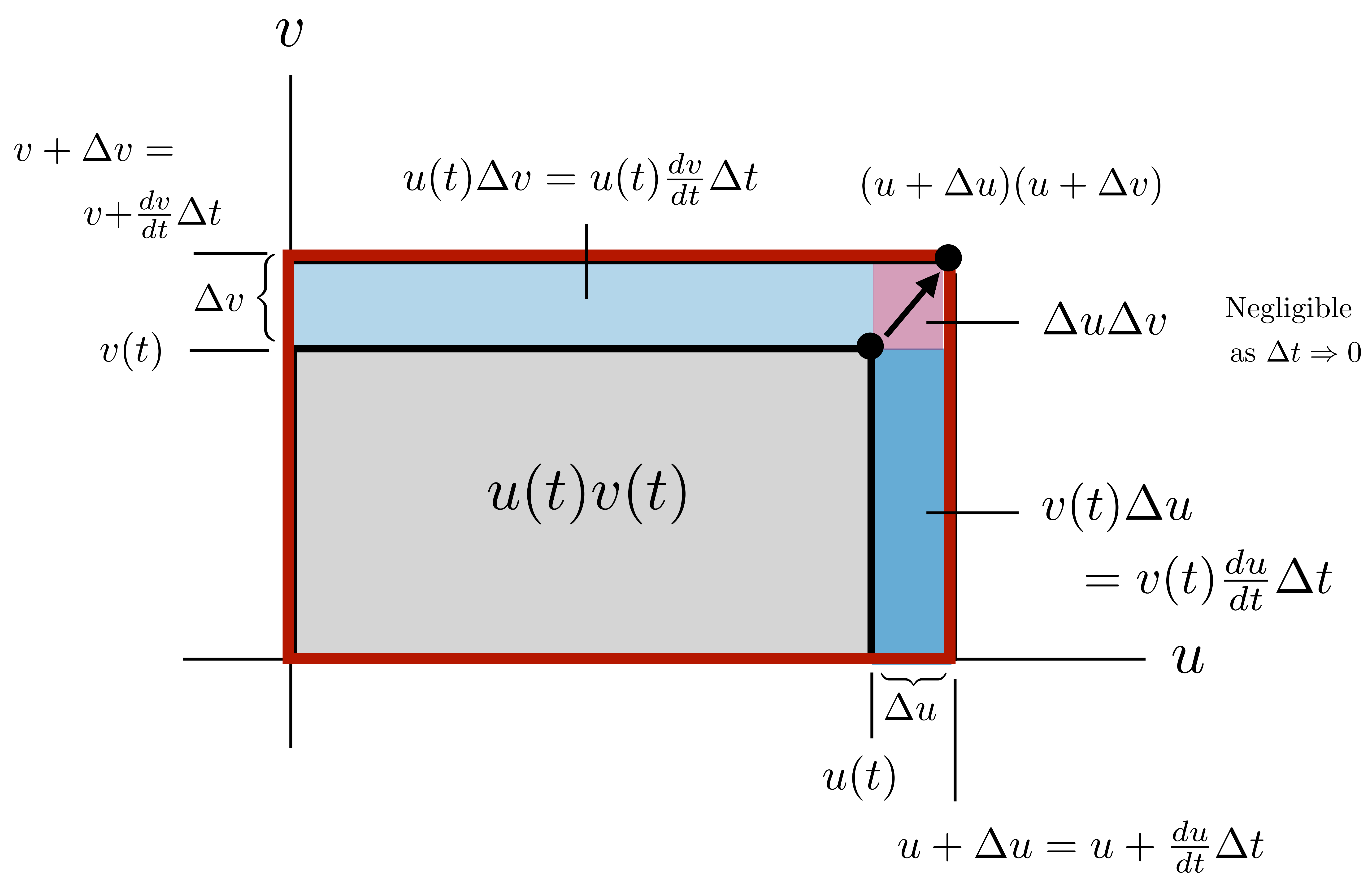
Calculus

Major sources:

Winter 2022 - Dan Calderone

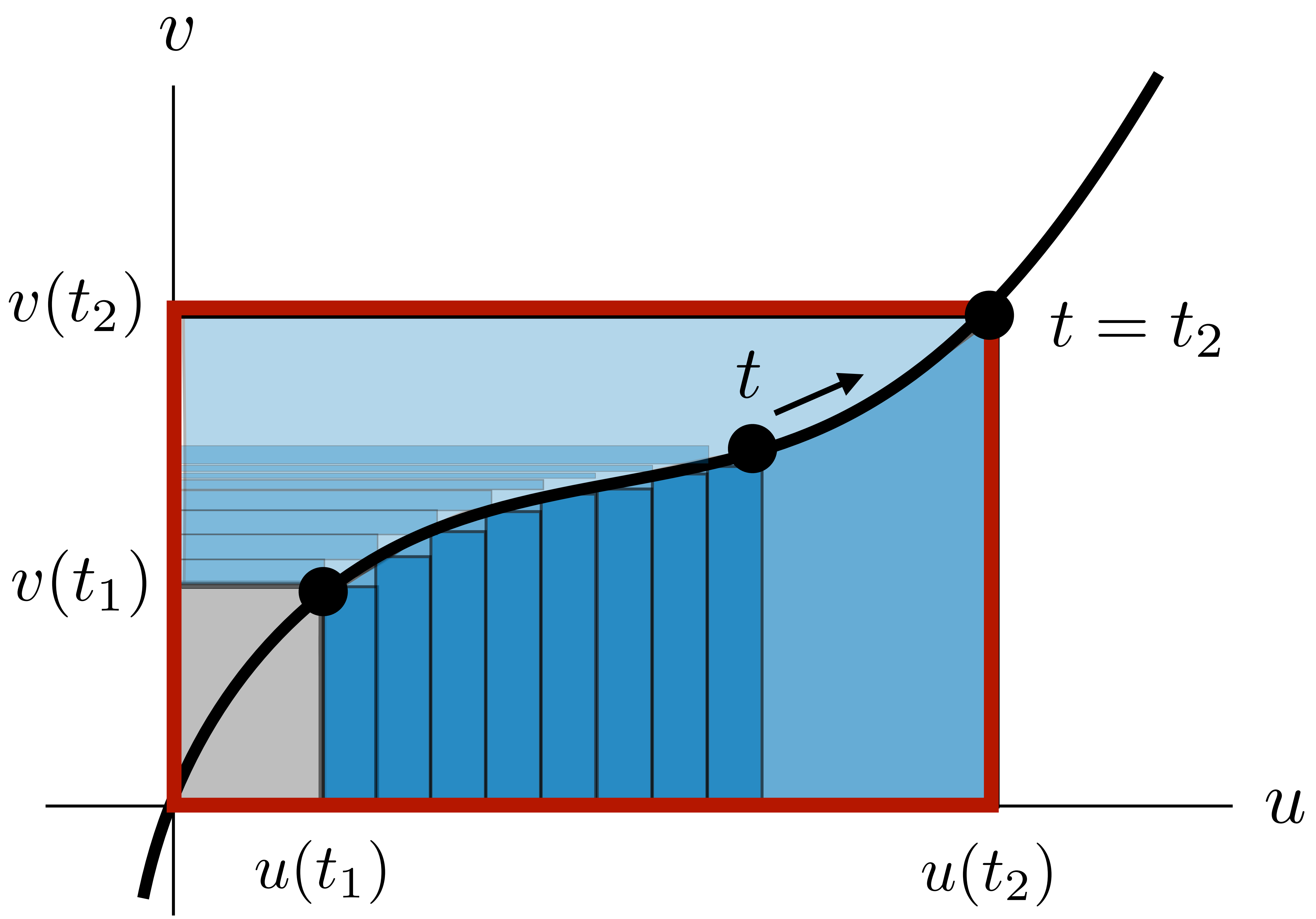
Product Rule



Product rule for derivatives:

$$\frac{d}{dt}\left(uv\right) = \frac{du}{dt}v + u\frac{dv}{dt}$$

Product rule to integration by parts



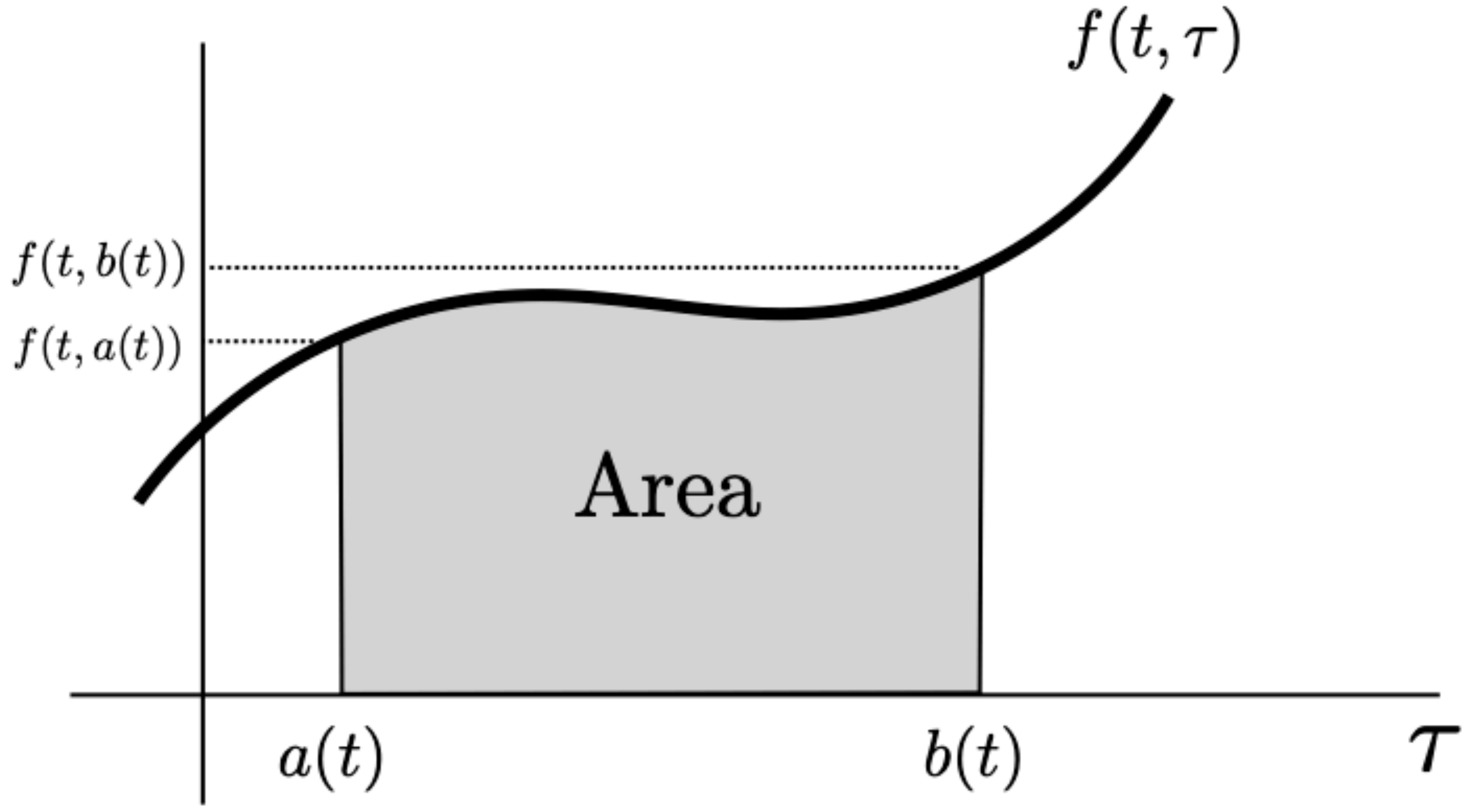
Integrating the product rule...

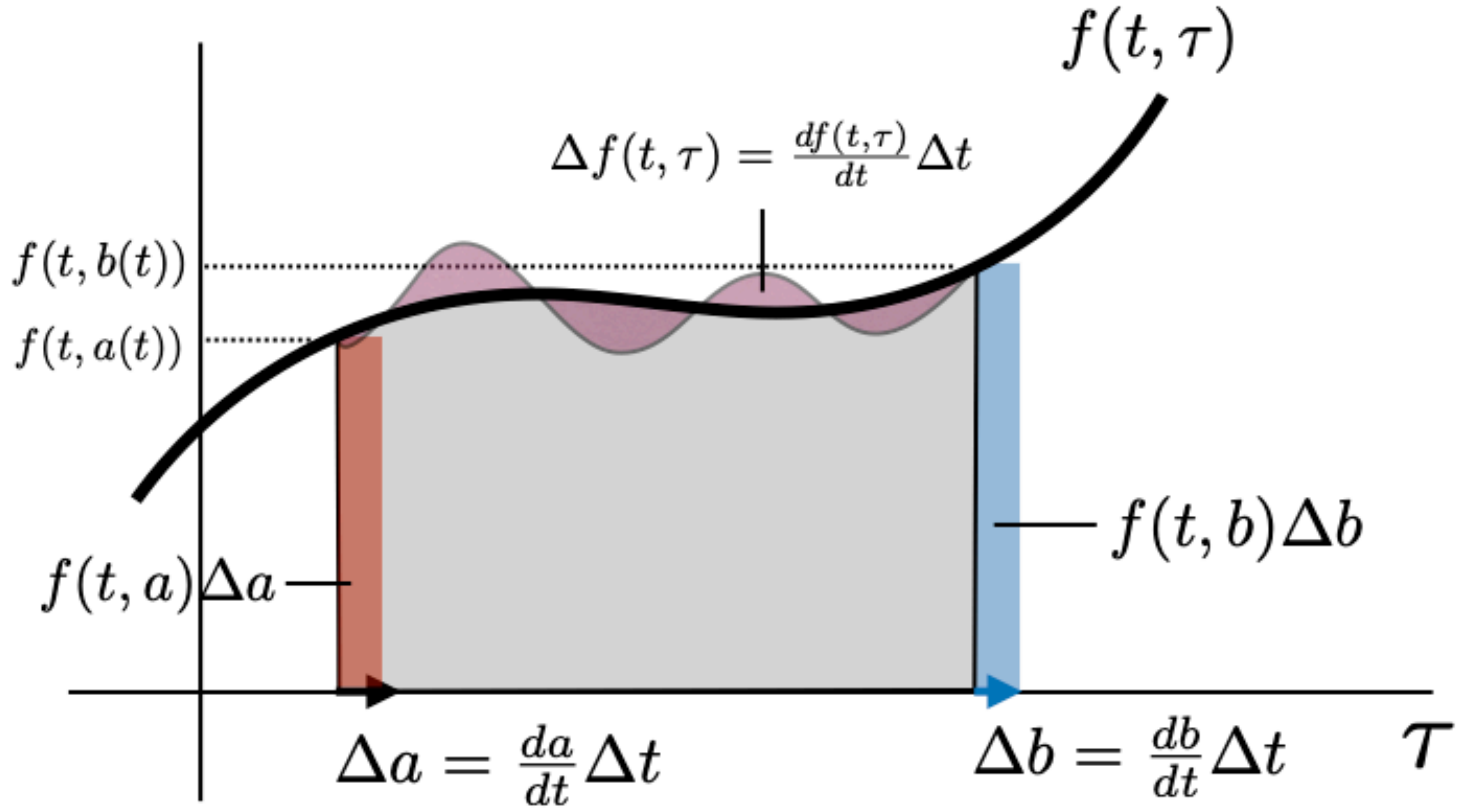
...to get integration by parts

$$\int_{t_1}^{t_2} \frac{d}{dt}\Big(uv\Big)\ dt = \int_{t_1}^{t_2} \frac{du}{dt}v + u\frac{dv}{dt}\ dt$$

$$\Rightarrow \quad \Big[uv\Big]_{t_1}^{t_2} = \int_{u(t_1)}^{u(t_2)} v\ du + \int_{v(t_1)}^{v(t_2)} u\ dv$$

Leibniz Integral Rule





$$\text{Area} = \int_{a(t)}^{b(t)} f(t,\tau)\ d\tau$$

$$\frac{d\text{Area}}{dt} = f(t,b)\frac{db}{dt} - f(t,a)\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{df(t,\tau)}{dt}\ d\tau$$

Vector Derivatives

Function: $f : \mathcal{X} \to \mathcal{Y}$ $\quad y = f(x)$

Derivative: linear map that estimates Δf given Δx

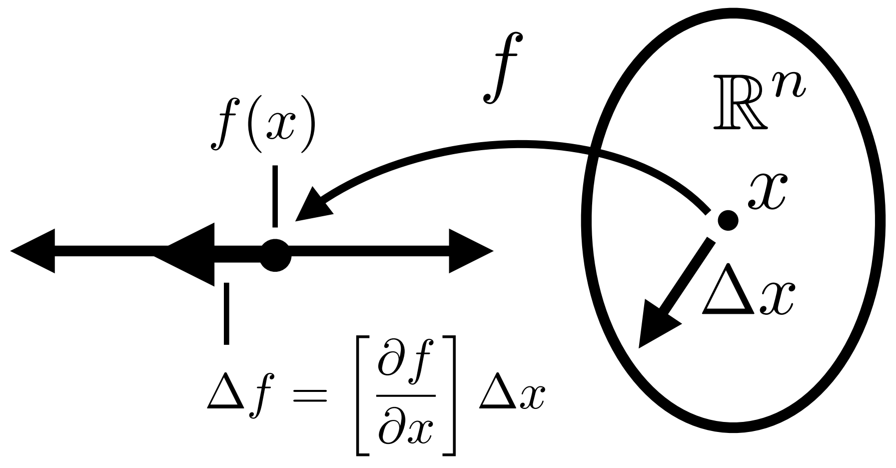
$$\Delta f = \left[\frac{\partial f}{\partial x}\right] \Delta x \qquad \Delta f = \frac{\partial f}{\partial x} \Delta x$$

Scalar Derivatives:

$$f : \mathbb{R} \to \mathbb{R} \qquad \Delta y = \Delta f = \frac{\partial f}{\partial x} \Delta x$$

Vector Derivatives: scalar functions

$$f : \mathbb{R}^n \to \mathbb{R}$$



Vector perturbation

$$\Delta f = \left[\frac{\partial f}{\partial x}\right] \Delta x = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

row vector

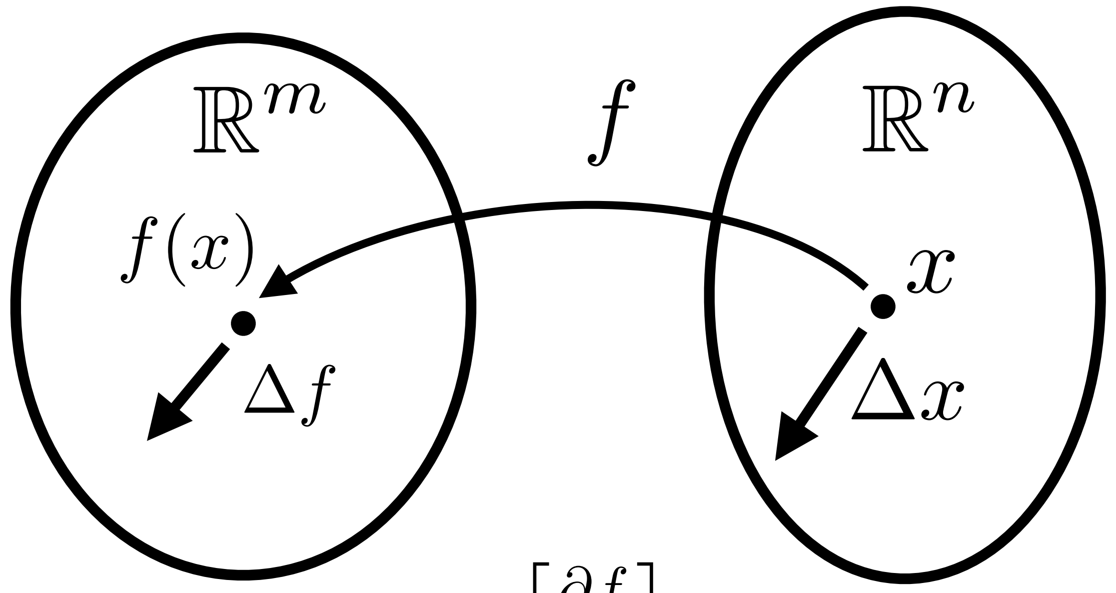
$$= \frac{\partial f}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f}{\partial x_n} \Delta x_n$$

…partial derivative rule

Vector Derivatives:

vector functions

$$f : \mathbb{R}^n \to \mathbb{R}^m$$



$$\Delta f = \left[\frac{\partial f}{\partial x}\right]_x \Delta x$$

$$\frac{\partial f_1}{\partial x} \qquad \frac{\partial f}{\partial x_1}$$

$$\Delta f = \left[\frac{\partial f}{\partial x}\right] \Delta x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$= \begin{bmatrix} | \\ \frac{\partial f}{\partial x_1} \\ | \end{bmatrix} \Delta x_1 + \cdots + \begin{bmatrix} | \\ \frac{\partial f}{\partial x_n} \\ | \end{bmatrix} \Delta x_n = \begin{bmatrix} \frac{\partial f_1}{\partial x} \Delta x \\ \vdots \\ \frac{\partial f_1}{\partial x} \Delta x \end{bmatrix}$$