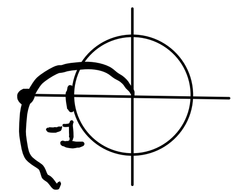
ROBUSTNESS MARGINS:



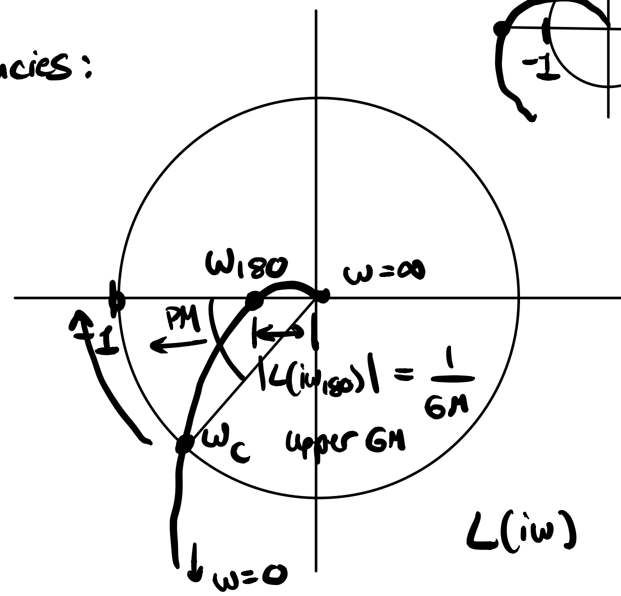
Two important frequencies:

$\omega_{gc} = \omega_c$

$|L(i\omega_c)| = 1$

$\omega_{pc} = \omega_{180}$

$\measuredangle L(i\omega_{180}) = -180^\circ$



GAIN MARGIN

$$GM = \frac{1}{|L(i\omega_{180})|}$$

$GM = 1$ BAD

$GM = 10, \infty$ GOOD

↳ we can't make sys unstable multiplying by pure gain

↳ Upper GM : if $|L(i\omega_{180})| < 1 \Rightarrow GM > 1$

↳ Lower GM if $|L(i\omega_{180})| > 1 \Rightarrow GM < 1$

PHASE MARGIN

$$PM = \measuredangle L(i\omega_c) - (-180^\circ)$$

pure phase shift : $e^{-i\phi}$ $\phi < PM$ to maintain stability time delay:

frequency dep. phase shift

: $e^{-\tau s}$ $\quad \tau \omega_c < PM$

$\Rightarrow$ if $\tau > \frac{PM}{\omega_c} \longrightarrow$ sys goes unstable.

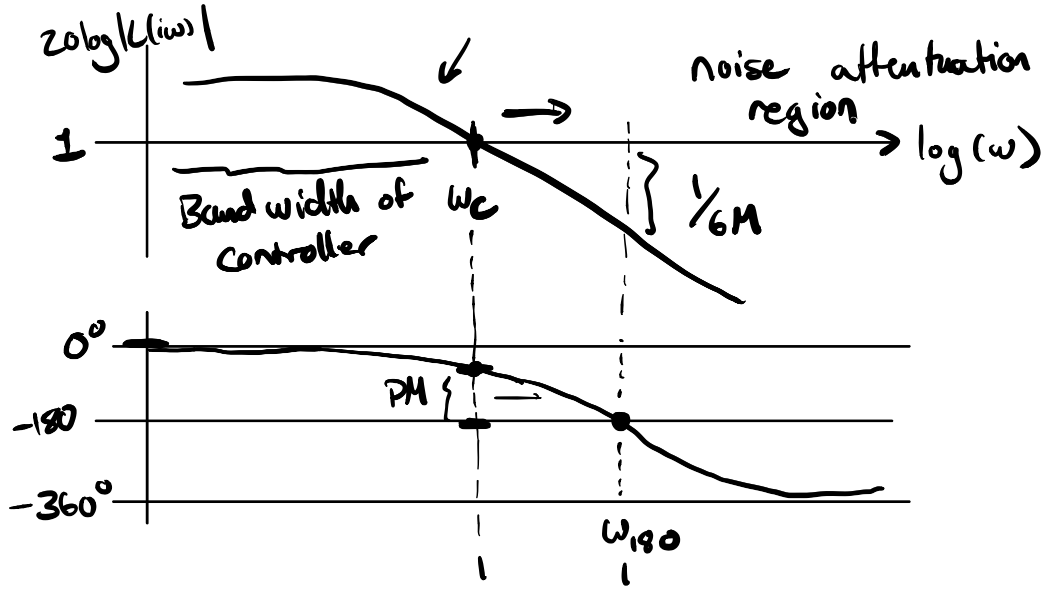
Typical PM: $30^\circ \rightarrow 60^\circ$

if $\omega_c = 1\,Hz = 2\pi \frac{rad}{s}$ $\quad PM = 45^\circ = \frac{\pi}{4}$

$\tau > \frac{\pi}{4} / 2\pi = \frac{1}{8} s \longrightarrow$ sys go unstable

$\omega_c = \underline{10\,Hz} \longrightarrow$ max time delay. $\underline{\frac{1}{80}}\, s$

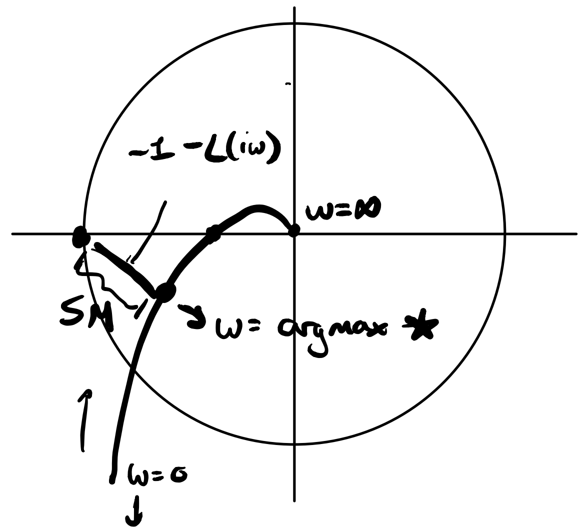
GAIN & PHASE MARGIN FROM BODE



$\omega_c < \omega_{180}$

Stability Margin: Combined Meas.

SM: closest distance on $L(i\omega)$ to -1

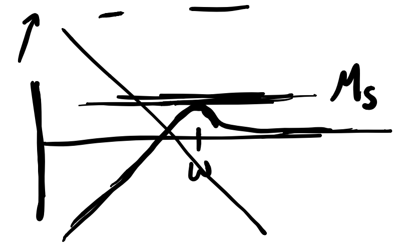


Sensitivity TF:

$$\rightarrow S(s) = \frac{1}{1+L(s)}$$

$$M_s = \max_{\omega}^{(\star)} |S(i\omega)| = \max_{\omega} \left| \frac{1}{-1-L(i\omega)} \right| = \frac{1}{SM}$$

plot Bode of $S(s)$



Relationships between margins

$$M_s = \max_{\omega} \left| \frac{1}{1+L(i\omega)} \right| \geq \left| \frac{1}{1+\underbrace{L(i\omega_{180})}_{-\frac{1}{GM}}} \right| = \left| \frac{1}{1-\frac{1}{GM}} \right|$$

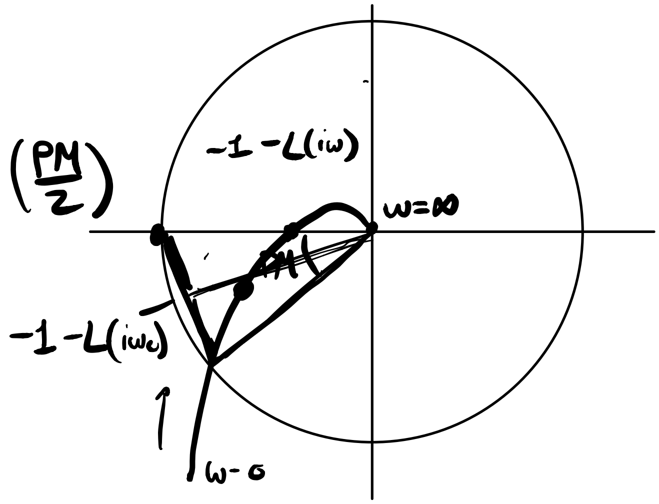
$$\frac{1}{M_s} \leq 1 - \frac{1}{GM}$$

$$-\frac{1}{M_s} + 1 \geq \frac{1}{GM}$$

$$\boxed{GM \geq \frac{1}{1-\frac{1}{M_s}} = \frac{M_s}{M_s - 1} = \frac{1}{1-SM}}$$

Phase Margin:

$$|-1-L(i\omega_c)| = 2\sin\left(\frac{PM}{2}\right)$$

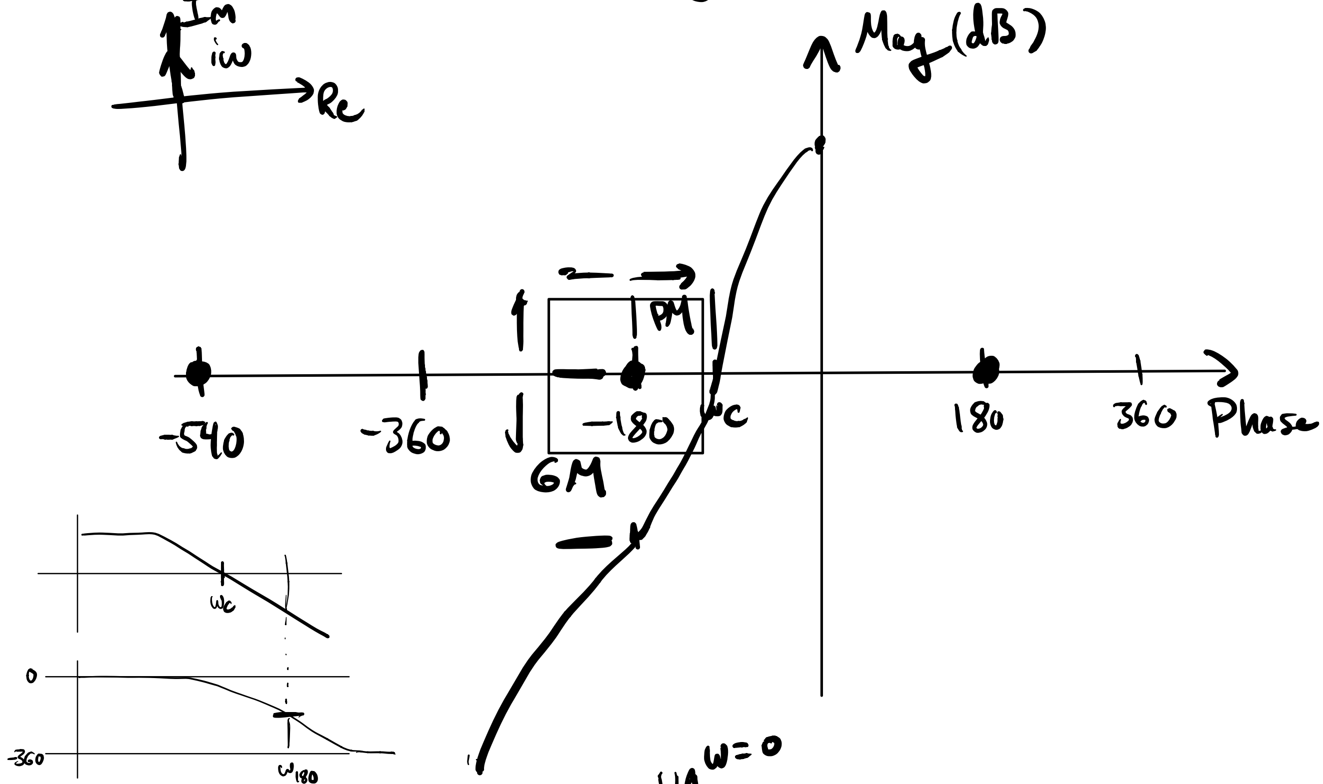


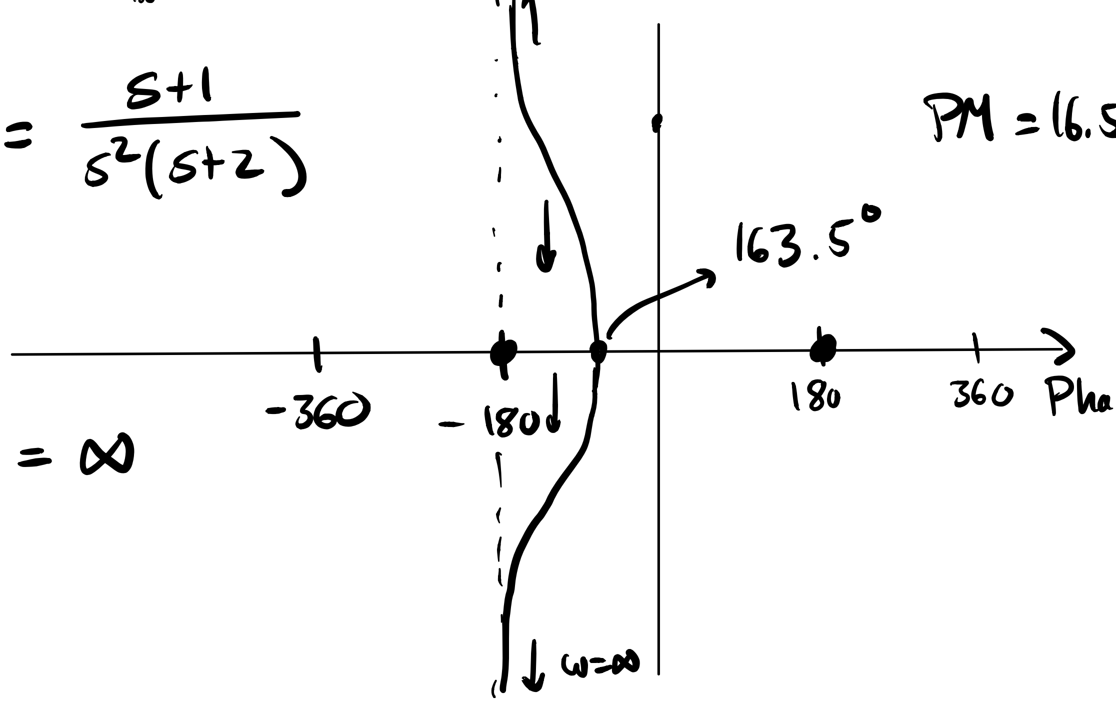
$$M_s = \max_{\omega}\left|\frac{1}{1+L(i\omega)}\right| \geq \left|\frac{1}{-1-L(i\omega_c)}\right| = \frac{1}{2\sin\left(\frac{PM}{2}\right)}$$

$$\frac{1}{M_s} \leq 2\sin\left(\frac{PM}{2}\right) \Longrightarrow$$

$$\boxed{\begin{aligned} PM &\geq 2\arcsin\left(\frac{1}{2M_s}\right) \geq \frac{1}{M_s} \\ &\geq 2\arcsin\left(\tfrac{1}{2}SM\right) \geq SM \end{aligned}}$$

Nichols Plot   Mag & phase





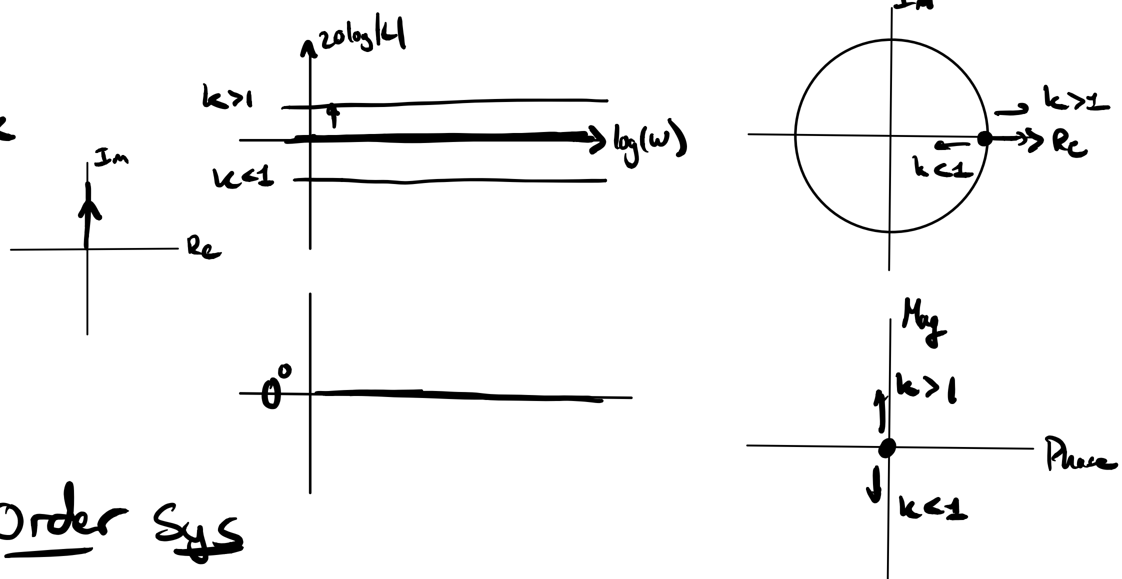
$$L(s) = \frac{s+1}{s^2(s+2)}$$

PM = 16.5°

GM = ∞

mathlets.org/mathlets/bode-nyquist-plots/

$L(s) = k$

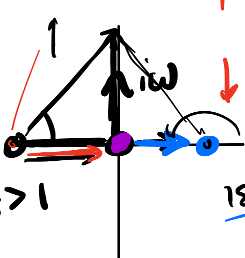


First Order Sys

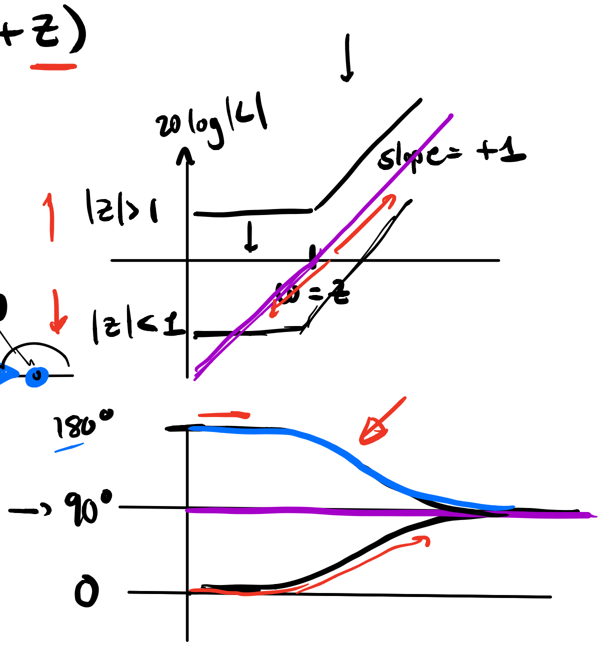
$L(s) = (s + z)$

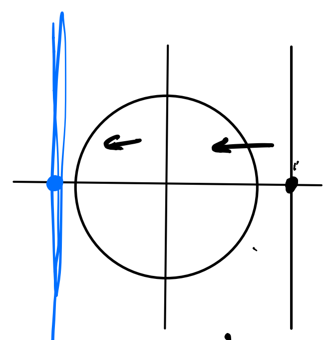
iw

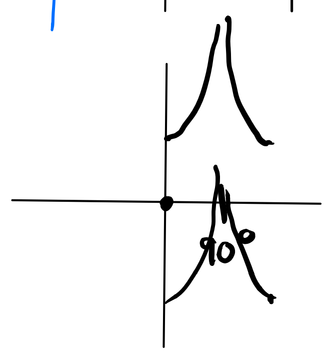
$\boxed{z = 0}$



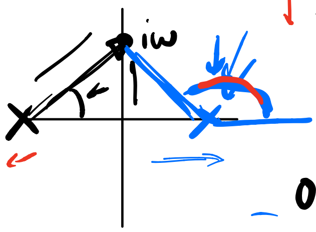
$L(s) = s$

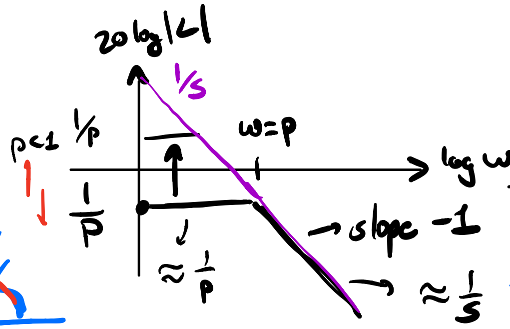


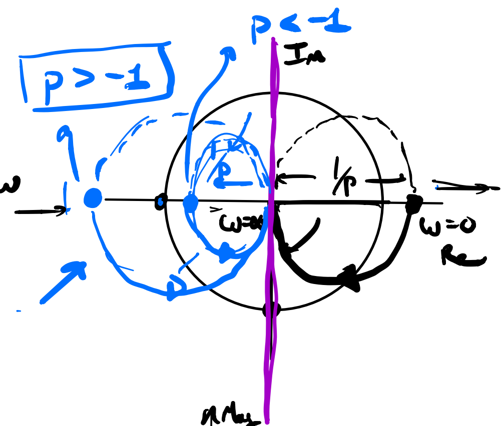




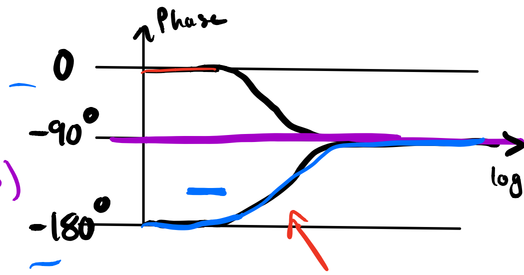
$L(s) = \frac{1}{s+p}$

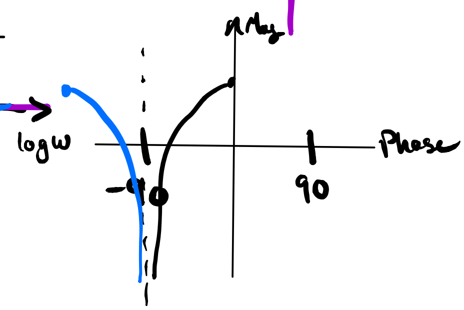






$L(s) = \frac{1}{s}$ (p=0)





CL stability

$1 + L(s) = 0$

$1 + \frac{1}{s+p} = 0 \Rightarrow s+p+1 = 0$

$p+1 > 0 \Rightarrow \boxed{p > -1}$

$p < -1$

2ND ORDER SYS:

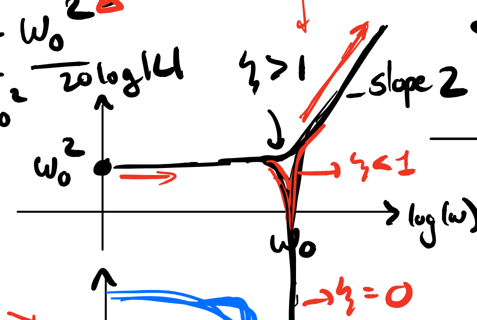
ω_0 = natural freq ζ : damping ratio

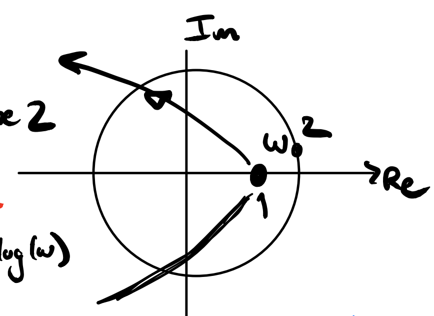
$L(s) = as^2 + bs + c$ $c > 0$

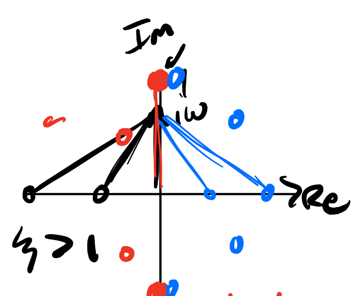
$= s^2 + 2\omega_0\zeta s + \omega_0^2$

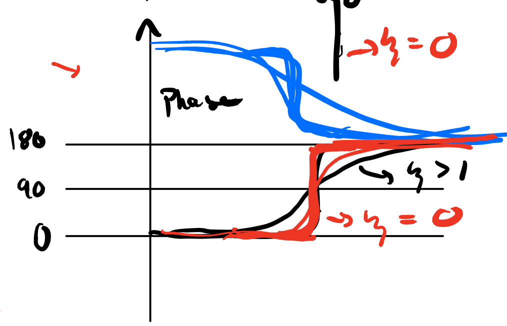
$z_{1,2} = -\omega_0\zeta \pm \sqrt{\omega_0^2\zeta^2 - \omega_0^2}$

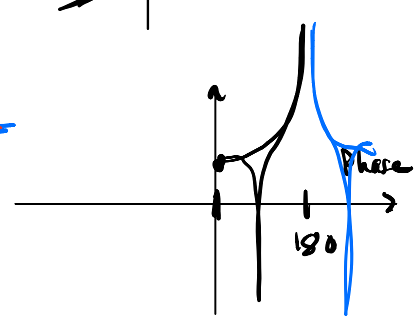
$= -\omega_0\zeta \pm \omega_0\sqrt{\zeta^2 - 1}$







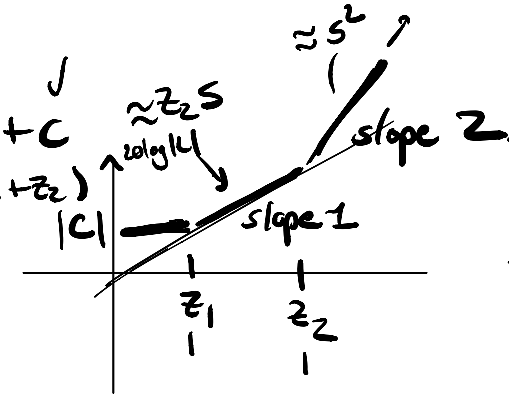


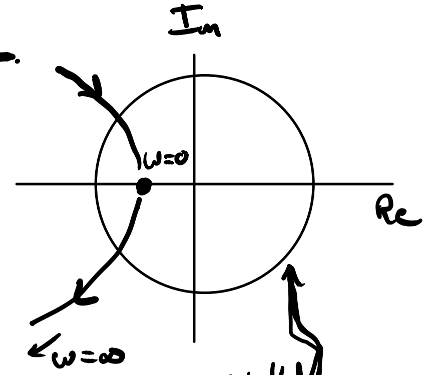


$\zeta < 1 \rightarrow$ underdamped

$\zeta = 0$ $\quad c < 0$

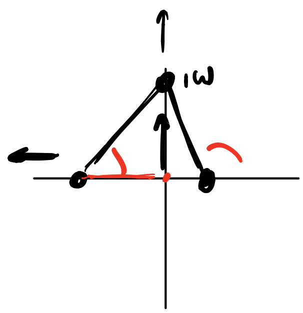
$$L(s) = s^2 + bs + c$$
$$= (s+z_1)(s+z_2)$$

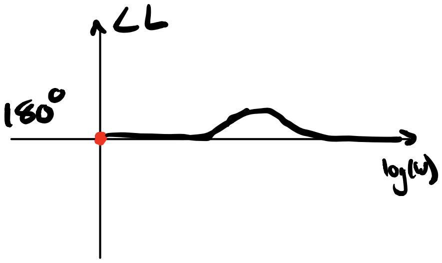


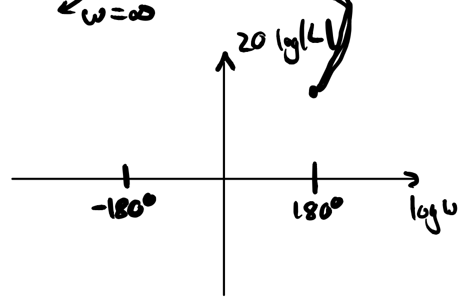


$z_1 > 0$, $z_2 < 0$

$|z_1| << |z_2|$



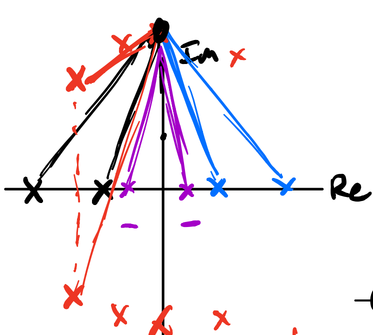


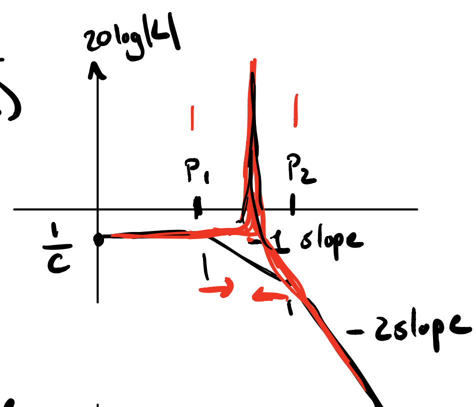


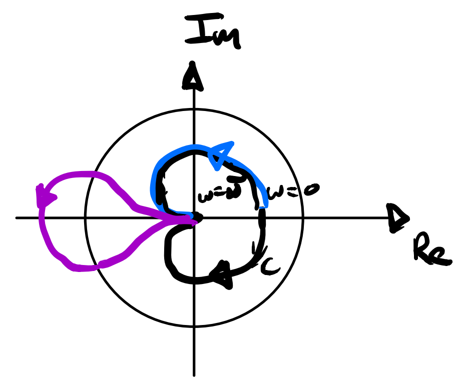
2 poles:

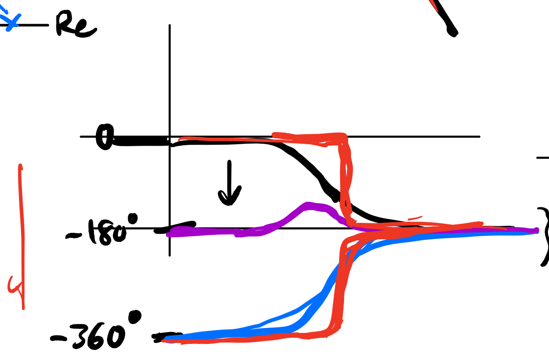
$$L(s) = \frac{1}{as^2 + bs + c}$$
$$= \frac{1}{(s+p_1)(s+p_2)}$$

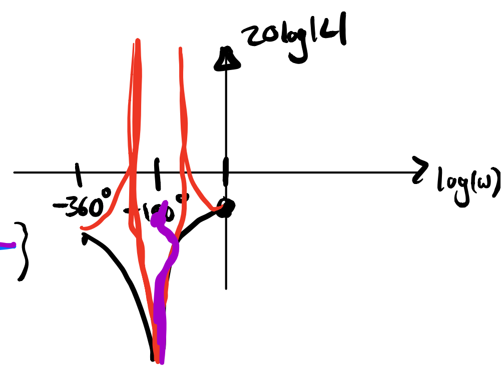
p_1, p_2 real $\quad p_1, p_2 < 0$





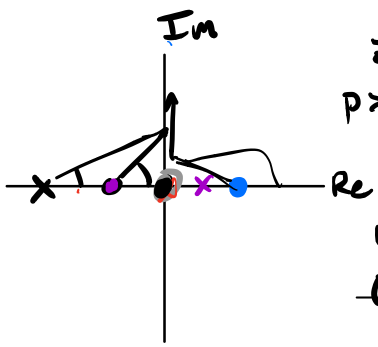


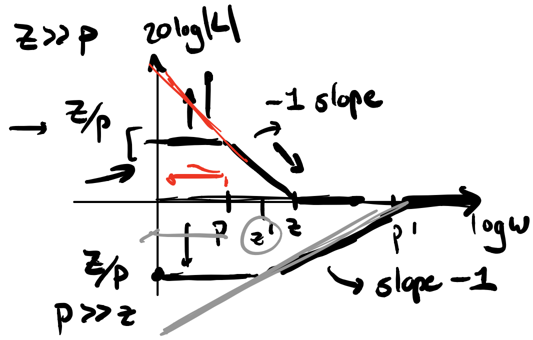


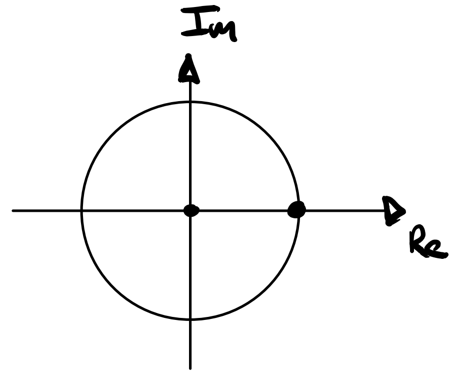


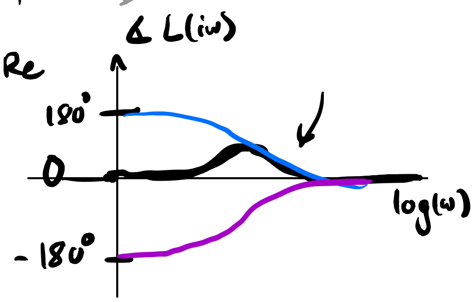
$$L(s) = \frac{s+z}{s+p}$$

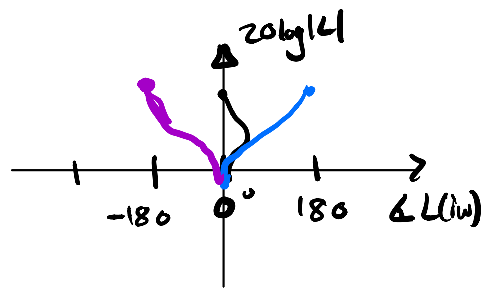
$z/p \quad \omega = 0 \qquad 1 \quad \omega = \infty$

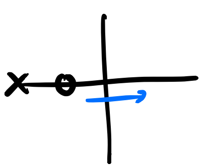


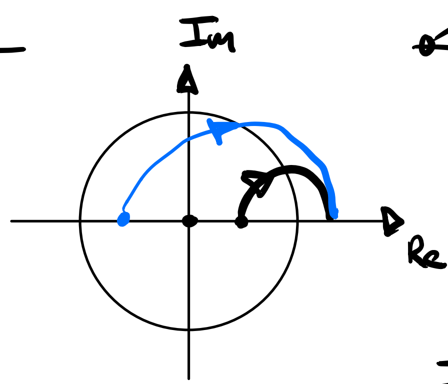


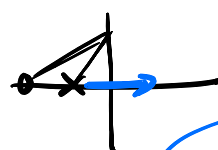


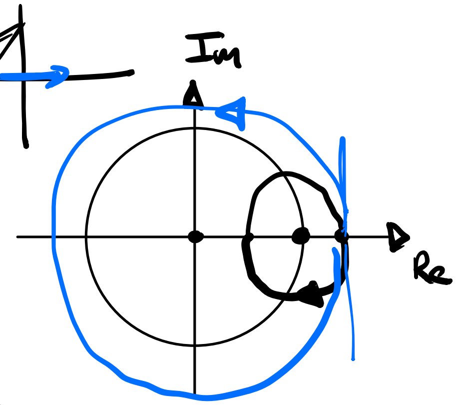


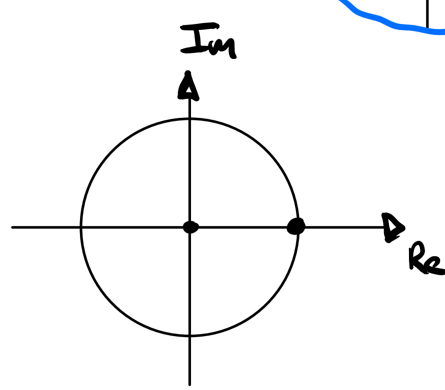












Transfer function
is <u>stable</u> - mini<u>mum</u> phase

$\Rightarrow$ all poles & zeros are in LHP