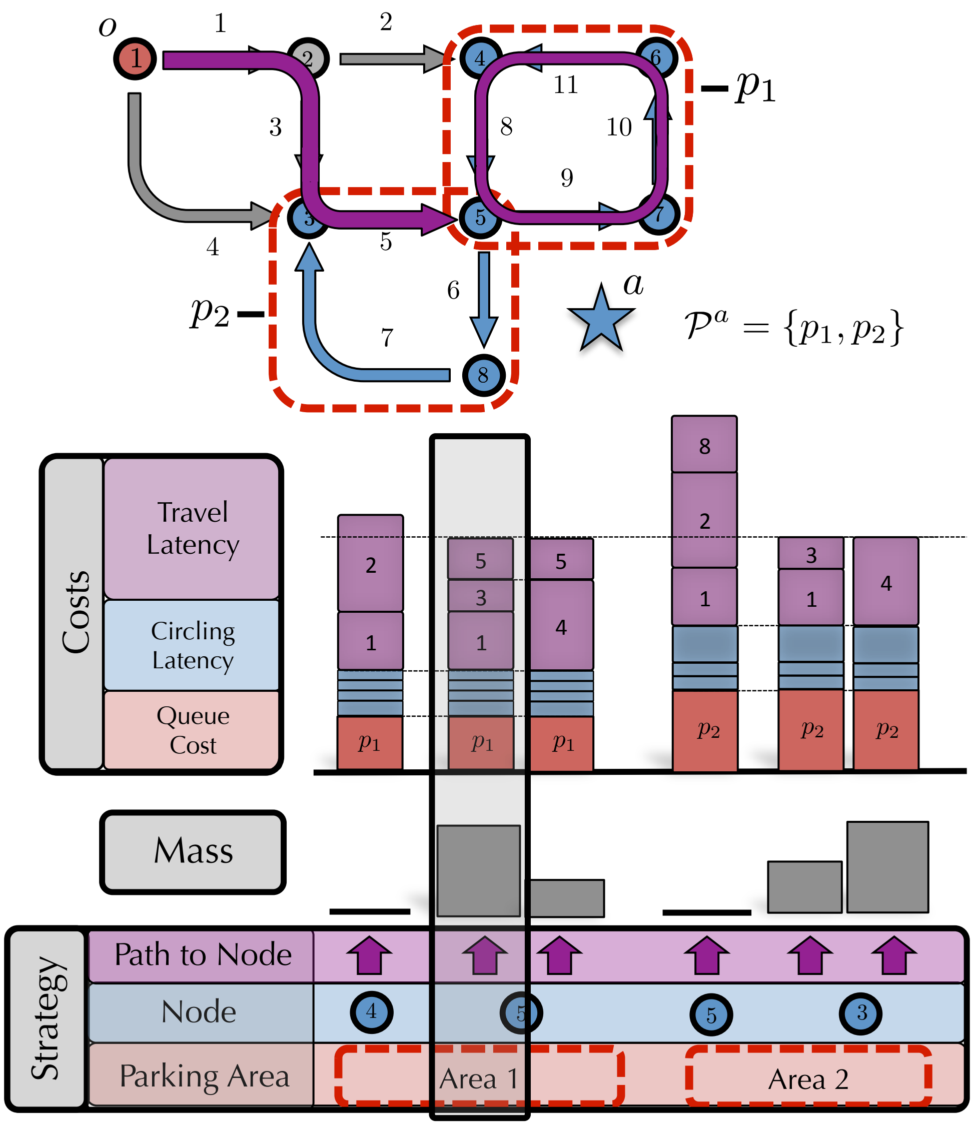
Edge Formulation



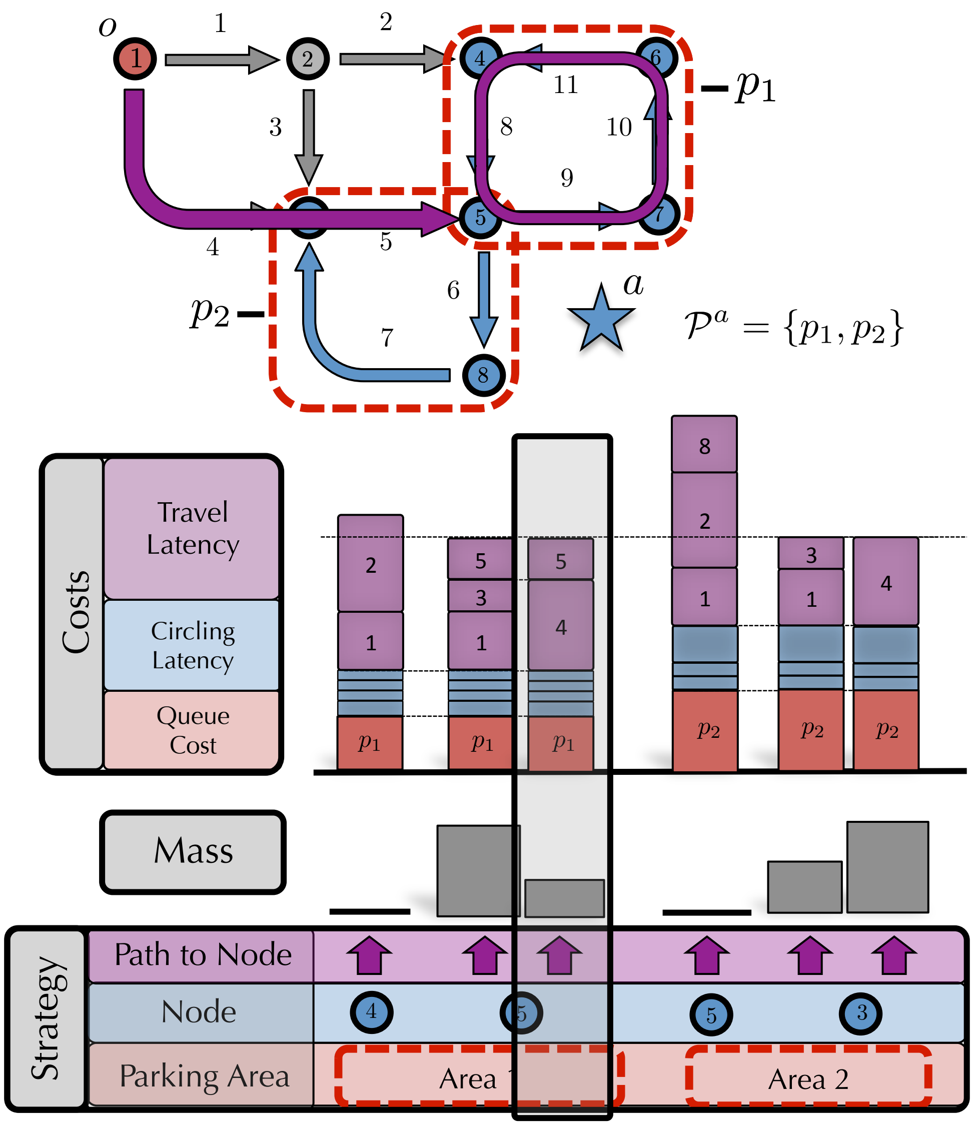
Edge Formulation



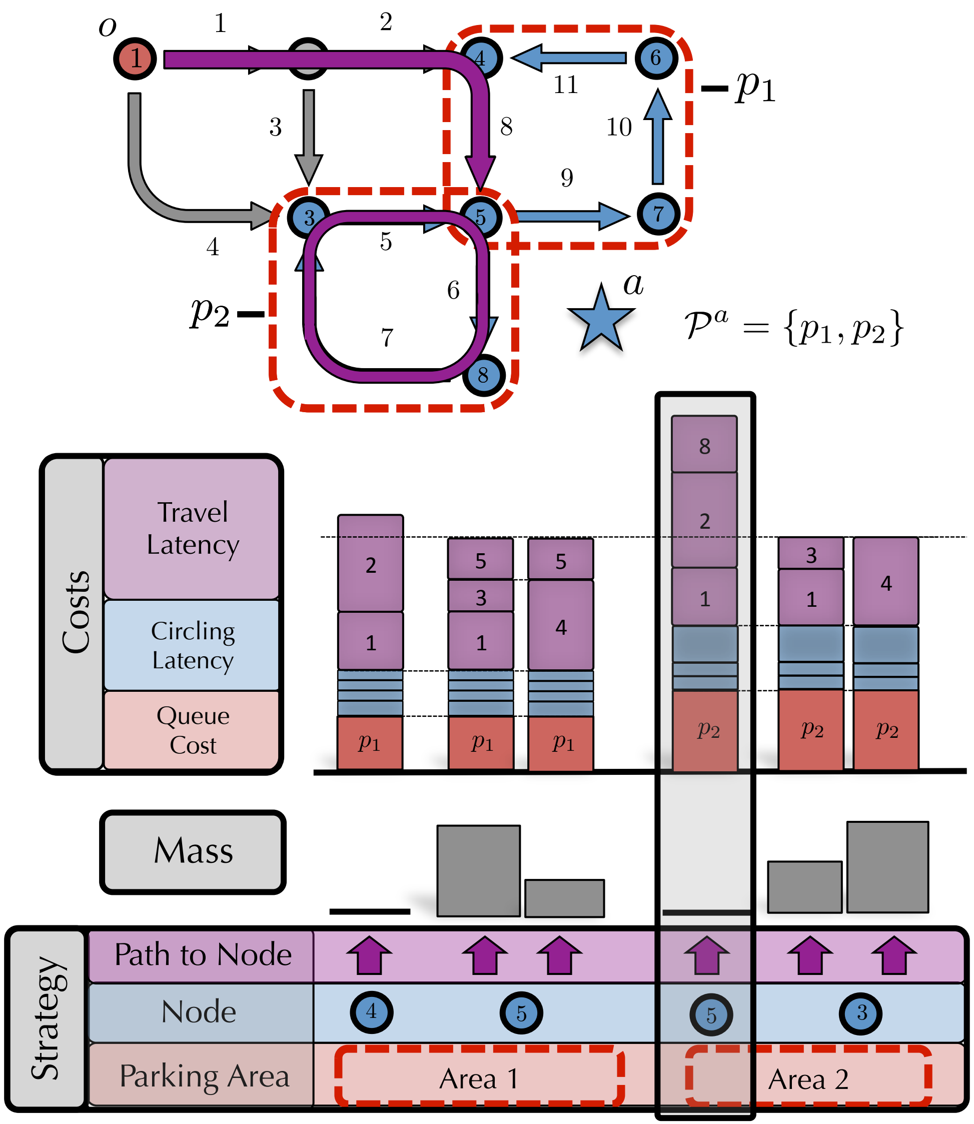
Edge Formulation



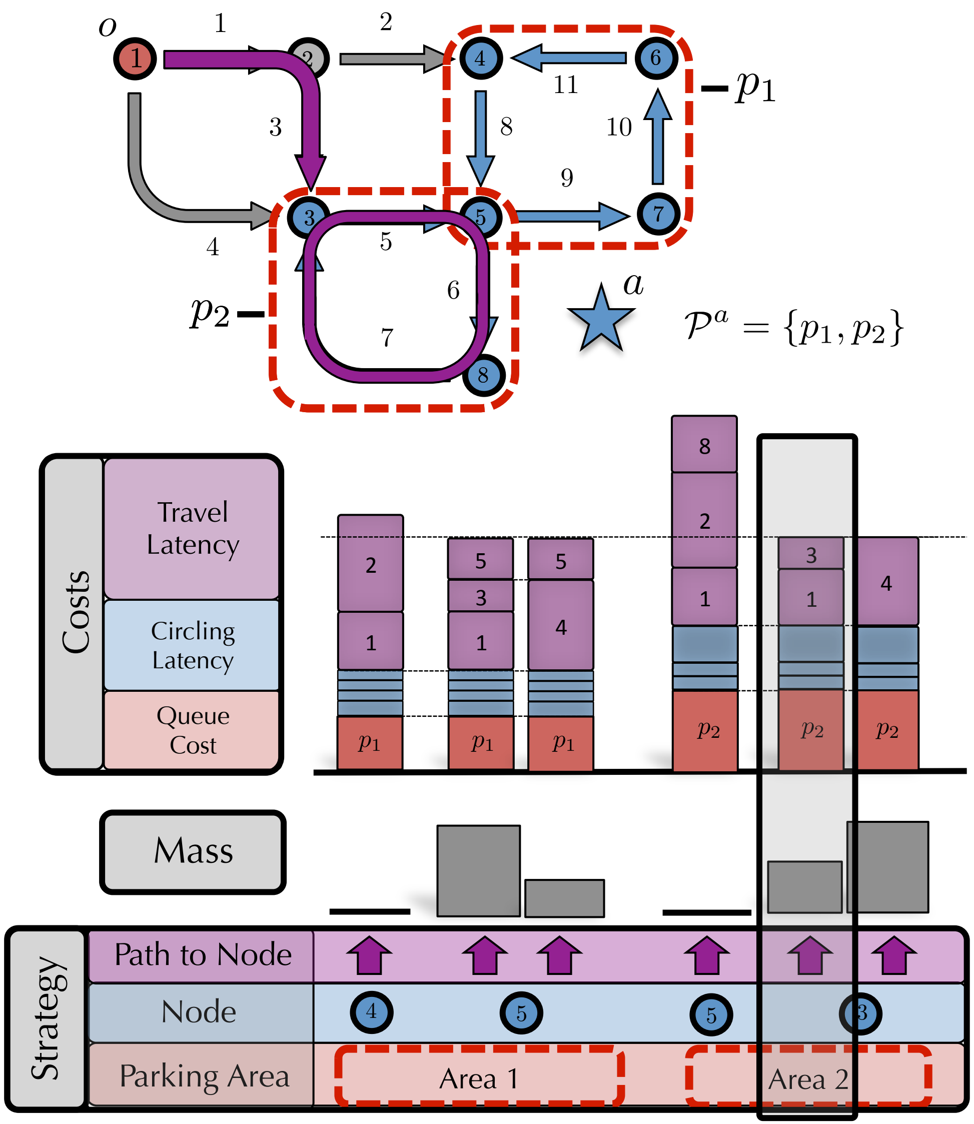
Edge Formulation



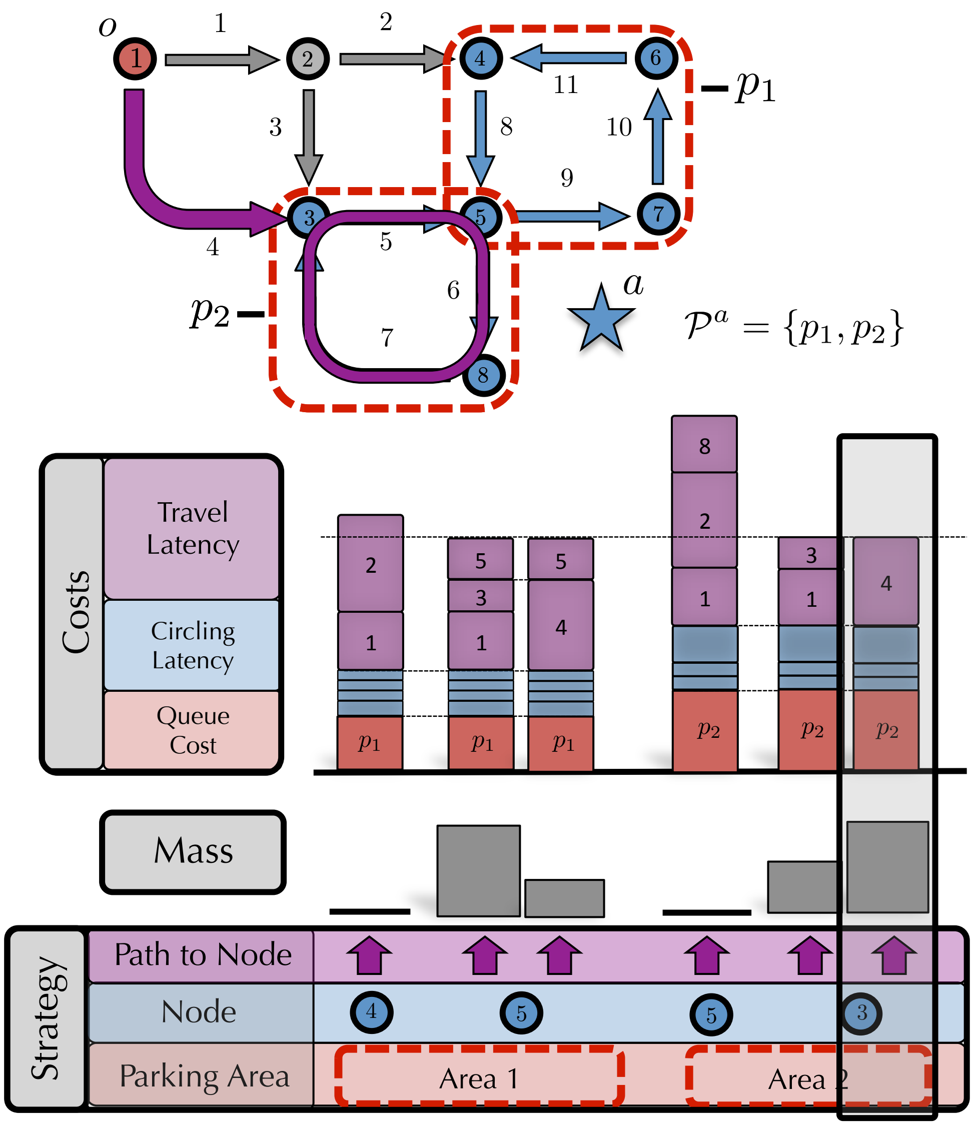
Edge Formulation



Edge Formulation



Edge Formulation



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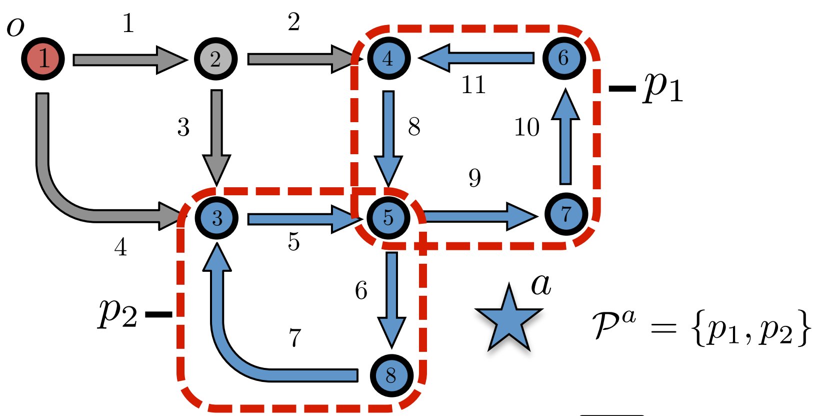
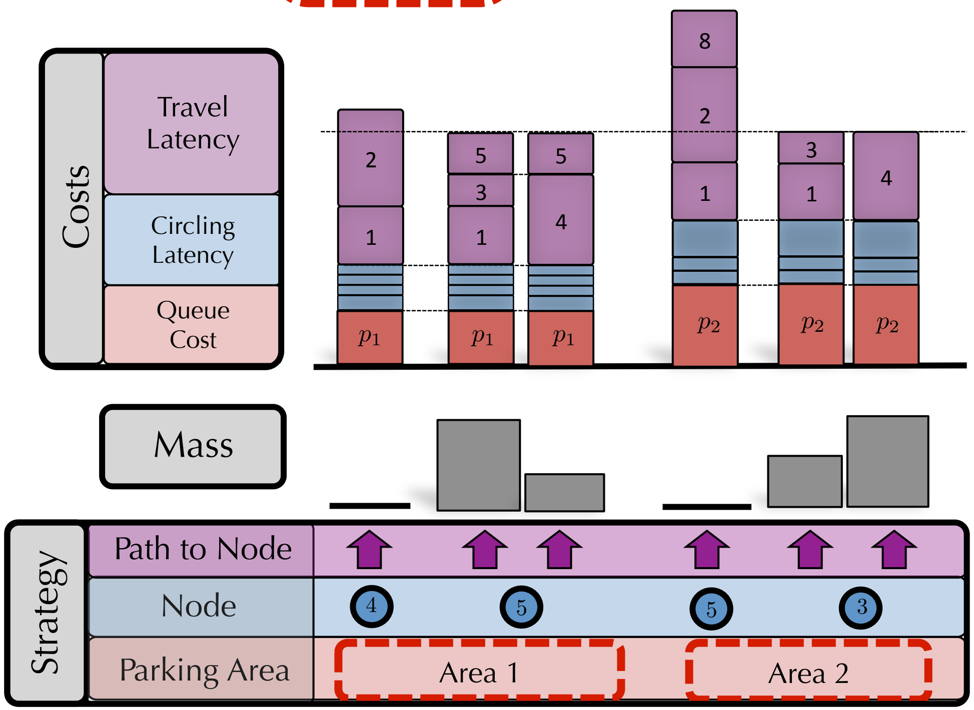


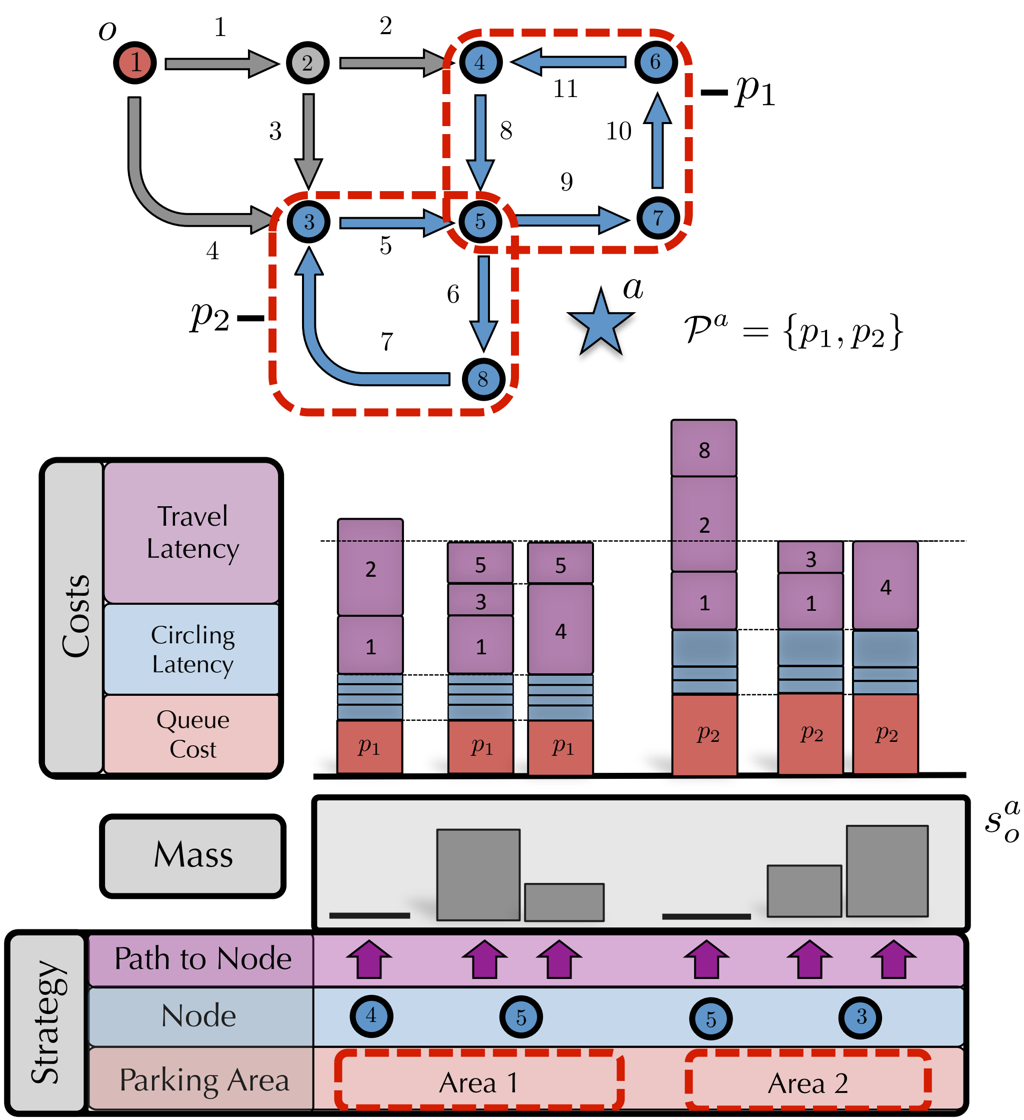


Equilibrium Condition

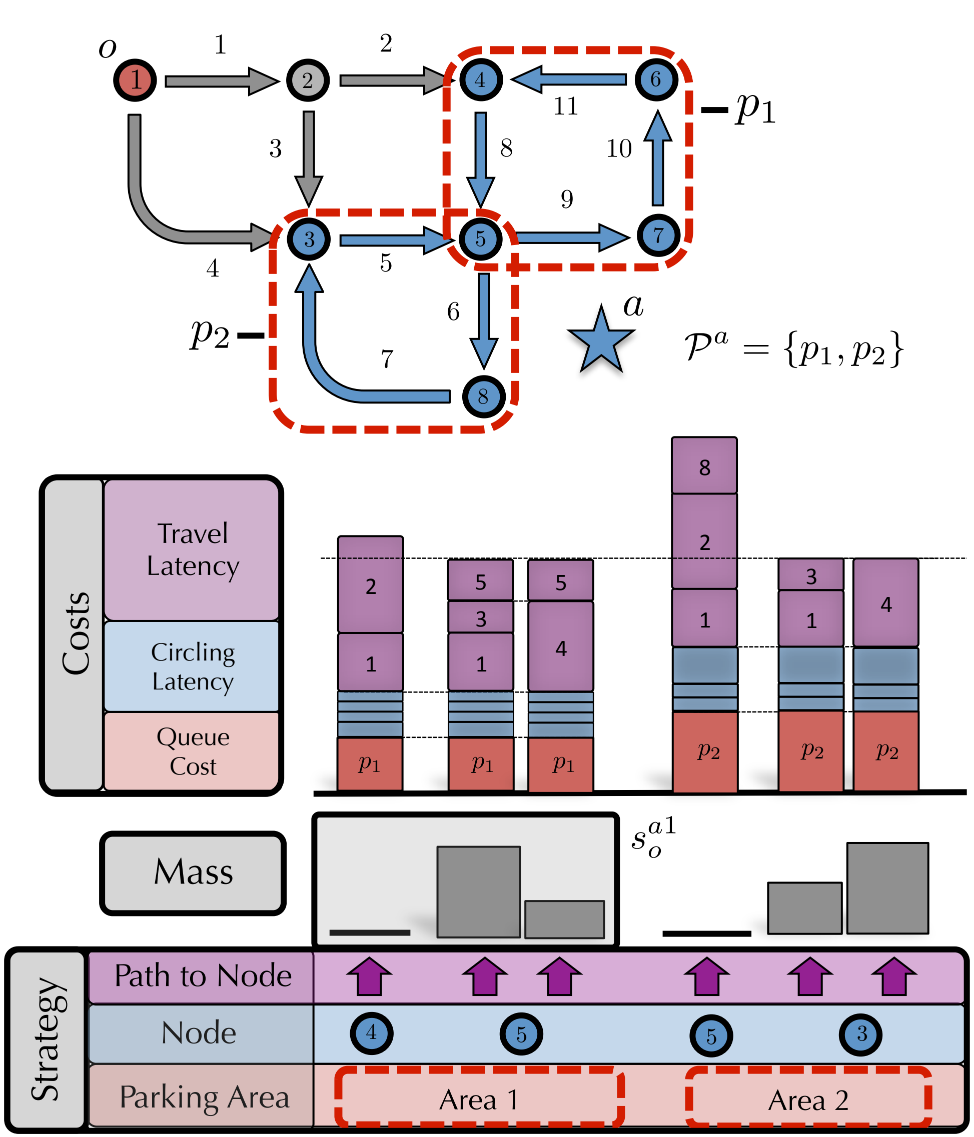
Queue-Routing Wardrop Equilibrium

Edge Formulation

s_o^a : Traffic from origin o to attraction a



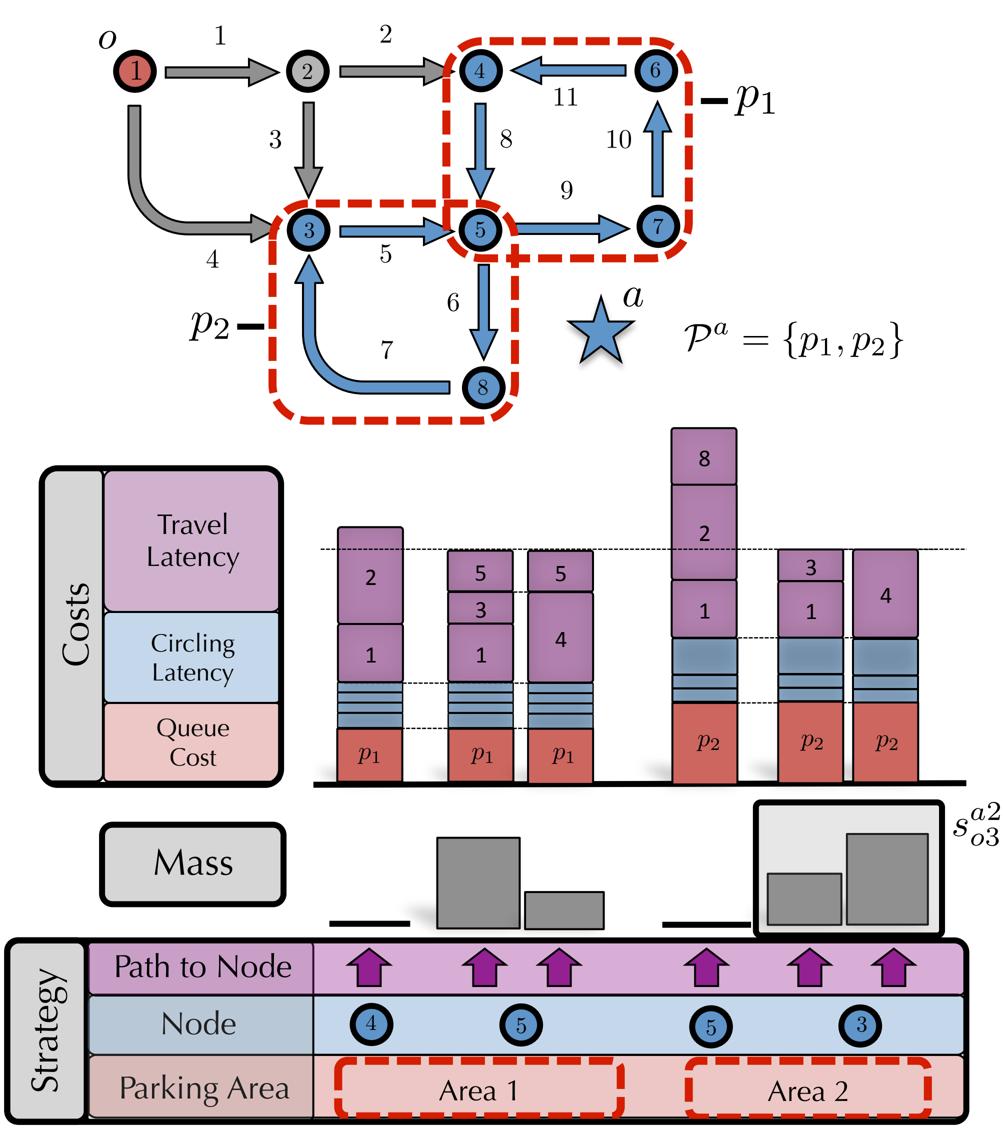
Edge Formulation



s_o^a : Traffic from origin o to attraction a

s_o^{ap} : …parking in area p $\quad s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap}$

Edge Formulation

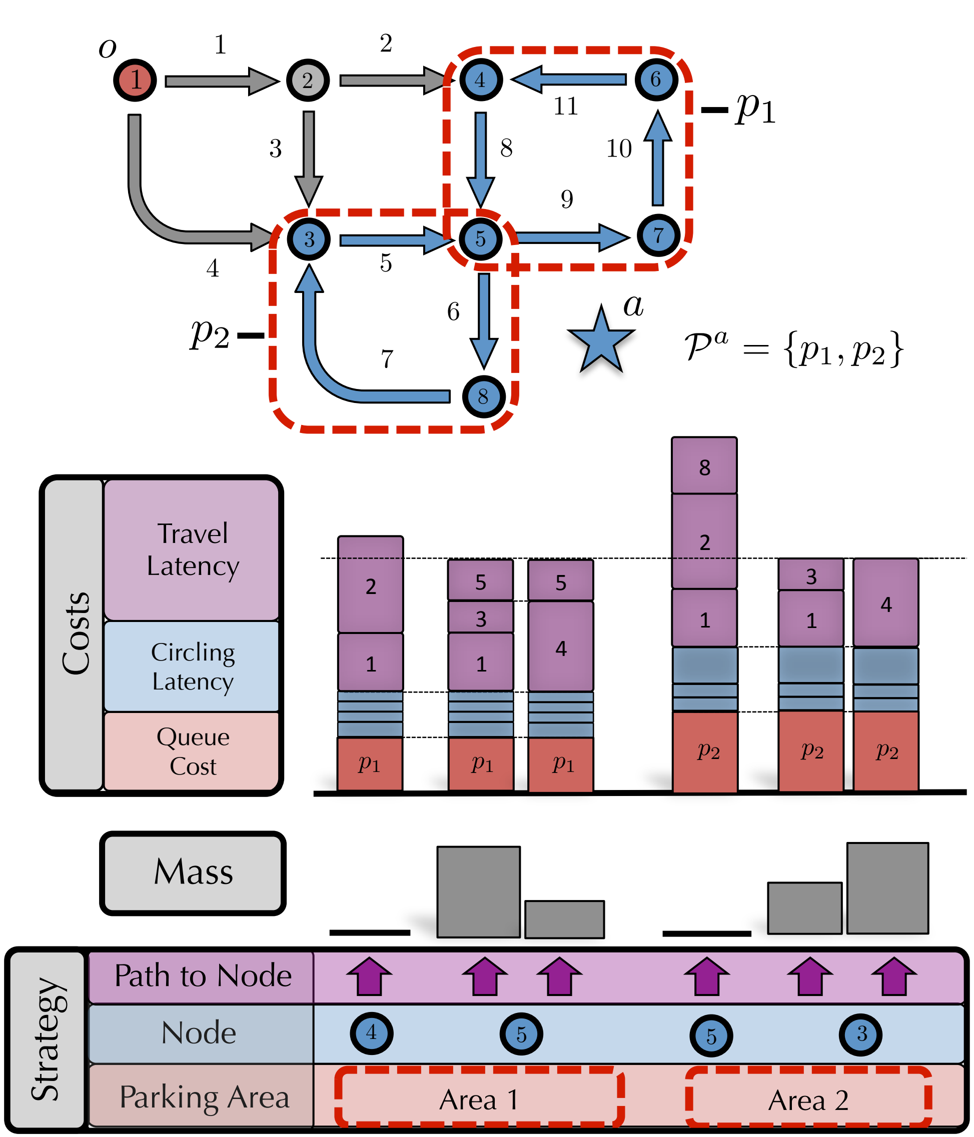


s_o^a : Traffic from origin o to attraction a

s_o^{ap} : …parking in area p $\quad s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap}$

s_{od}^{ap} : …entering thru node d $\quad s_o^{ap} = \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$

Edge Formulation



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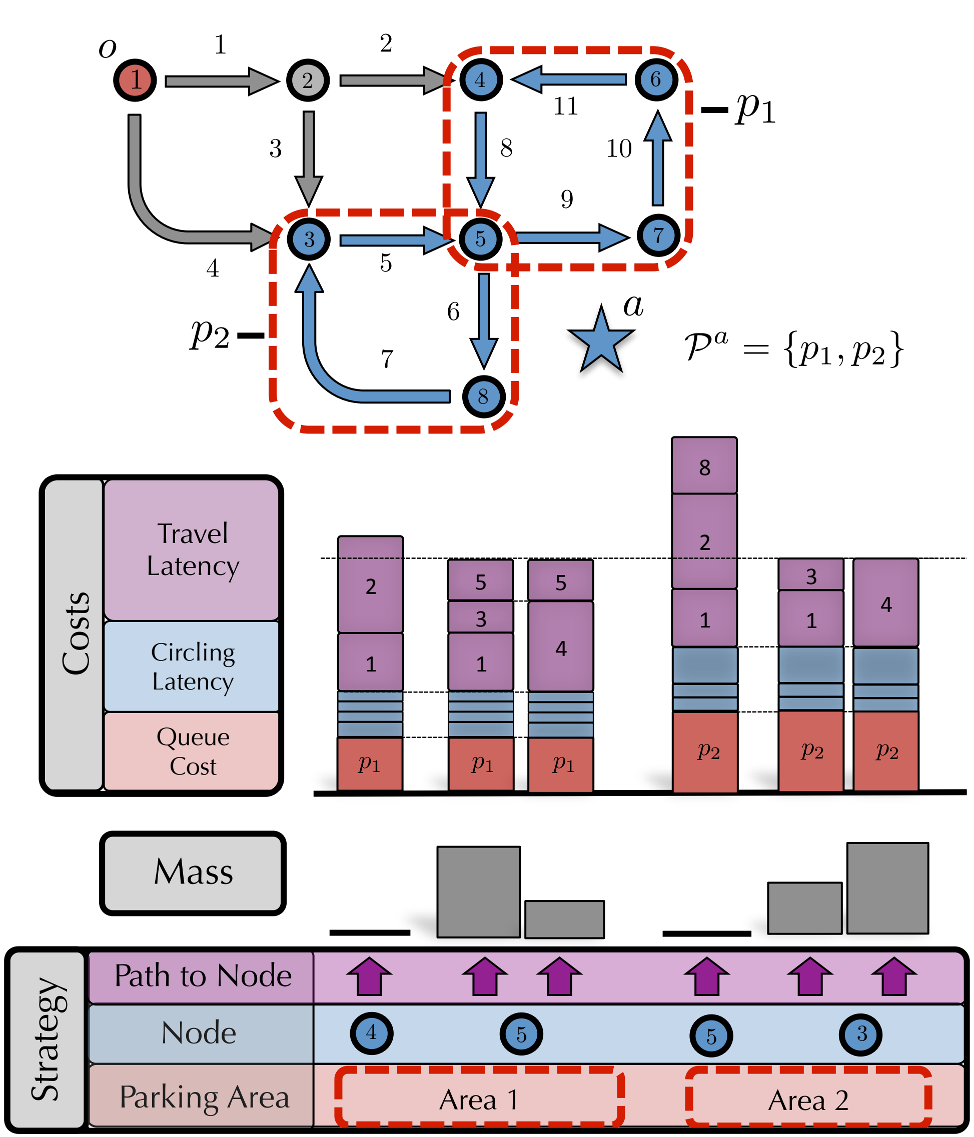
s_{od}^{ap} : …entering thru node d $\qquad s_o^{ap} = \sum_{d\in\mathcal{N}^p} s_{od}^{ap}$

s^p : Total traffic in area p $\qquad s^p = \sum_{o,a} \sum_{d\in\mathcal{N}^p} s_{od}^{ap}$

Edge Formulation



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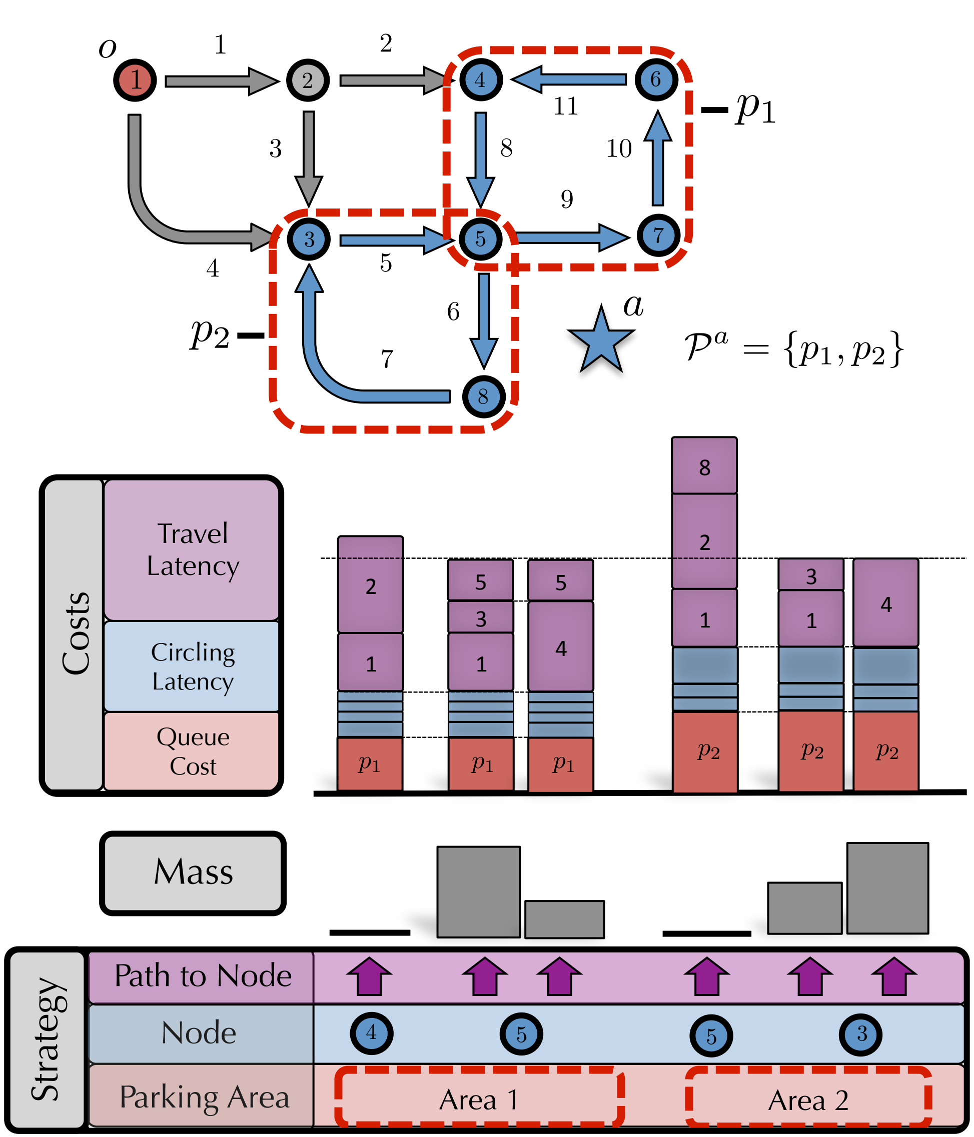
---

x_{od}^{ap} : Edge flows from population s_{od}^{ap}

$$Gx_{od}^{ap} = S_{od}^{ap} \qquad (S_{od}^{ap})_i = \begin{cases} s_{od}^{ap} & ;\ \text{if } i = o \\ -s_{od}^{ap} & ;\ \text{if } i = d \\ 0 & ;\ \text{otherwise} \end{cases}$$

Edge Formulation



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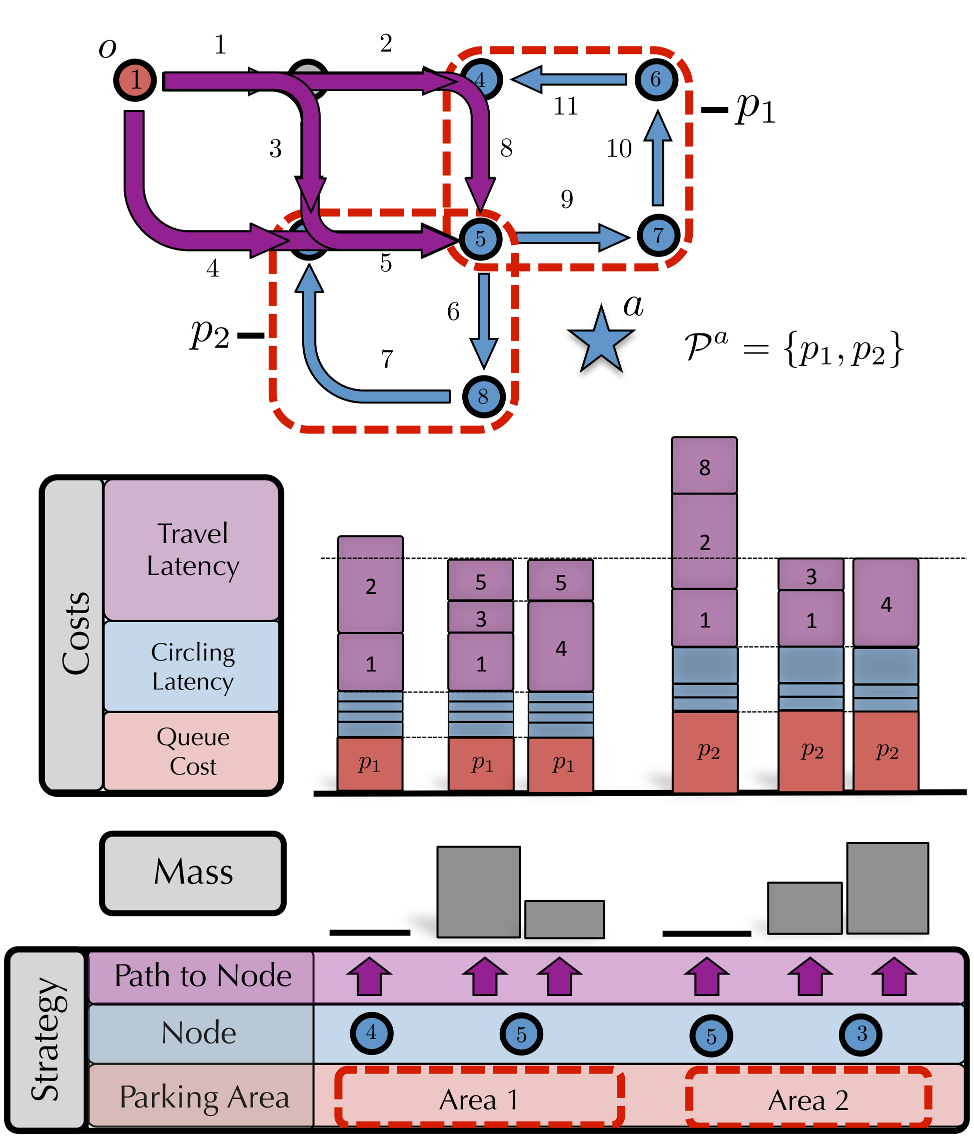
x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

Edge Formulation



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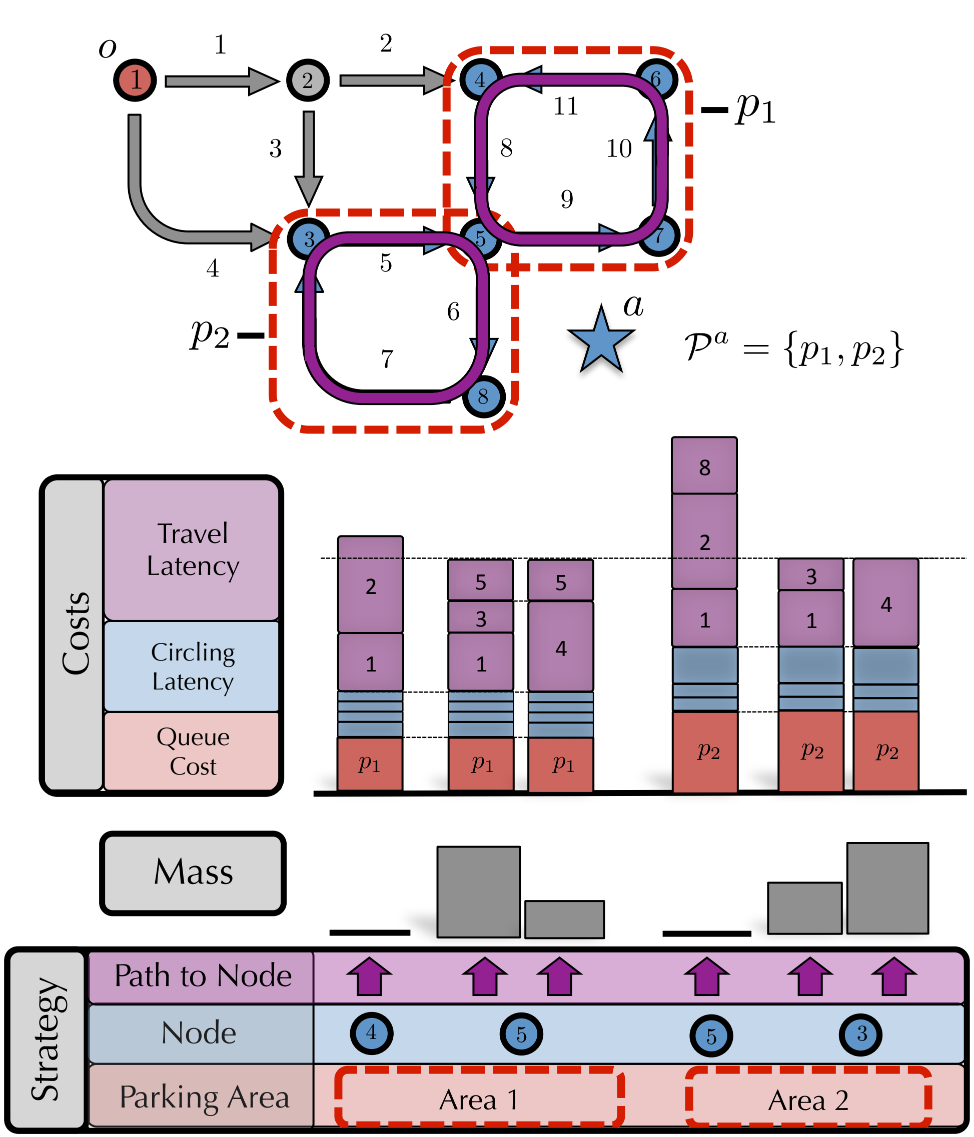
Edge Formulation



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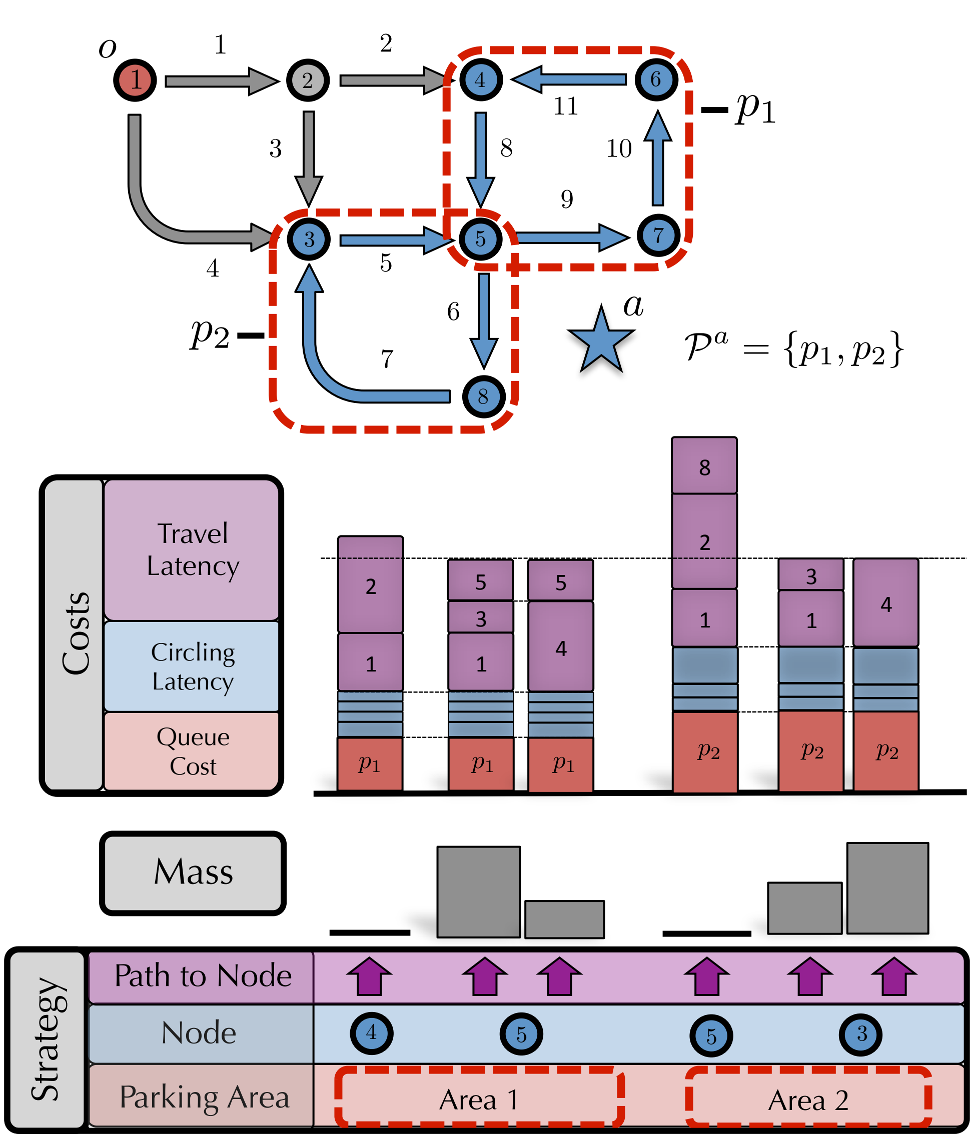
Edge Formulation



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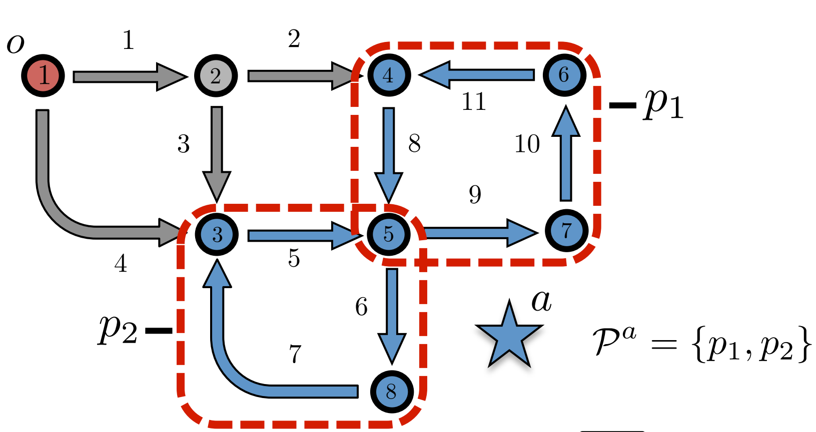
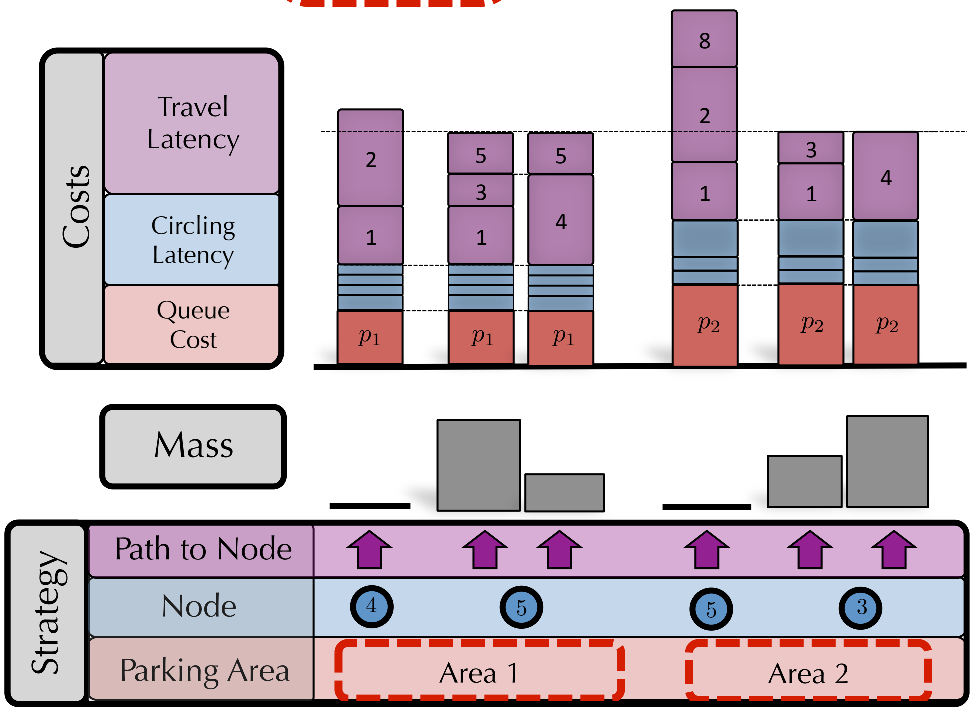
Potential Function

$$P(x) = \sum_e \int_0^{x_e} l_e(u)\ du \ + \sum_p \int_0^{s^p} -R_p + C_p u\ du$$

Edge Formulation





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x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

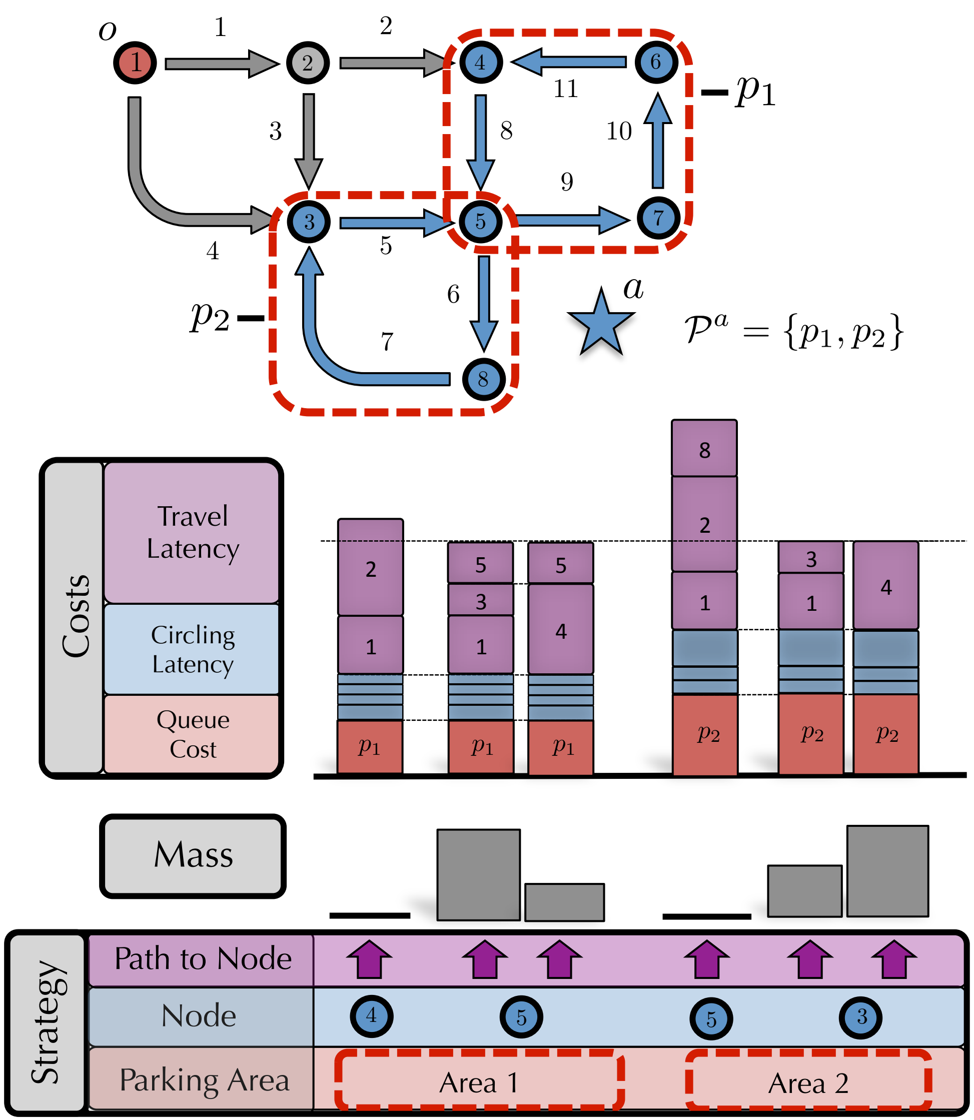
Optimization Problem

$$\min_{x_{od}^{ap},\, s_{od}^{ap}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u\, du$$

$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad G x_{od}^{ap} = S_{od}^{ap} \qquad \forall\, o, d, a, p$$

$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \qquad \forall\, o, d, a, p$$

Edge Formulation



$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p\left[\sum_{d\in\mathcal{N}^p} s_{od}^{ap}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap},\, s_{od}^{ap}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u\, du$$

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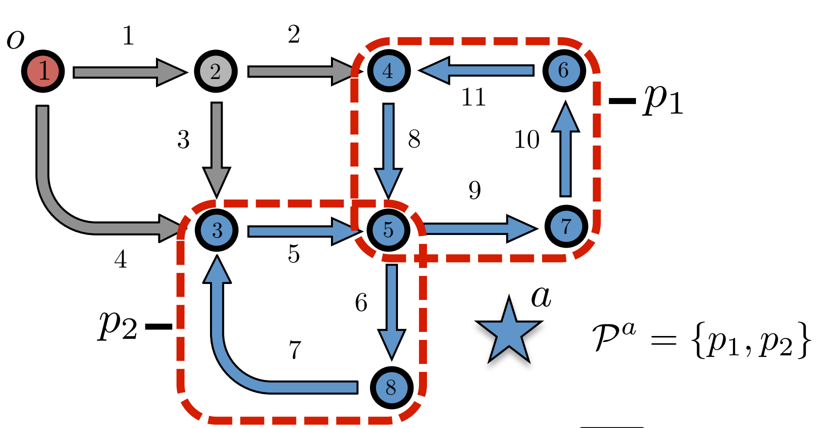
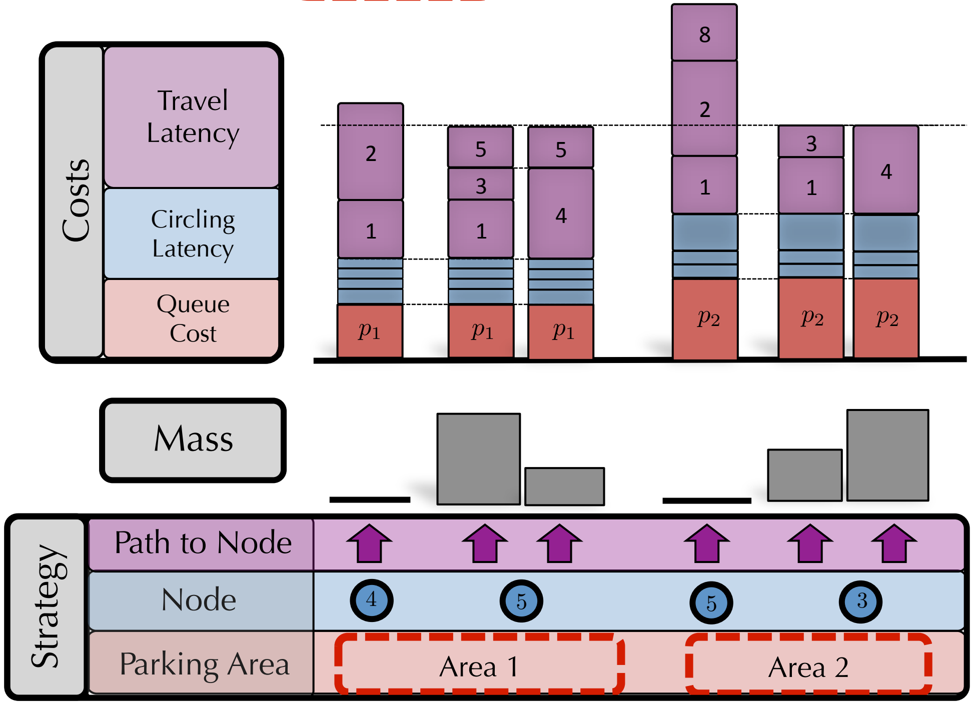
Edge Formulation





$$x = \sum_{o,a,p,d} x^{ap}_{od} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p\left[\sum_{d\in\mathcal{N}^p} s^{ap}_{od}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s^{ap}_{od}$$

Optimization Problem

$$\min_{x^{ap}_{od},\, s^{ap}_{od}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u\, du$$

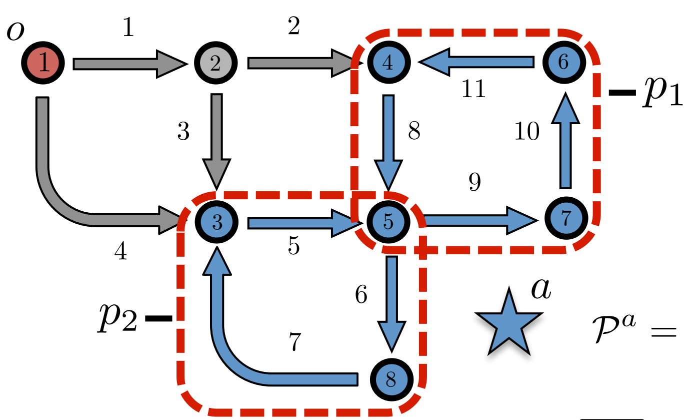
$$\text{s.t.} \quad x^{ap}_{od} \geq 0, \quad G x^{ap}_{od} = S^{ap}_{od} \qquad \forall\, o, d, a, p$$

$$s^{ap}_{od} \geq 0, \quad \sum_p \sum_{d\in\mathcal{N}^p} s^{ap}_{od} = s^a_o \qquad \forall\, o, d, a, p$$

Gradient… $\nabla_{x^{ap}_{od}} P = l(x)$

Travel Latency

Edge Formulation



$$\mathcal{P}^a = \{p_1, p_2\}$$

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \qquad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap},\, s_{od}^{ap}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u \, du$$

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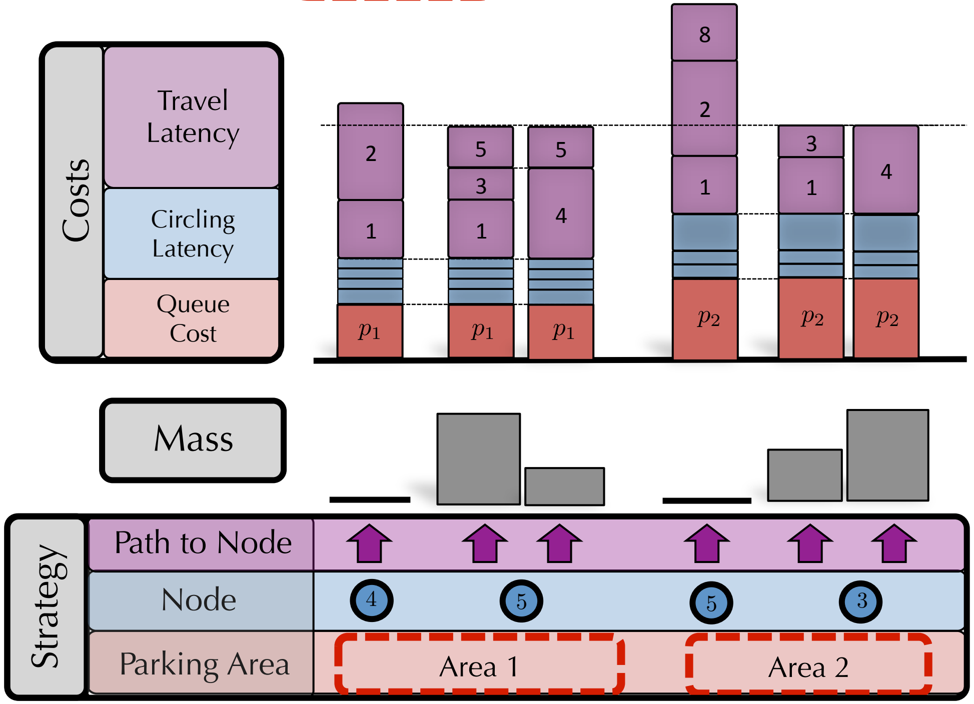
Gradient… $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + -R_p + C_p s^p$$

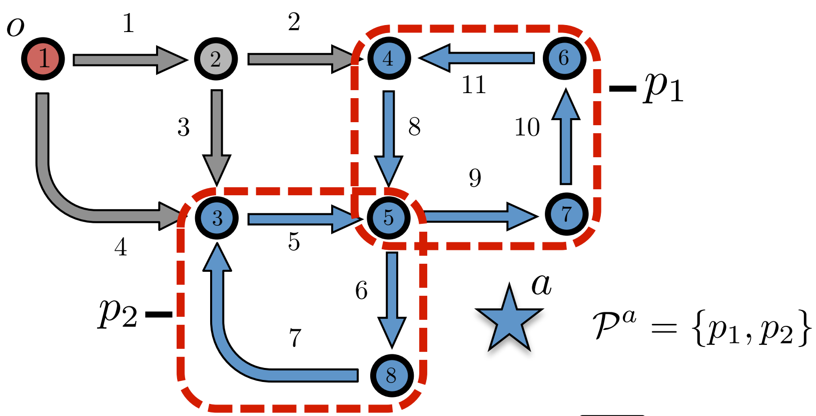
Travel Latency

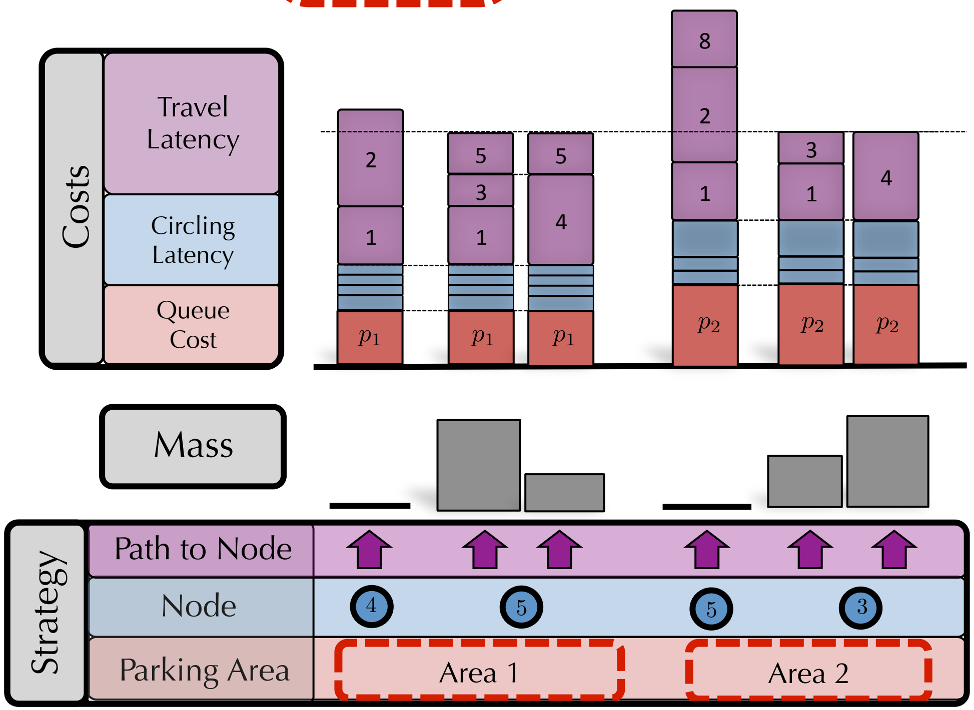
Circling Latency

Queue Cost



Edge Formulation





$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d\in\mathcal{N}^p} s_{od}^{ap}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap},\, s_{od}^{ap}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u\, du$$

$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad G x_{od}^{ap} = S_{od}^{ap} \qquad \forall\ o,d,a,p$$

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Gradient… $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x)\mathbf{E}^p + -R_p + C_p s^p$$

Travel Latency

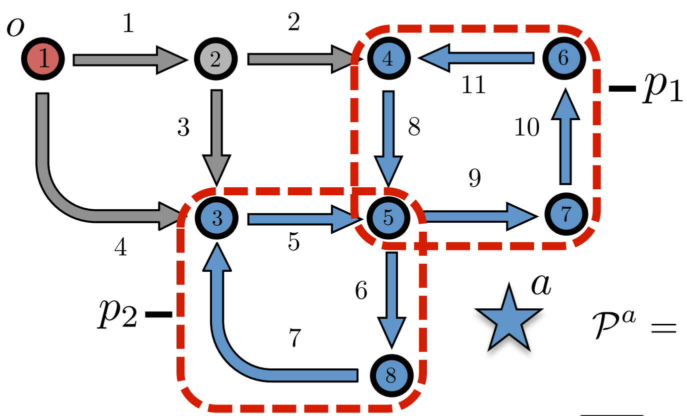
Circling Latency

Queue Cost

First Order Optimality…

$$\mathcal{L}(x, s, \pi, \lambda, \nu, \mu) = P(x) + - \sum_{o,d,a,p} (\pi_{od}^{ap})^T (G x_{od}^{ap} - S_{od}^{ap}) - \sum_{o,d,a,p} (\nu_{od}^{ap})^T x_{od}^{ap} - \sum_{o,a} \lambda_o^a \left[\Sigma_p \Sigma_{d\in\mathcal{N}^p} s_{od}^{ap} - s_o^a\right] - \sum_{o,d,a,p} (\mu_{od}^{ap})^T s_{od}^{ap}$$

Edge Formulation



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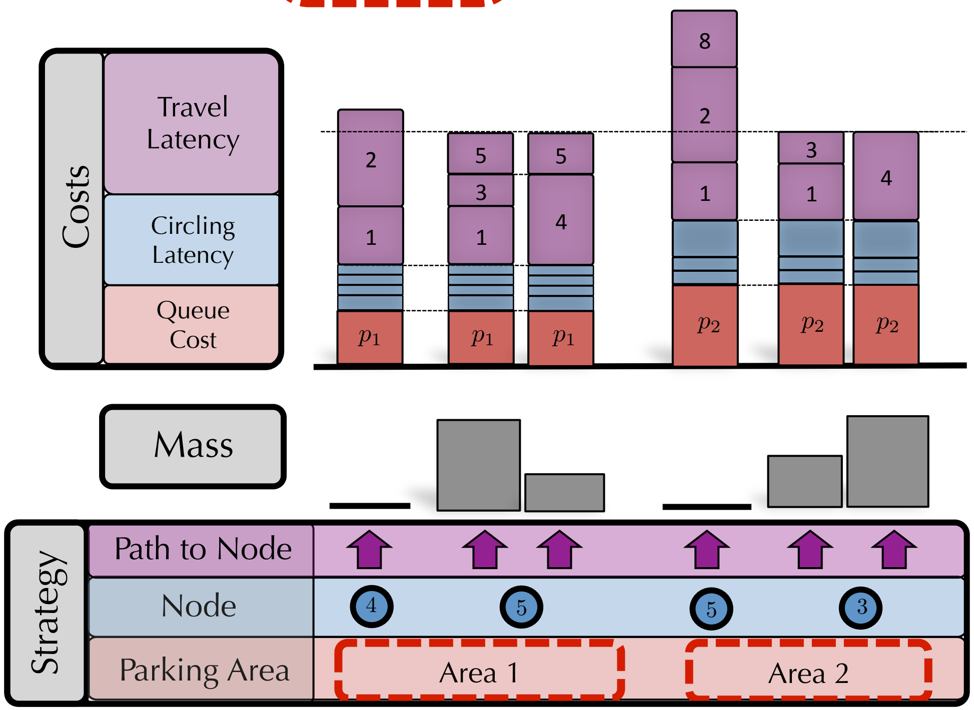
Optimization Problem

$$\min_{x^{ap}_{od},\, s^{ap}_{od}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u \, du$$

$$\text{s.t.} \quad x^{ap}_{od} \geq 0, \quad G x^{ap}_{od} = S^{ap}_{od} \qquad \forall\, o,d,a,p$$

$$s^{ap}_{od} \geq 0, \quad \sum_p \sum_{d\in\mathcal{N}^p} s^{ap}_{od} = s^a_o \qquad \forall\, o,d,a,p$$



Gradient… $\nabla_{x^{ap}_{od}} P = l(x)$

$$\nabla_{s^{ap}_{od}} P = \frac{1}{|\mathcal{E}^P|} l(x)\mathbf{E}^p + -R_p + C_p s^p$$

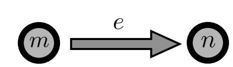
Travel Latency

Circling Latency

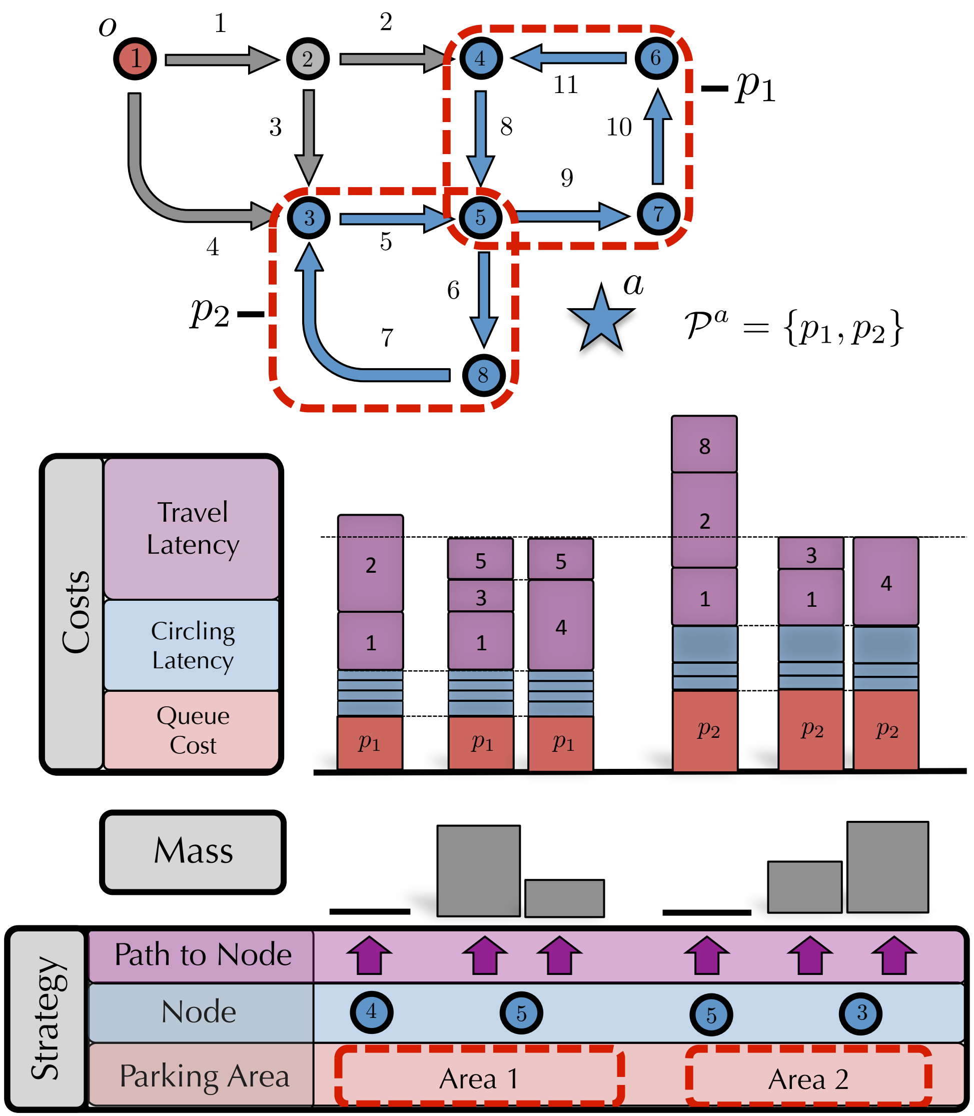
Queue Cost

First Order Optimality…

$$\frac{\partial \mathcal{L}}{\partial (x^{ap}_{od})_e} : \quad l_e(x_e) = (\pi^{ap}_{od})_n - (\pi^{ap}_{od})_m + (\nu^{ap}_{od})_e$$



Edge Formulation



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Optimization Problem

$$\min_{x^{ap}_{od},\, s^{ap}_{od}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u\, du$$

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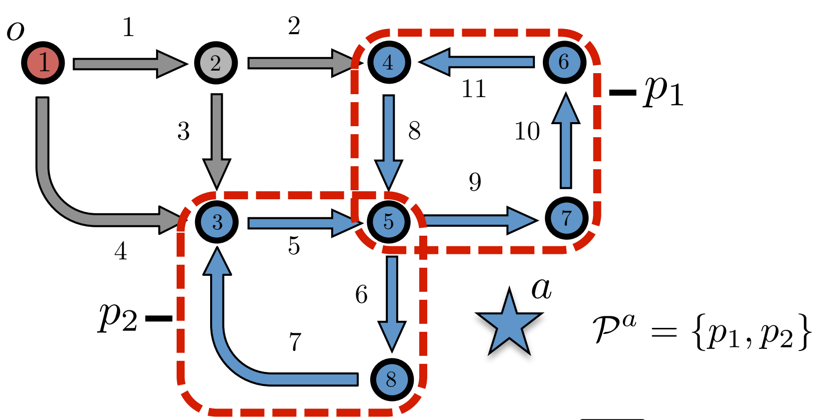
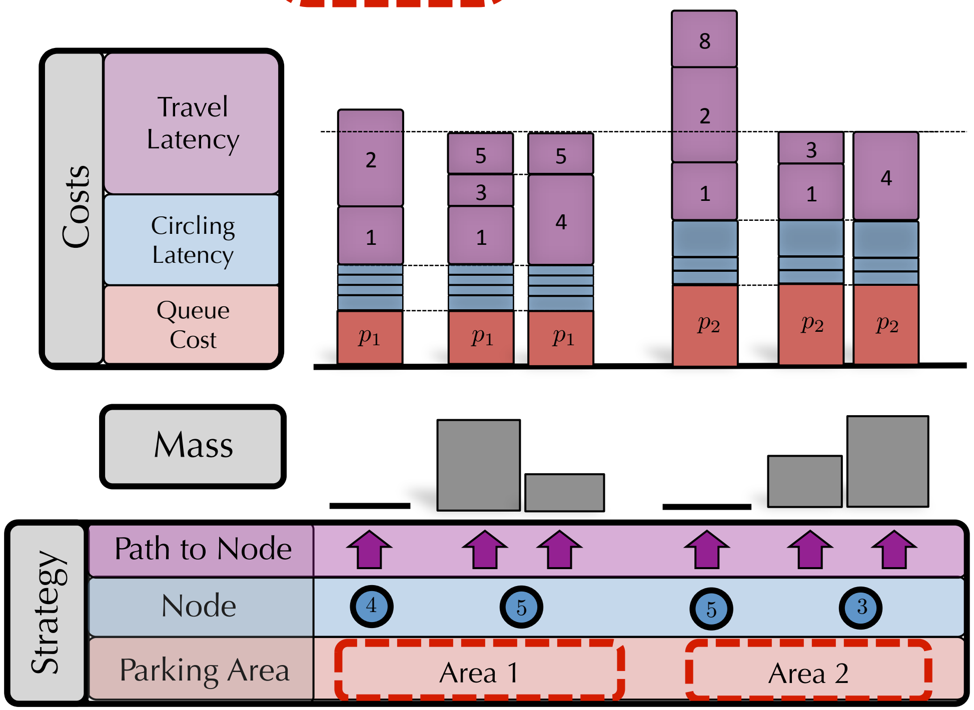
$$m \xrightarrow{e} n$$

summing over path from o to d....

Edge Formulation





$$x = \sum_{o,a,p,d} x^{ap}_{od} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s^{ap}_{od} \right] \qquad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s^{ap}_{od}$$

Optimization Problem

$$\min_{x^{ap}_{od},\, s^{ap}_{od}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u \, du$$

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$$\nabla_{s^{ap}_{od}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + -R_p + C_p s^p$$

Travel Latency

Circling Latency

Queue Cost

First Order Optimality…

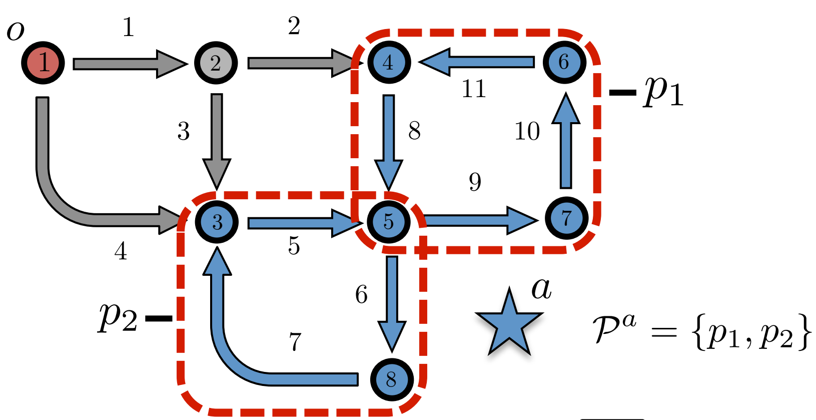
$$\frac{\partial \mathcal{L}}{\partial (x^{ap}_{od})_e} : \quad l_e(x_e) = (\pi^{ap}_{od})_n - (\pi^{ap}_{od})_m + (\nu^{ap}_{od})_e$$

$$m \xrightarrow{e} n$$

summing over path from o to d….

$$\sum_{e \in r} l_e(x_e) = (\pi^{ap}_{od})_d - (\pi^{ap}_{od})_o + \sum_{e \in r} (\nu^{ap}_{od})_e$$

Edge Formulation



$$x = \sum_{o,a,p,d} x^{ap}_{od} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s^{ap}_{od} \right] \qquad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s^{ap}_{od}$$

Optimization Problem

$$\min_{x^{ap}_{od},\, s^{ap}_{od}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\ du \ + \sum_p \int_0^{s^p} -R_p + C_p u\ du$$

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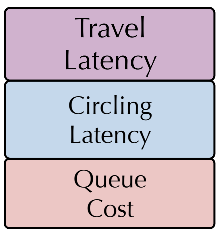
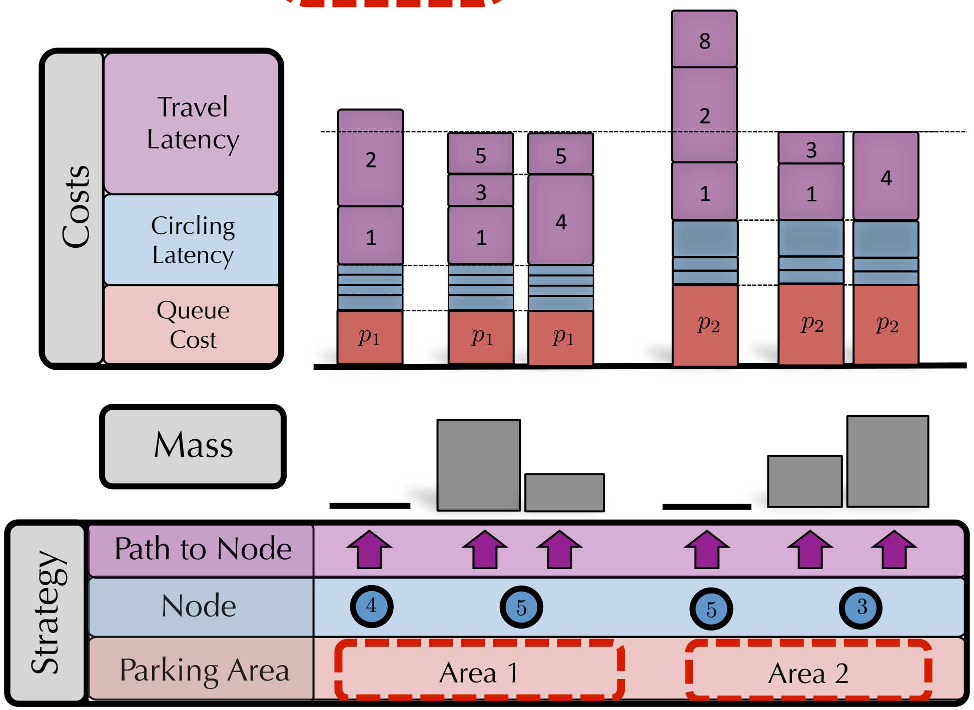


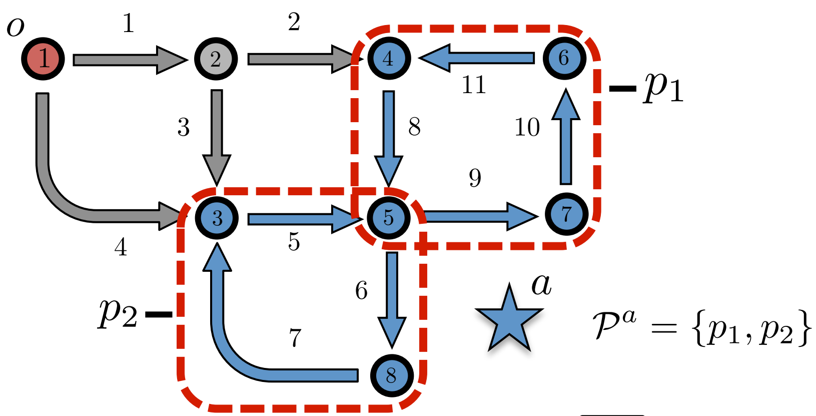
First Order Optimality…

$$\frac{\partial \mathcal{L}}{\partial (x^{ap}_{od})_e} : \quad \boxed{\sum_{e \in r} l_e(x_e) = (\pi^{ap}_{od})_d - (\pi^{ap}_{od})_o + \sum_{e \in r} (\nu^{ap}_{od})_e}$$

$$\frac{\partial \mathcal{L}}{\partial s^{ap}_{od}} :$$



Edge Formulation



$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d\in\mathcal{N}^p} s_{od}^{ap}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap},\, s_{od}^{ap}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\; du \;+\; \sum_p \int_0^{s^p} -R_p + C_p u\; du$$

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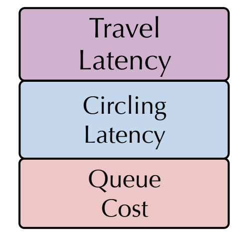
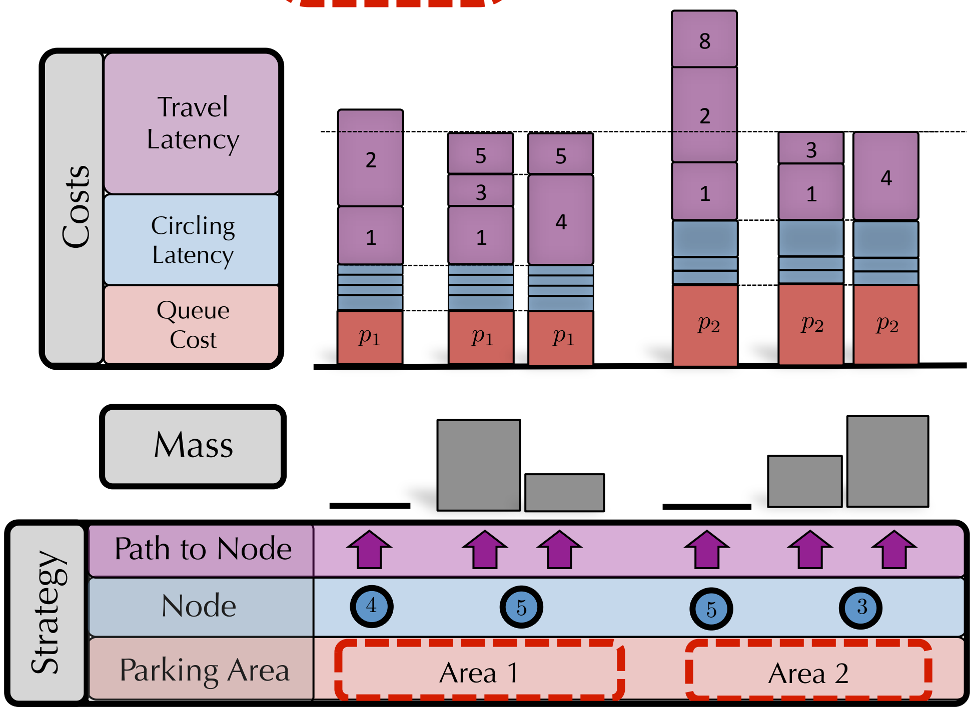
$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x)\mathbf{E}^p \; + \; -R_p + C_p s^p$$



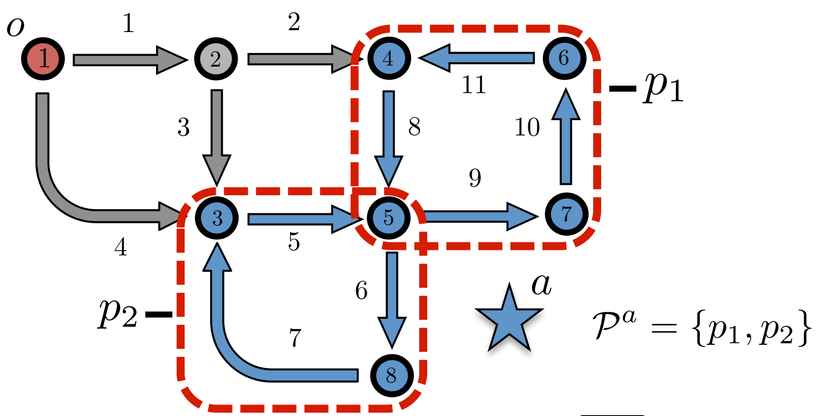
First Order Optimality…

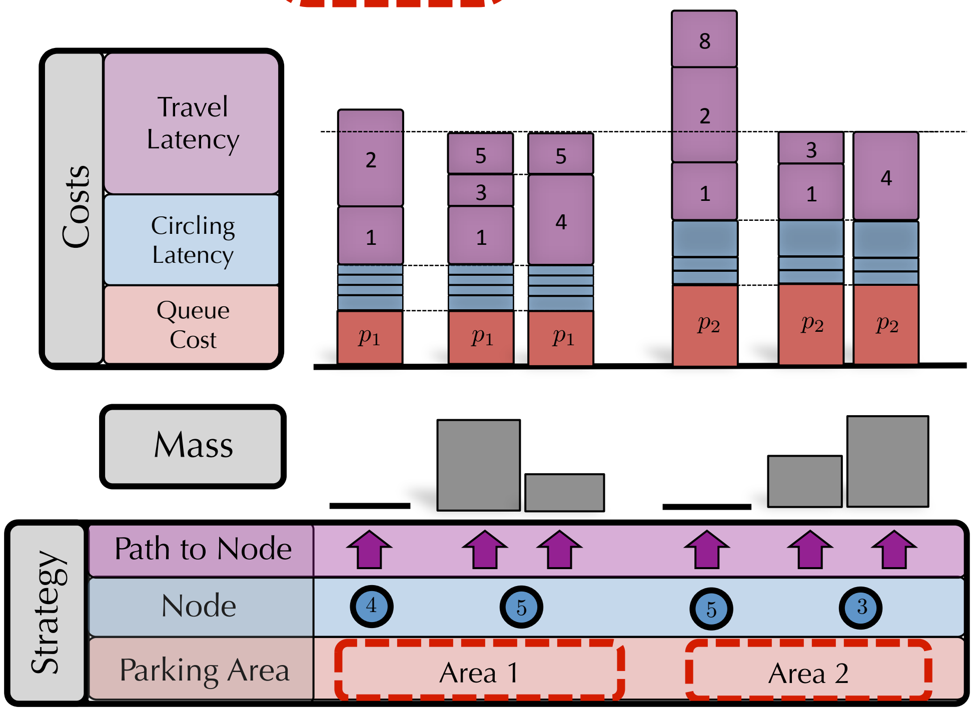
$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \; \sum_{e\in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e\in r} (\nu_{od}^{ap})_e$$

$$\frac{\partial \mathcal{L}}{\partial s_{od}^{ap}} : \; \frac{1}{|\mathcal{E}^P|} l(x)\mathbf{E}^p - R_p + C_p s^p = (\pi_{od}^{ap})_o - (\pi_{od}^{ap})_d + \lambda_o^a + \mu_{od}^{ap}$$



Edge Formulation





$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d\in\mathcal{N}^p} s_{od}^{ap}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap},\, s_{od}^{ap}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\ du \ + \sum_p \int_0^{s^p} -R_p + C_p u\ du$$

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$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x)\mathbf{E}^p \ + \ -R_p + C_p s^p$$

Travel Latency
Circling Latency
Queue Cost

First Order Optimality…

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad \sum_{e\in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e\in r} (\nu_{od}^{ap})_e$$

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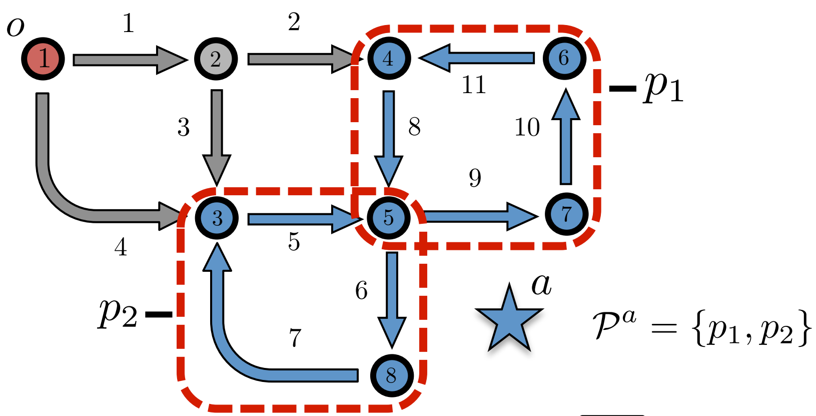
$$\sum_{e\in r} l_e(x_e) + \frac{1}{|\mathcal{E}^p|} l(x)\mathbf{E}^p - R_p + C_p s^p = \lambda_o^a + \sum_{e\in r} (v_{od}^{ap})_e + \mu_{od}^{ap}$$

Edge Formulation



$$x = \sum_{o,a,p,d} x^{ap}_{od} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p \left[\sum_{d\in\mathcal{N}^p} s^{ap}_{od}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s^{ap}_{od}$$

Optimization Problem

$$\min_{x^{ap}_{od},\, s^{ap}_{od}} \quad P(x) = \sum_e \int_0^{x_e} l_e(u)\, du + \sum_p \int_0^{s^p} -R_p + C_p u \, du$$

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Travel Latency

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First Order Optimality…

$$\frac{\partial \mathcal{L}}{\partial (x^{ap}_{od})_e} : \quad \sum_{e\in r} l_e(x_e) = (\pi^{ap}_{od})_d - (\pi^{ap}_{od})_o + \sum_{e\in r} (\nu^{ap}_{od})_e$$

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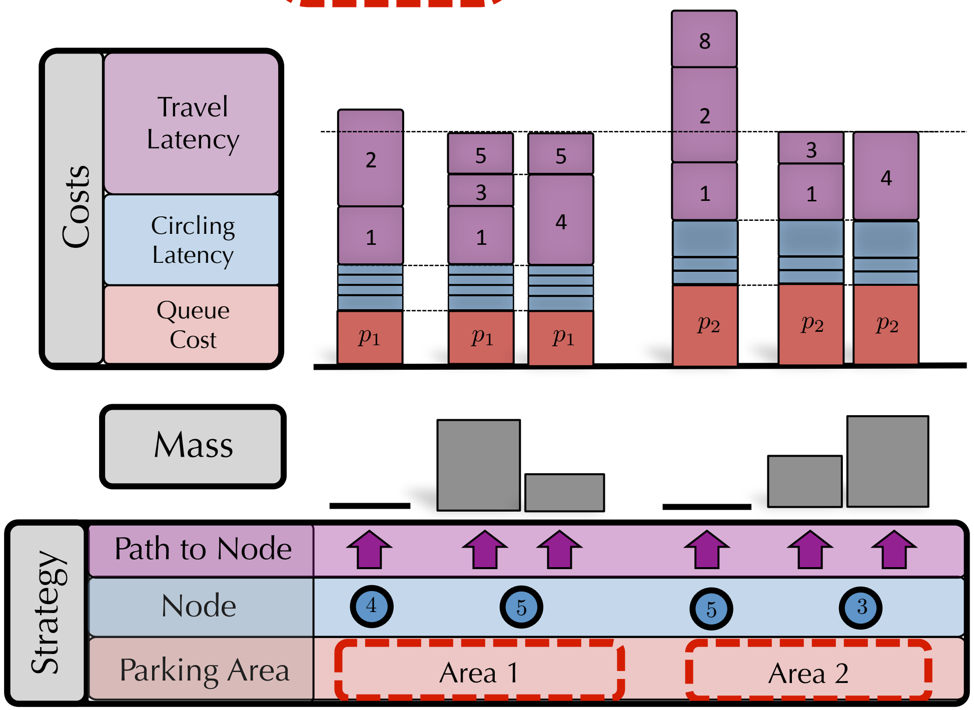
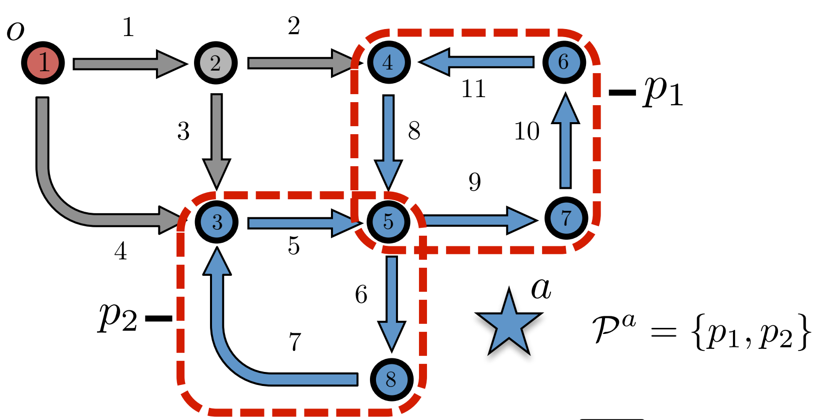
$$\underbrace{\sum_{e\in r} l_e(x_e)}_{\text{Traveling Cost}} + \underbrace{\frac{1}{|\mathcal{E}^p|} l(x)\mathbf{E}^p}_{\text{Circling Cost}} \underbrace{- R_p + C_p s^p}_{\text{Queue Cost}} = \lambda^a_o + \sum_{e\in r} (v^{ap}_{od})_e + \mu^{ap}_{od}$$



Edge Formulation



$$\mathcal{P}^a = \{p_1, p_2\}$$

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|}\mathbf{E}^p\left[\sum_{d\in\mathcal{N}^p} s_{od}^{ap}\right] \qquad s^p = \sum_{o,a}\sum_{d\in\mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

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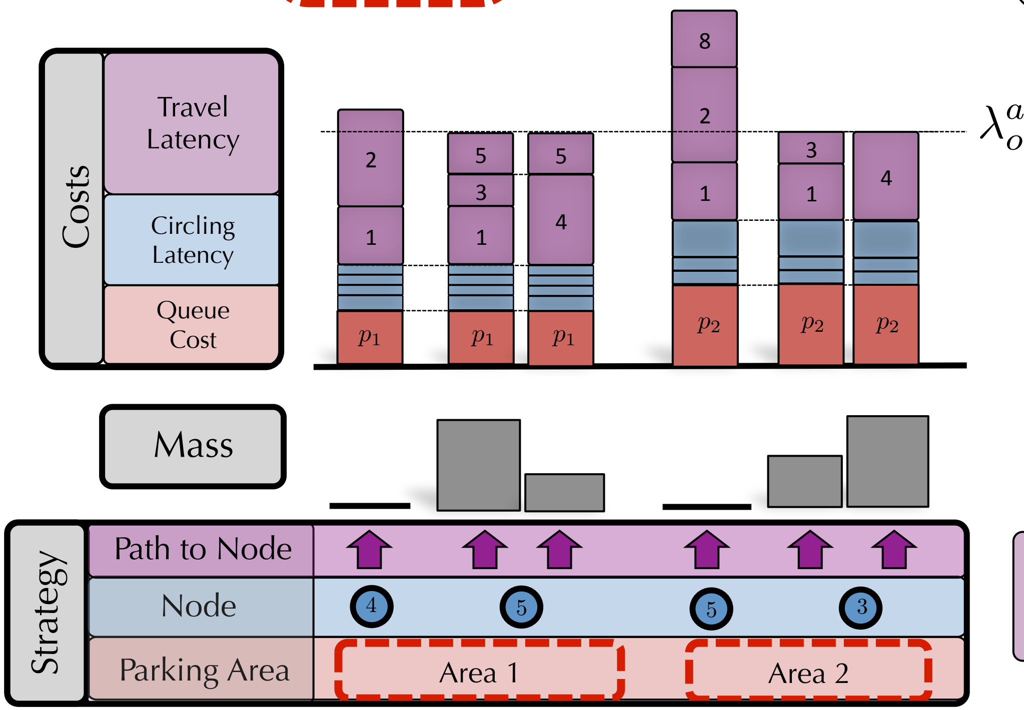

Gradient… $\nabla_{x_{od}^{ap}} P = l(x)$

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Travel Latency

Circling Latency

Queue Cost

First Order Optimality…

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad \sum_{e\in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e\in r} (\nu_{od}^{ap})_e$$

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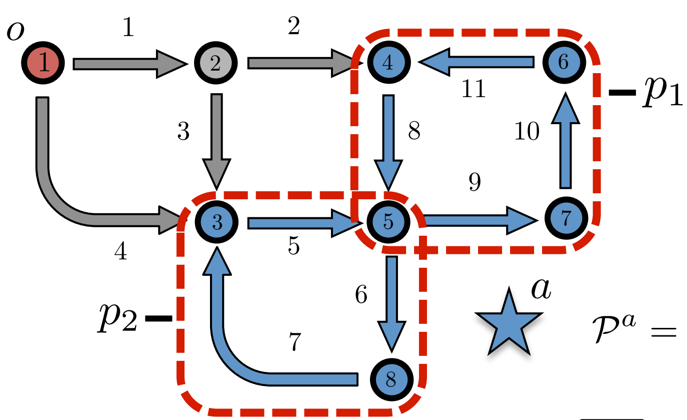
$$\underbrace{\sum_{e\in r} l_e(x_e)}_{\text{Traveling Cost}} + \underbrace{\frac{1}{|\mathcal{E}^p|} l(x)\mathbf{E}^p}_{\text{Circling Cost}} \underbrace{- R_p + C_p s^p}_{\text{Queue Cost}} = \underbrace{\lambda_o^a + \sum_{e\in r} (v_{od}^{ap})_e + \mu_{od}^{ap}}_{\textbf{Total Cost}}$$

Edge Formulation



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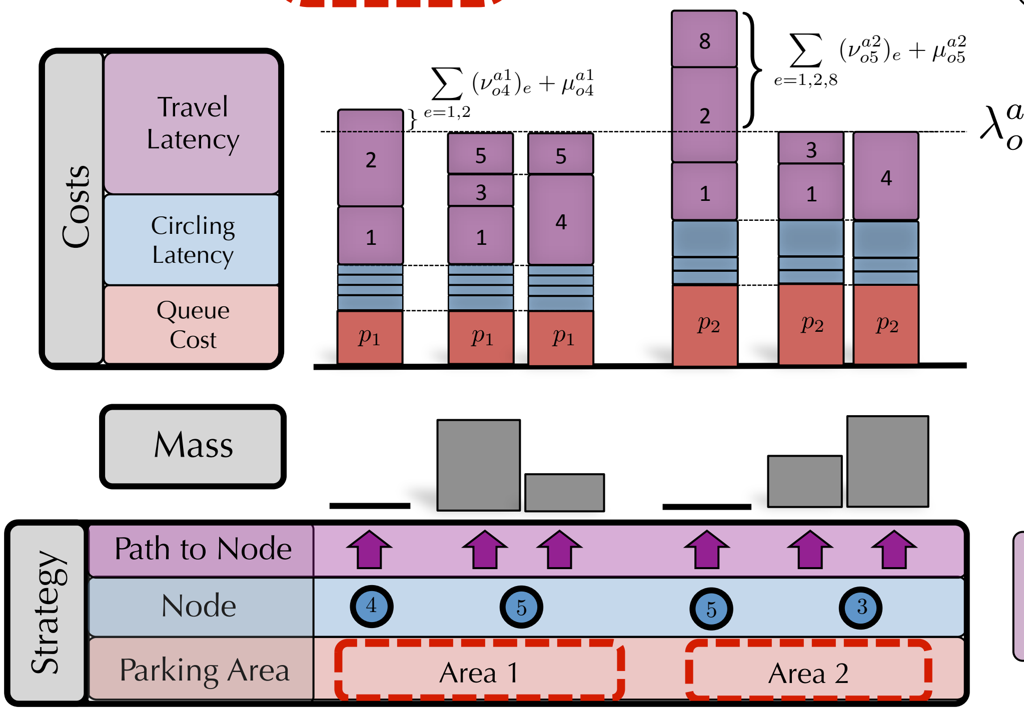

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$\sum_{e\in r} l_e(x_e)$	$+ \frac{1}{\vert\mathcal{E}^p\vert} l(x)\mathbf{E}^p$	$- R_p + C_p s^p$	$= \lambda_o^a$	$+ \sum_{e\in r} (v_{od}^{ap})_e + \mu_{od}^{ap}$
Traveling Cost	Circling Cost	Queue Cost	**Total Cost**	Extra Cost