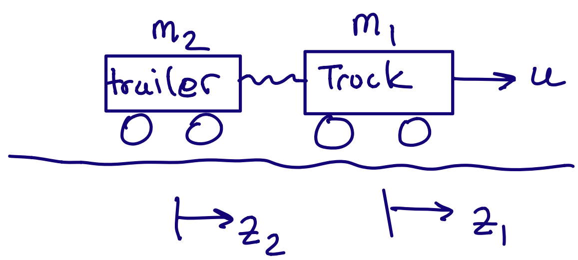
AA447 - Feedback Control - Spring 2021

Homework 1

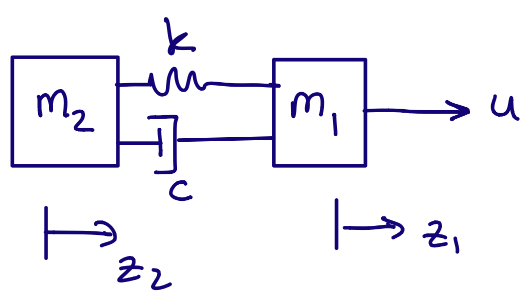
Due Date: Saturday, Apr 10^{th}, 2021 at 11:59 pm

1. **Dynamics**

   Consider the model of a truck pulling a trailer.



- **(PTS: 0-2)** Draw free body diagrams for the truck and the trailer in the above system. You can model each as a mass connected by a spring and a damper and ignore the effects of friction and drag.



- **(PTS: 0-2)** Use Newton's 2nd law to write equations for the acceleration of the truck and trailer.
- **(PTS: 0-2)** Combine these equations into a state space model with four states, the position and velocities of both the truck and trailer.

2. **State-space models**

   Consider the state space model of the LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

   where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and $D \in \mathbb{R}$.

- **(PTS: 0-2)** Write an expression for the transfer function $G(s)$ such that the Laplace transform $Y(s)$ is given by $Y(s) = G(s)U(s)$ where $U(s)$ is the Laplace transform of the control input $u(t)$.
- **(PTS: 0-2)** How does the denominator of the transfer function relate to the matrix A? How do the roots of the denominator relate to A?
- **(PTS: 0-2)** Rewrite the state-space model by applying the coordinate system $x = Tx'$ for some invertible coordinate transformation $T \in \mathbb{R}^{n \times n}$. Show that the state-space model in this new coordinate system gives the same transfer function $G(s)$.

3. **Laplace Transforms**

Compute the Laplace transforms for the following time signals. If the time signal is in terms of arbitrary functions $f(t)$ and/or $g(t)$, your answer can be in terms of their Laplace transforms $\mathcal{L}\{f(t)\} = F(s)$ and/or $\mathcal{L}\{g(t)\} = G(s)$

- **(PTS: 0-2) 1st Derivative:** $\mathcal{L}\left\{\frac{df}{dt}\right\} =?$
- **(PTS: 0-2) Integral:** $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} =?$
- **(PTS: 0-2) Convolution:** $\mathcal{L}\left\{\int_0^t g(t-\tau)f(\tau)\, d\tau\right\} =?$
- **(PTS: 0-2) Time delay:** $\mathcal{L}\{f(t-a)\} =?$

Compute the inverse Laplace transforms for the following frequency signals.

- **(PTS: 0-2)** $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} =?$
- **(PTS: 0-2)** $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} =?$
- **(PTS: 0-2)** $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega^2)}\right\} =?$

4. **Anatomy of a Transfer Function**

Consider the transfer function of the form

$$G(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)\cdots(s-z_k)}{(s-\lambda_1)\cdots(s-\lambda_n)} \tag{1}$$

where $z_1, \ldots z_k$ are the *zeros*, the roots of the numerator $N(s)$ and $\lambda_1, \ldots \lambda_n$ are the *poles*, the roots of the denominator or characteristic polynomial.

The frequency response for a given frequency ω is given by the complex number $G(j\omega)$ with magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$.

For a given frequency ω, we can write the terms $j\omega - z_k$ and $j\omega - \lambda_\ell$ in polar form, ie.

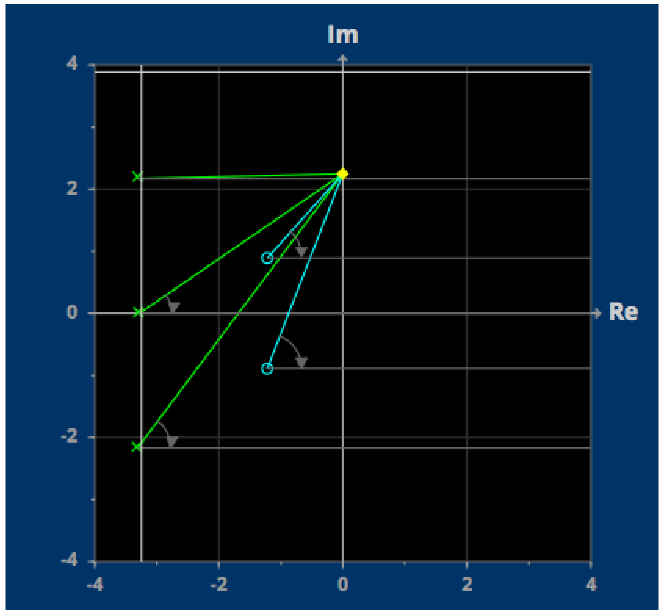
$$j\omega - z_k \triangleq \alpha_k(\omega)e^{j\phi_k(\omega)} \;\; \forall k, \qquad j\omega - \lambda_\ell \triangleq \beta_\ell(\omega)e^{j\theta_\ell(\omega)} \;\; \forall \ell$$

Note that $\alpha_k, \phi_k, \beta_\ell, \theta_\ell$ all depend on ω.

- **(PTS: 0-2)** Consider the transfer function

$$G(s) = \frac{(s-z_1)(s-z_2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)}$$

where the poles and zeros are shown as x's and o's respectively in the following diagram.



Label

$$\alpha_1(\omega), \alpha_2(\omega), \qquad \phi_1(\omega), \phi_2(\omega)$$
$$\beta_1(\omega), \beta_2(\omega), \beta_2(\omega), \qquad \theta_1(\omega), \theta_2(\omega), \theta_3(\omega)$$

for the $j\omega$ shown as the yellow diamond in the diagram above.

- **(PTS: 0-2)** For the general transfer function given in Equation (1) write $G(j\omega)$ in terms of $\alpha_k(\omega), \phi_k(\omega), \beta_\ell(\omega), \theta_\ell(\omega)$. How do $\alpha_k(\omega)$ and $\beta_\ell(\omega)$ affect $|G(j\omega)|$? How do $\phi_k(\omega)$ and $\theta_\ell(\omega)$ affect $\angle G(j\omega)$?
- **(PTS: 0-2)** Experiment with the transfer function in the visualizer given at

  `https://mathlets.org/mathlets/bode-and-nyquist-plots/`

  for different values of ω. Make sure to check the $i\omega$ check box shown. What happens to $|G(j\omega)|$ as λ_ℓ approaches the $j\omega$ axis? Why? What happens to $\angle G(j\omega)$ if you move z_k far right in the right half plane? Why?