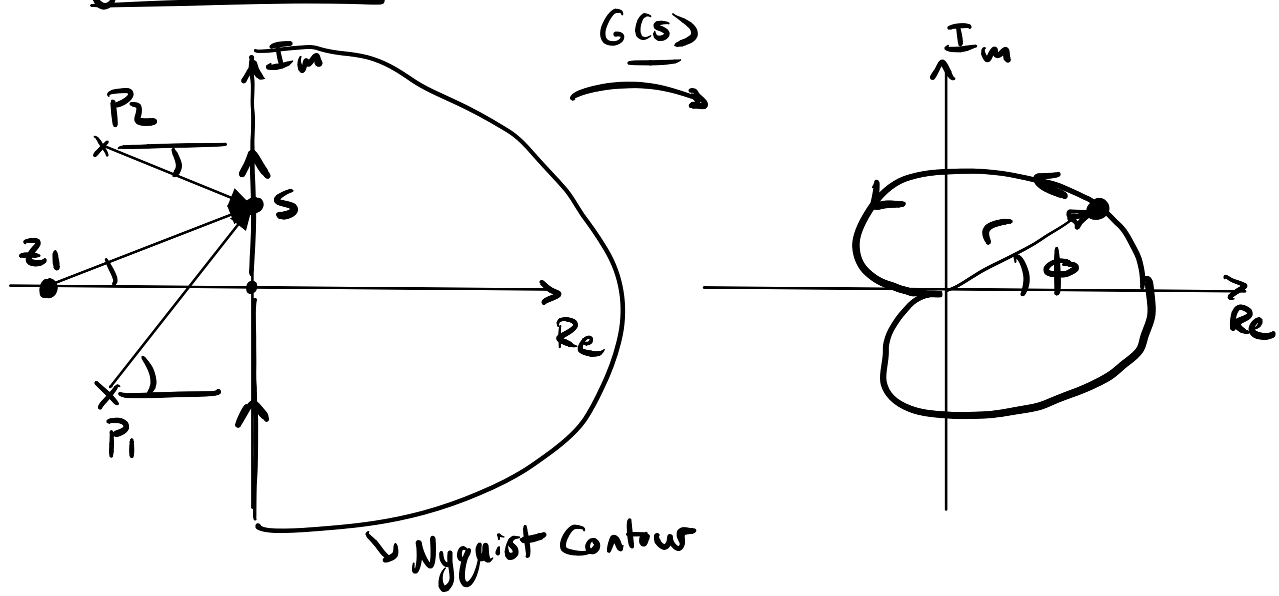
Nyquist Plots



$$G(s) = \frac{\beta_n s^m + \beta_{n-1} s^{m-1} + \cdots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

proper (causal) TF $m \leq n$

$$= \frac{(s-z_1) \cdots (s-z_m)}{(s-p_1) \cdots (s-p_n)}$$

$$= \underbrace{\frac{\prod_k |s-z_k|}{\prod_{k'} |s-p_{k'}|}}_{r} \, e^{i \overbrace{\left(\sum_k \angle(s-z_k) - \sum_{k'} \angle(s-p_k)\right)}^{}}$$

$$\underbrace{\sum_k \angle(s-z_k) - \sum_{k'} \angle(s-p_k)}_{\phi}$$

Polar form

Some Notes:

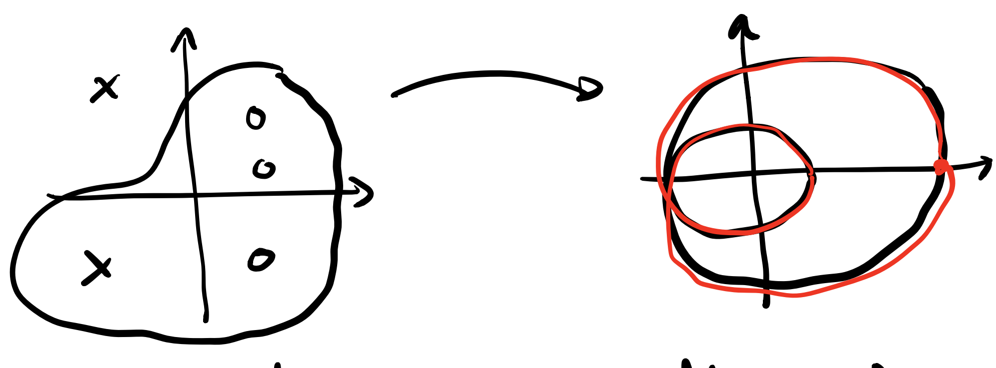
Magnitude:

- going close to a pole causes magnitude of Nyquist to blow up to ∞
- contour out at ∞ not hard
  - if $n > m$ $\longrightarrow$ magnitude goes to 0
  - if $n = m$ $\longrightarrow$ mag. goes to $\frac{\beta_n}{\alpha_n}$

Phase

- $(s - z_k)$: positive (cw) contribution to the phase
- $(s - p_k)$: negative (cw) contribution to the phase

Intuition



- if we circle a zero, adds 360° to phase...
- if we circle a pole, subtracts 360° from phase

# of CW encirclements of 0 of Nyquist plot = # of zeros in the contour − # of poles in the contour

Want to assess stability of closed loop transfer functions

$$\frac{L(s)}{1+L(s)}, \quad \frac{G(s)}{1+L(s)}, \text{ etc....}$$

poles of these are zeros of $1 + L(s) = 0$

plant

$$G(s) = \frac{num_G(s)}{den_G(s)}$$

Controller

$$C(s) = \frac{num_C(s)}{den_C(s)}$$

OPEN LOOP TF : $L(s) = G(s)C(s) = \dfrac{num_G \; num_C}{\boxed{den_G \; den_C}}$

$$1 + L(s) = 1 + GC$$

$$= 1 + \frac{num_G}{den_G}\frac{num_C}{den_C} = 0$$

roots

$$\frac{\boxed{den_G\, den_C + num_G\, num_C}}{\boxed{den_G\, den_C}} = 0$$

applying Nyquist...

\# RHP zeros of $den_G\, den_C + num_G\, num_C$ = \# CW encirclements of 0 of Nyquist of $\underline{1+L(s)}$ + \# RHP of poles of $1+L(s)$ (zeros of $den_G\, den_C$)

draw Nyquist of $L(s)$

this is just Nyquist of $1+L(s)$ shifted to the left by -1.

$$\underbrace{\text{\# RHP zeros of } 1+L(s) \text{ (poles of the closed loop system)}}_{=0} = \text{\# CW encirclements of } -1 \text{ of Nyquist of } L(s) + \text{\# RHP of poles of } \underline{1+L(s)}$$

poles of the open loop sys.

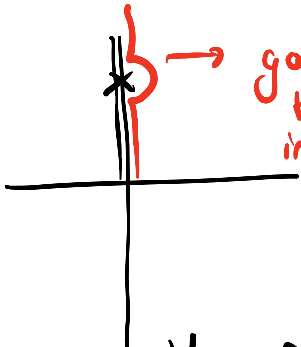
Closed Loop Stability:

no poles in RHP

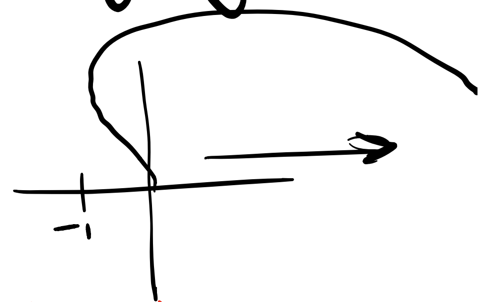
Nyquist Stability:

\# CCW encirclements of -1 of Nyquist of $L(s)$ = \# of RHP poles of the open loop $L(s)$

How to deal w poles on the imaginary axis

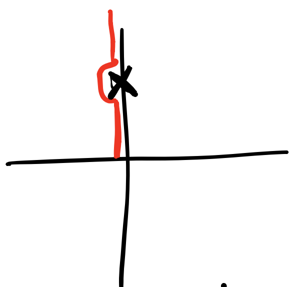


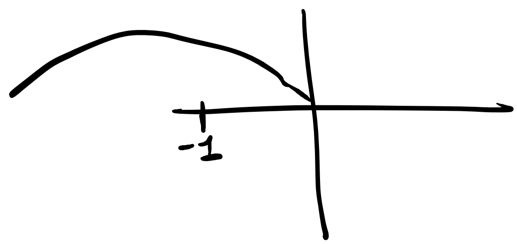
go around the pole in a small arc.



close to the pole (on right hand side)

- magnitude of Nyquist will be large
- phase of Nyquist will be 0





close to the pole (on left hand side)

- mag. large
- phase will be 180°

pole on iw axis ⟺ marginal stable

Sketching Nyquist Plots

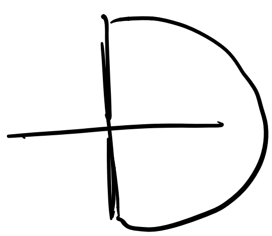
Important Points

- $\omega = 0$
- $\omega = \infty$
- where Nyquist cross real & imag axes

min/max Real part

min/max Imag part

$G(s)$ $\quad$ $G(i\omega)$



Ex. $L(s) = \dfrac{1}{s+1}$

$$L(i\omega) = \frac{1}{i\omega + 1}\,\frac{(-i\omega+1)}{(-i\omega+1)} = \frac{-i\omega+1}{-i^2\omega^2 + 1^2} = \frac{-i\omega+1}{\omega^2+1}$$

$$= |i\omega + 1|^2$$

$$= \underbrace{\frac{1}{\boxed{\omega^2+1}}}_{x} + \underbrace{\frac{-\omega}{\boxed{\omega^2+1}}}_{y} i$$

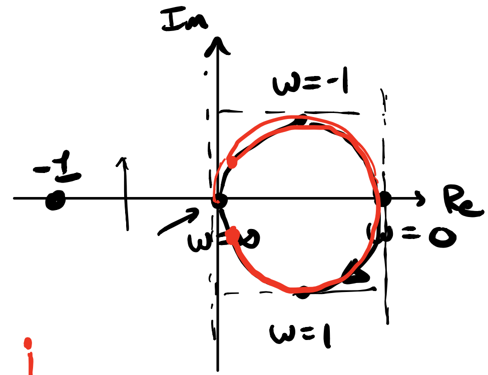
x = Real part

y = Imag part

$$L(i\omega) = x + yi$$

$w = 0 \qquad x = 1, \quad y = 0$

$w = \infty \qquad x = 0, \quad y = 0$

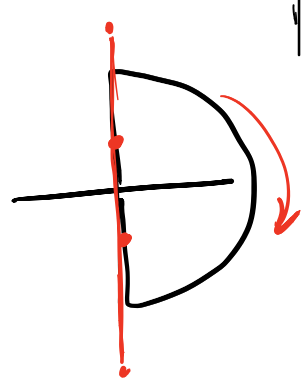


$x = 0 \longrightarrow$ only when $w = \infty$

(cross imag axis)

$y = 0 \longrightarrow w = \infty$, $w = 0$

(cross real axis)



min/max x: $\dfrac{dx}{dw} = 0$

$$\frac{dx}{dw} = \frac{-1}{(w^2+1)^2}(2w) \Longrightarrow w = 0$$

min/max y: $\dfrac{dy}{dw} = \boxed{0}$

$$\frac{dy}{dw} = \frac{d}{dw}\left(\frac{-w}{w^2+1}\right) = \frac{+w(2w)}{(w^2+1)^2} - \frac{1}{w^2+1}$$

$$= \frac{+2w^2 - w^2 - 1}{(w^2+1)^2} = \frac{w^2 - 1}{(w^2+1)^2}$$

$$= \frac{(w+1)(w-1)}{(w^2+1)^2}$$

w^2

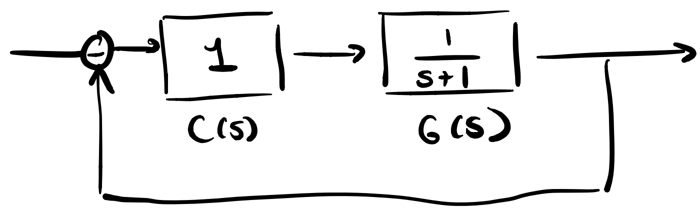
w^4

$\rightarrow w = \infty$

$w = \pm 1$

$w = 1 \quad \frac{1}{2} - \frac{1}{2}i \qquad w = -1 \quad \frac{1}{2} + \frac{1}{2}i$

$L(s) = \frac{1}{s+1}$



$1 + L(s) = 1 + \frac{1}{s+1} = \frac{s+2}{s+1} \leftarrow$ stable

stable w̄ no RHP poles...

0 ccw circles around −1 = 0 RHP poles of L(s) ✓ stable

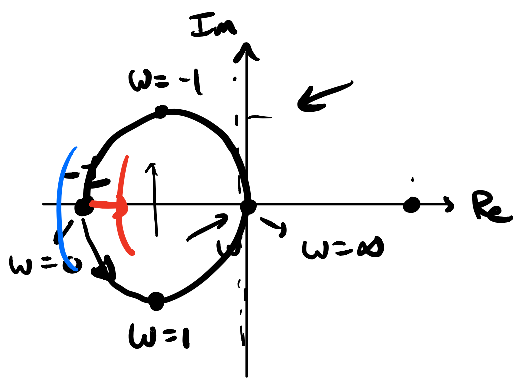
Ex. $L(s) = \frac{1}{s-\lambda}$

$L(i\omega) = \frac{1}{-\lambda + i\omega} = \frac{-\lambda - i\omega}{\lambda^2 + \omega^2}$

$L(i\omega) = \frac{-\lambda}{\lambda^2 + \underline{\omega^2}} - \frac{\omega}{\lambda^2 + \omega^2} i$

1 open loop RHP pole of L(s) at $\boxed{1}$

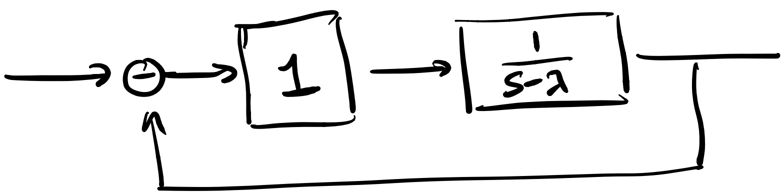
need to circle −1 once



Consider x : at $\omega = 0$ (not closed loop stable)

$\boxed{\lambda > 1 :}$ $\quad x = \dfrac{-\lambda}{\lambda^2} > -1$ $\quad$ doesn't circle -1

$\boxed{\lambda < 1}$ $\quad x = \dfrac{-\lambda}{\lambda^2} < -1$ $\quad$ does circle -1 (closed loop stable)



$$1 + L(s) = 1 + \frac{1}{s-\lambda} = \frac{s \boxed{-\lambda + 1}}{s-\lambda}$$

roots

$\lambda = 1$: $\dfrac{s}{s-\lambda}$ $\quad$ root at 0 $\quad$ marginally stable

Ex. $L(s) = \dfrac{1}{(s+1)^2}$

$$L(i\omega) = \frac{1}{(i\omega+1)^2} = \frac{1}{(1-\omega^2)+2i\omega} = \frac{1-\omega^2-2i\omega}{\underbrace{(1-\omega^2)^2+4\omega^2}_{1-2\omega^2+\omega^4+4\omega^2}}$$

$$= \frac{1-\omega^2-2i\omega}{(\omega^2+1)^2}$$

$$= \frac{1-\omega^2}{(\omega^2+1)^2} + \left(\frac{-2\omega}{(1+\omega^2)^2}\right) i$$

$$\underbrace{\qquad}_{x} \qquad \underbrace{\qquad}_{y}$$

$\omega = 0$: $\quad x = 1 \quad y = 0$

$\omega = \infty$: $\quad x = 0 \quad y = 0$

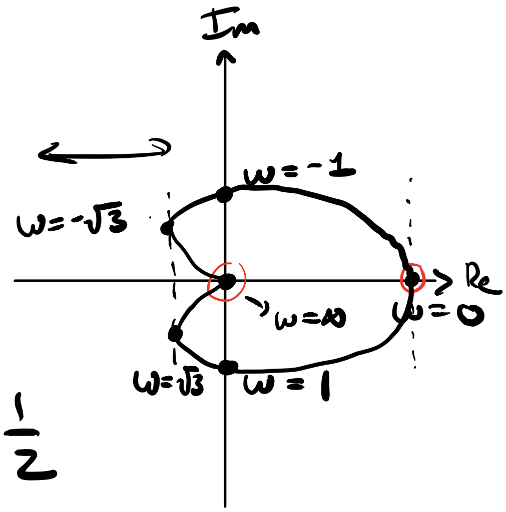
crossings

$x = 0$ (crossing imag axis)

$$\frac{1-\omega^2}{(\omega^2+1)^2} = 0$$

$\omega = \infty$

$\omega = \pm 1$



$\omega = \pm 1$: $\quad y = \left(\frac{\mp 2}{4}\right) = \mp \frac{1}{2}$

$y = 0$ $\quad \omega = 0$ $\quad \omega = \infty$

$$\frac{1-\omega^2}{(\omega^2+1)^2} + \left(\frac{-2\omega}{(1+\omega^2)^2}\right) i$$

min/max x:

$$\frac{dx}{d\omega} = \frac{-2\omega}{(\omega^2+1)^2} + \frac{-(1-\omega^2)\,2(2\omega)}{(1+\omega^2)^3}$$

$$= \frac{-2\omega - 2\omega^3 - 4\omega + 4\omega^3}{(1+\omega^2)^3}$$

$$= \frac{2\omega(-1-2+\omega^2)}{(1+\omega^2)^3}$$

$\omega = 0 \qquad \omega = \pm\sqrt{3}$

$\omega = \pm\sqrt{3} \qquad x = \dfrac{1-3}{8} = -\dfrac{1}{4}$

$y = \mp\dfrac{2\sqrt{3}}{8} = \mp\dfrac{\sqrt{3}}{4}$

Summary:
computing these few critical points can give us a sense for how the Nyquist plot looks.

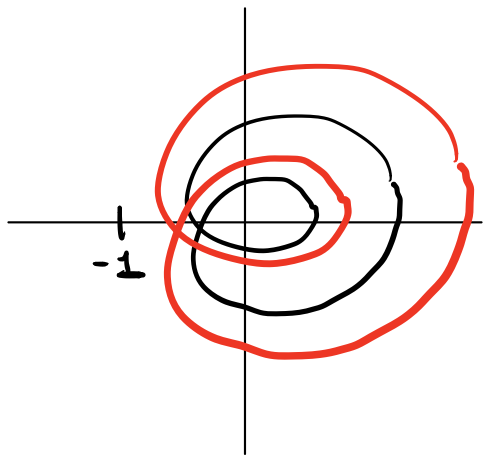
Note: Nyquist plot is symmetric about real line if $L(s)$ has only real coeffs

Gain & Phase Margin

Nyquist Plots of $L(s)$

Increasing Gain:

$L(s)$ $\quad$ $\alpha L(s)$ $\quad$ $\alpha > 1$



shift phase

$L(s)$ $\quad$ $e^{i\phi} L(s)$