# Stability Margins

Nyquist stability : $Z = P + N$

- $Z$ → poles of CL TF in RHP
- $P$ → poles of OLTF in RHP
- $N$ → CW encirclements of $-1$

used to determine # of roots of

$$1 + L(s) = 0$$

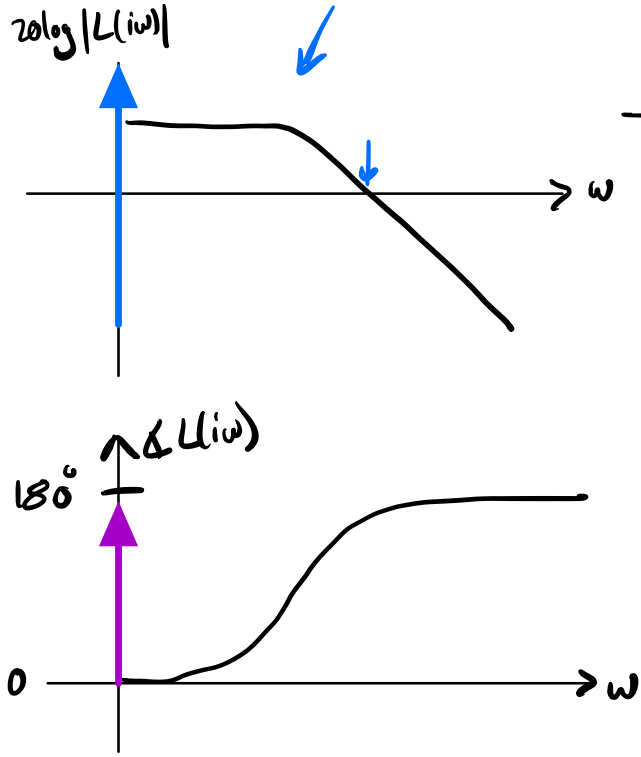
$L(s) = C(s)\,G(s)$

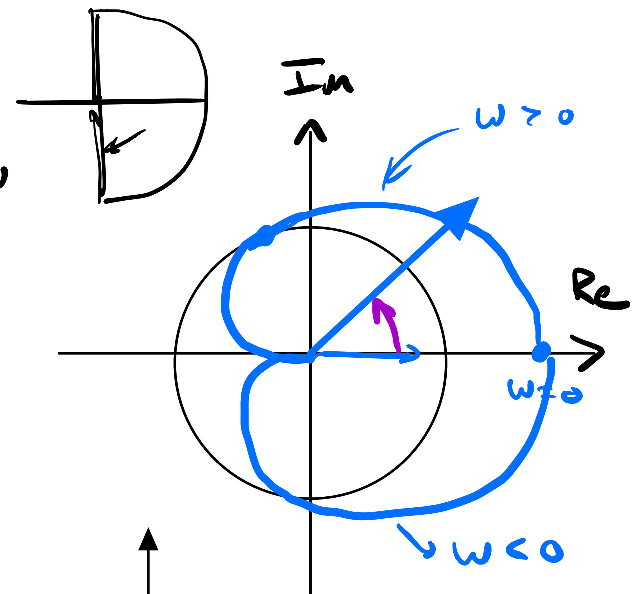
$C(s)$ → controller analytically determined **design**

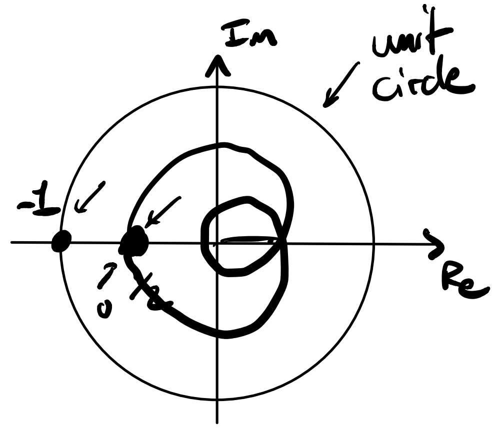
$G(s)$ → Plant
- model
- could be determined experimentally from data

Nyquist stability works even if $G(s)$ is only determined numerically.

# Connections between Nyquist & Bode:




## Stability Margin Example

$$L(s) = \frac{s-1}{(s+1)(s+2)}$$

$N=0$, $P=0 \Rightarrow Z=0 \Rightarrow$ BIBO stable

Suppose there is a pure delay in the system

$e^{-\tau s}$ delay: $\tau$ seconds

Can this cause the closed-loop system to become unstable?

In general: yes...
but for this TF....?

Delayed Loop TF: time delay: $\tau$ seconds

$$e^{-\tau s} L(s)$$

phase shift: $\tau\omega$ phase shift

time freq of input

on imag axis

$$|\underline{e^{-i\tau\omega}}\,\underline{L(i\omega)}| = |\underbrace{e^{-i\tau\omega}}_{1}||L(i\omega)| = |L(i\omega)|$$

$$\angle e^{i\tau\omega} L(i\omega)| = \angle e^{-i\tau\omega} + \angle L(i\omega)$$

$$= -\tau\omega + \angle L(i\omega)$$

Since Nyquist plot is inside the unit circle

$\Rightarrow$ cannot be destabilized by adding a phase shift or a pure delay

$\Rightarrow$ Extremely robust to time delay.

$\Rightarrow$ has <u>infinite</u> <u>delay</u> margin

⇒ has infinite phase margin

No matter how much delay or phase we add we can't destabilize the sys.

It is possible to destabilize the sys by multiplying by a constant $k$

By examining Nyquist plot:

the TF $kL(s)$ is closed loop unstable

if $k \geq 2$ or $k \leq -3$ unstable

if $k \in (-3, 2)$ stable

$\Rightarrow$ The gain margin is $20 \log 2 \simeq 6\,dB$

if Nyquist plot goes through $s = -1$

ie. there exists $\omega$ s.t. $L(i\omega) = -1$

$\Rightarrow$ $s = \pm i\omega$ are closed loop poles

$\Rightarrow$ closed loop sys is not BIBO stable

$$kL(s) = \frac{k(s-1)}{(s+1)(s+2)}$$

$$kL(s) = -1$$

$$\frac{k(s-1)}{(s+1)(s+2)} = -1 \Rightarrow s^2 + 3s + 2 + ks - k = 0$$

$$s^2 + (3+k)s + 2 - k = 0$$

quadratic: $3 + k > 0 \Rightarrow k > -3$

$2 - k > 0 \qquad k < 2$

$3.14 = 180°$

Ex. $L(s) = \frac{4(s+1)}{(s-1)(s+3)}$

$L(s) = -1 \Rightarrow s^2 + 2s - 3 + 4s + 4 = 0$

$s^2 + 6s + 1 = 0$

is stable

roots $= \frac{-6 \pm \sqrt{36-4}}{2}$

$= -3 \pm \sqrt{8} \longrightarrow$ stable

what values of $k$ would cause $kL(s)$ to be unstable

$$kL(s) = -1$$

$$4k(s+1) = -(s-1)(s+3)$$

$$s^2 + (\underline{2+4k})s + \underline{4k-3} = 0$$

$$2+4k > 0 \implies k > -\tfrac{1}{2}$$

$$4k-3 > 0 \implies k > \tfrac{3}{4}$$

$\implies$ for BIBO stability $k > 3/4$

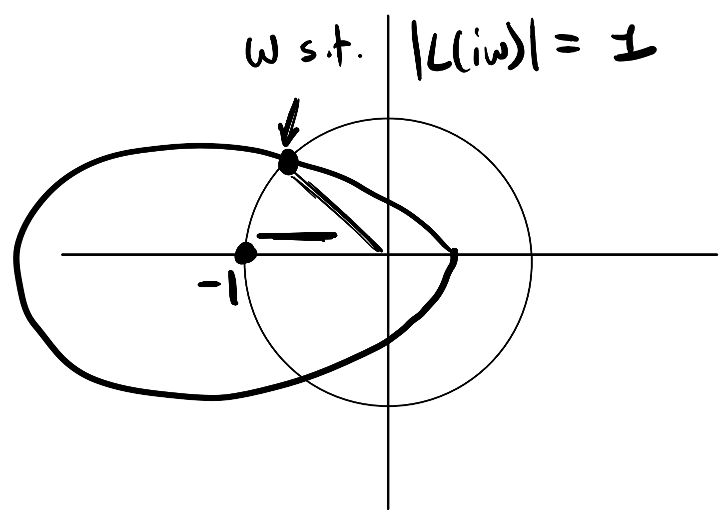
$k = 1$ nominally, $k > 3/4$

$$\implies \underbrace{20\left|\log\tfrac{3}{4}\right| \approx 2.5\,\text{dB}}$$

- <u>Delay margin</u> & <u>phase margin</u>    gain margin

Gain crossover frequency

$\omega$ s.t. $|L(i\omega)| = 1$



$$|L(i\omega)| = \left|\frac{4(s+1)}{(s-1)(s+3)}\right| = \frac{4|i\omega+1|}{|i\omega-1||i\omega+3|} \leftarrow$$

$$= \frac{4\cancel{|i\omega+1|}}{\cancel{|i\omega-1|}|i\omega+3|} = \frac{4\sqrt{\omega^2+1}}{\sqrt{\omega^2+(-1)^2}\,|i\omega+3|}$$

$$|L(i\omega)| = 4\frac{1}{|i\omega+3|} = 1$$

$$|i\omega+3|^2 = 4^2 \Rightarrow \omega^2 + 9 = 16$$

$$\omega^2 = 7$$

$$\boxed{\omega_{gc} = \pm\sqrt{7} \text{ rad/s}}$$

$\omega_{gc}$ : gain crossover frequency

- $e^{-\tau s}L(s)$ plug in $s = i\omega_{gc}$

$$e^{-\tau i\omega_{gc}} L(i\omega_{gc}) = -1 = e^{-i\pi}$$

$$\Rightarrow \underbrace{\angle e^{-\tau i\omega_{gc}}}_{-\sqrt{7}\tau} + \underbrace{\angle L(i\omega_{gc})}_{-1.44} = -\pi$$

$$\Rightarrow -\sqrt{7}\tau = -\pi + 1.44 \text{ rad}$$

$\Rightarrow$ $\boxed{\tau \approx 0.641 \text{ secs}}$ → delay margin

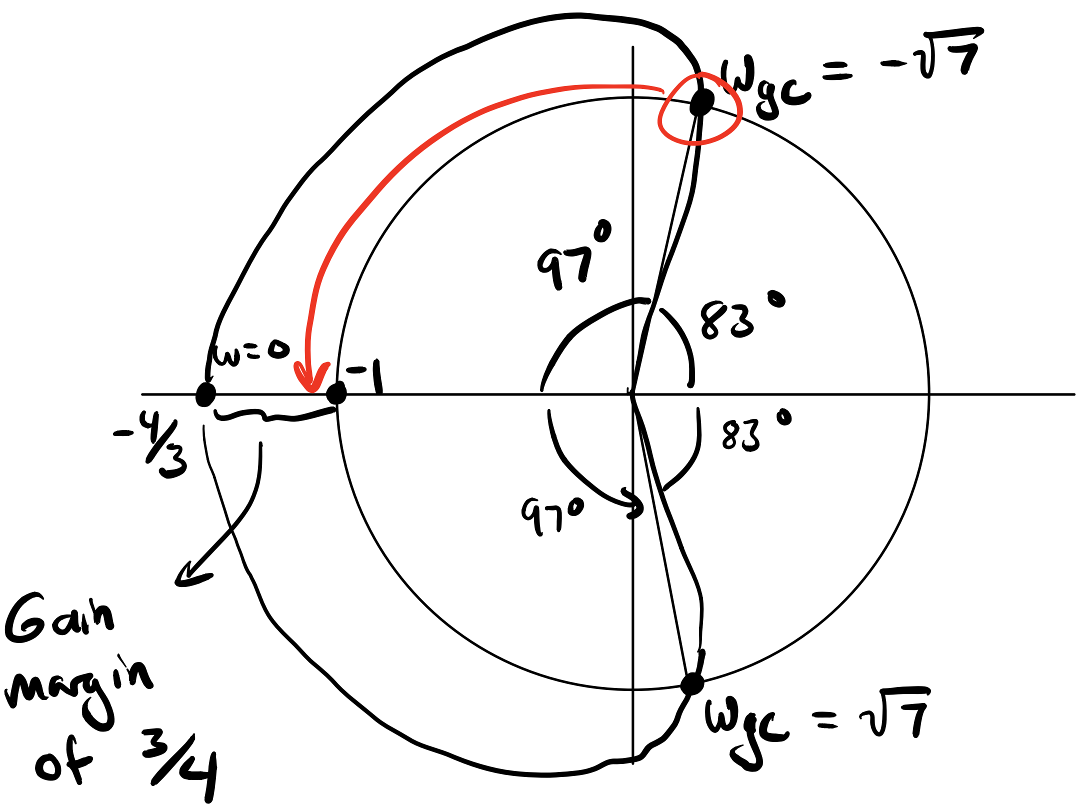
amount of delay you can tolerate while still having stability.

phase margin

$$\tau \omega_{gc} = 0.641\,s \,\sqrt{7}\,\frac{rad}{s}$$

$$= 1.7 \text{ rad} = 97°$$

$$\boxed{\text{Phase margin} = 97°}$$



For a BIBO stable system:

- Gain margin (GM) is the adjustment of $|L(i\omega)|$ s.t. the closed-loop system loses BIBO stability.
- Phase margin (PM) is the adjustment of $\measuredangle L(i\omega)$ s.t. the closed-loop system loses BIBO stability.
- GM and PM are linked to the polar form of a complex number

$$L(i\omega) = \underbrace{|L(i\omega)|}_{\text{mag.}} e^{i \underbrace{\measuredangle L(i\omega)}_{\text{angle}}}$$