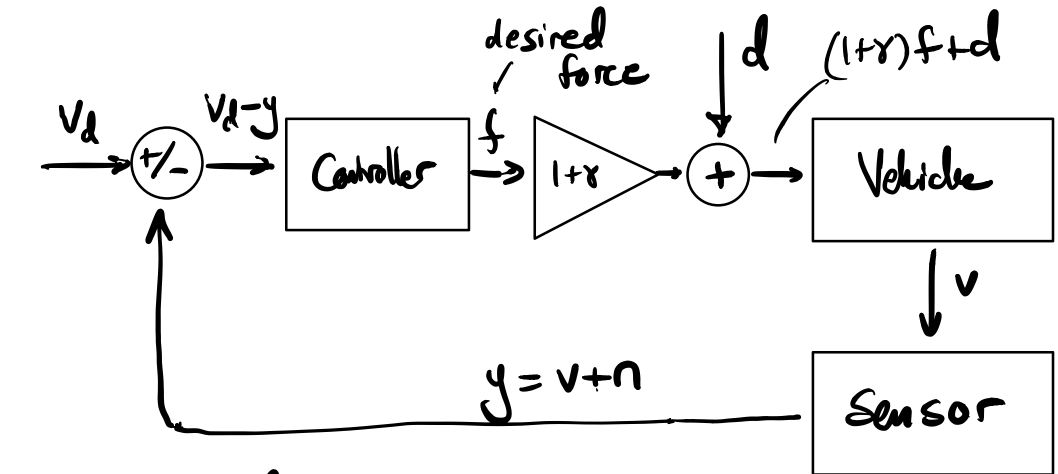


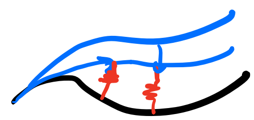
Proportional Controller:

$$f = -K_p (y - v_d)$$

Note: PID controller

$$f = -k_p x - k_D \dot{x} - k_I \int x$$

$$f = k \Delta x$$



$$x = v - v_d$$

$$\dot{x} = \dot{v} - \dot{v}_d = -\frac{k_p}{m}(v - v_d) - \dot{v}_d$$

tracking error dynamics

assuming that v_d = constant.

$$\dot{x} = -\underbrace{\frac{k_p}{m}}_{\alpha_p} x$$

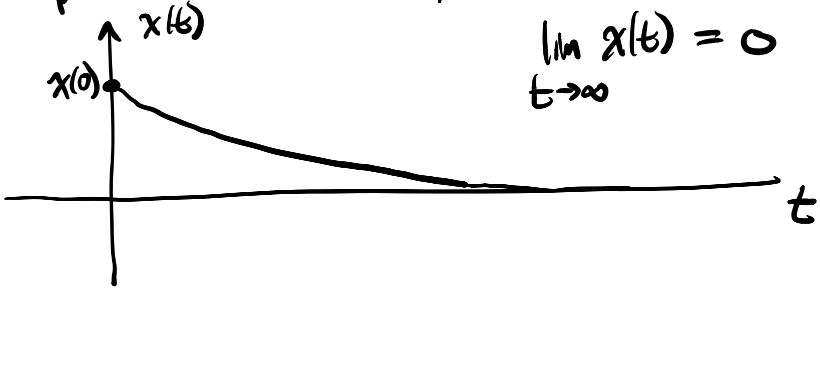
Solution:

$$x(t) = e^{-\alpha_p t} x(0)$$

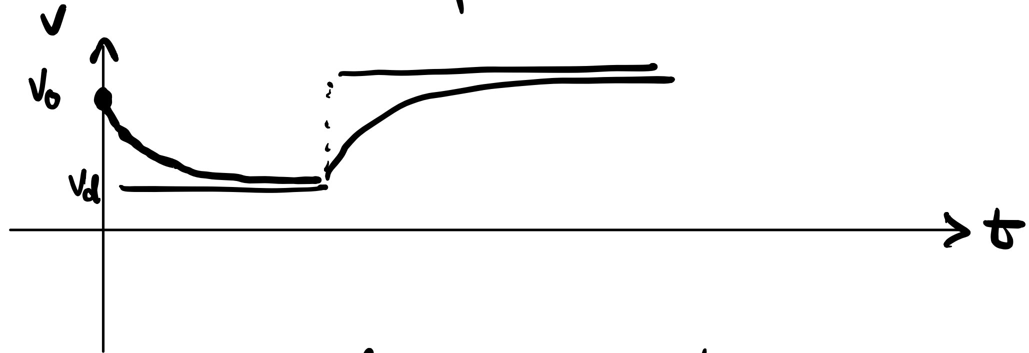
$\alpha_p > 0$

if $\alpha_p(t)$: $\alpha_p(t) \geq \alpha_{min} > 0$

$$\lim_{t \to \infty} x(t) = 0$$



since α_p depends on m ...
This control algorithm is robust to uncertainty due to mass variation as long as v_d is constant (piecewise constant in time)



- How big should we make k_p?

$$x(t) = e^{-\alpha_p t} x(0) \qquad \alpha_p = \frac{k_p}{m} \qquad k_p \uparrow \quad \alpha_p \uparrow$$

- Why not large k_p? large α_p quick decay
  - Very uncomfortable (jerky)
  - Actuator limits
  - Over correction
  - Too sensitive to noise

Add in the effect of disturbance d.

Dynamics:

$$\underline{m} \dot{v} = -k_p (v - v_d) + d$$

Newton:

$\underline{f} = \underline{m}\underline{a}$

$\underline{a} = \frac{f}{\underline{m}} \leftarrow$

for simplicity: $d =$ constant.

$$\dot{x} = -\alpha_p x + \frac{d}{m} \to \sigma$$

$$= -\alpha_p x + \sigma$$

$M\vec{a} = f$

$\vec{a} = M^{-1} f$

$$\begin{bmatrix} \cdot & 0 \\ 0 & \cdot \end{bmatrix}$$

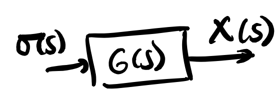
Laplace Transform:

$$s X(s) - x_0 = -\alpha_p X(s) + \sigma(s)$$

$$X(s) = \underbrace{\frac{x_0}{s+\alpha_p}}_{\text{init cond. contribution}} + \underbrace{\frac{\sigma(s)}{s+\alpha_p}}_{\text{disturbance contribution}}$$

if $x_0 = 0$ $\quad X(s) = \frac{1}{s+\alpha_p}\sigma(s)$

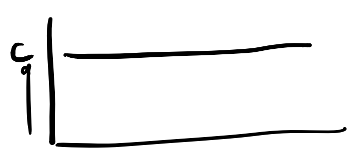
$G(s) = \frac{1}{s+\alpha_p}$ transfer function



if σ = constant

$\sigma = c$

$\sigma(s) = c\frac{1}{s}$

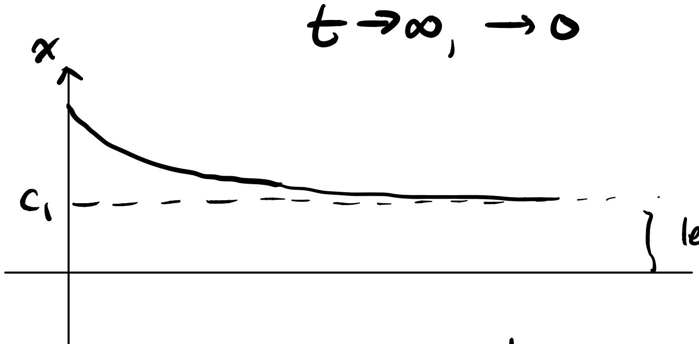


$$\Rightarrow X(s) = \frac{x_0}{s+\alpha_p} + \underbrace{\frac{c}{(s+\alpha_p)s}}_{\frac{c_1}{s} + \frac{c_2}{s+\alpha_p}}$$

for some c_1, c_2

$$\Rightarrow X(s) = \frac{x_0 + c_2}{s+\alpha_p} + \frac{c_1}{s}$$

$$\Rightarrow x(t) = \underbrace{e^{-\alpha_p t}(x_0 + c_2)}_{t \to \infty,\ \to 0} + \underline{c_1 u_{step}(t)}$$



left over error trajectory

$$G(s) = \frac{X(s)}{\sigma(s)} = \frac{1}{s + \alpha_p}$$

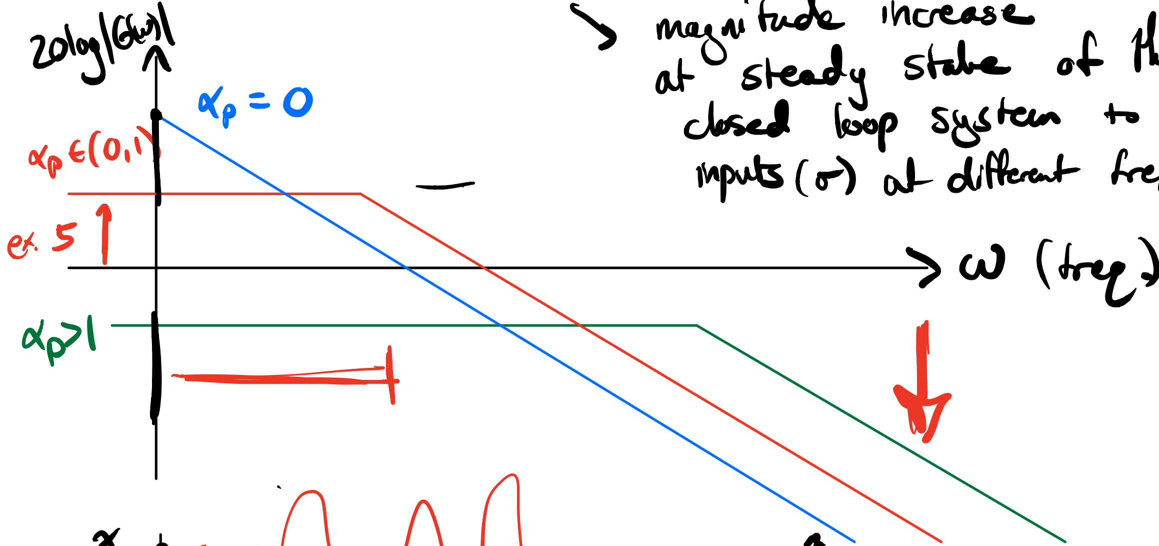
$\alpha_p > 0$
↓
BIBO stable
(bounded input, bounded output)

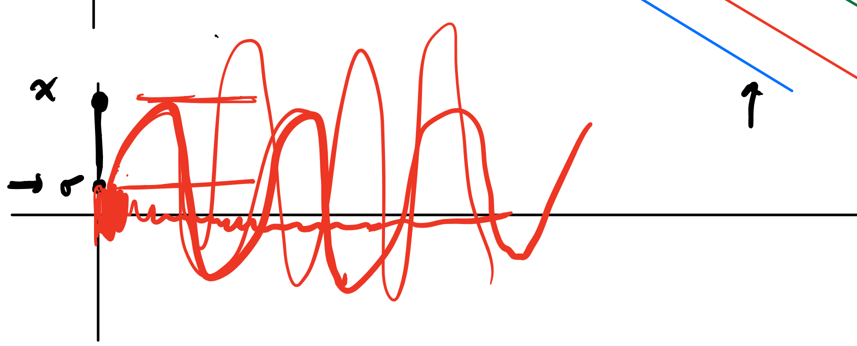
Pole is at $s = \underline{-\alpha_p} < 0$

Magnitude of TF $G(s)$

Bode magnitude plot

→ magnitude increase at steady state of the closed loop system to inputs (σ) at different frequencies





if there is no control: $\alpha_p = 0$

$$\dot{x} = \underline{\sigma} \Rightarrow x(t) = \sigma t$$

$$\lim_{t\to\infty} x(t) = \infty \quad \text{unacceptable}$$

Suppose v_d is not constant...

$$\Rightarrow \dot{x} = \dot{v} - \underline{\dot{v}_d} = -\alpha_p x + \underbrace{\sigma - \dot{v}_d}_{\text{new disturbance}}$$

want to design controllers for <u>disturbance rejection</u>

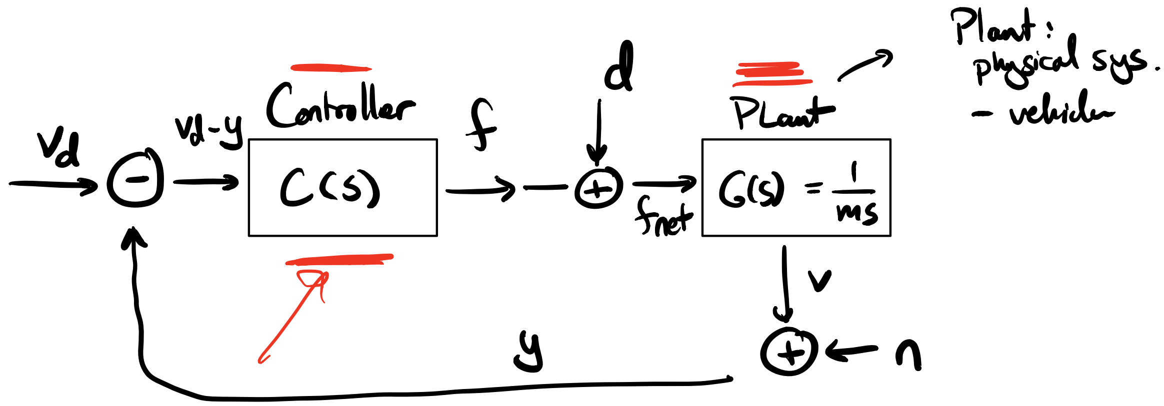
$\dot{v}_d \rightarrow$ part of disturbance

- Another source of trouble: measurement noise

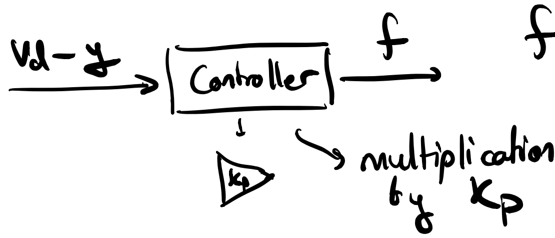
$$y = v + n$$

$$\Rightarrow \dot{x} = -\alpha_p x + \underbrace{\sigma - \dot{v}_d + \alpha_p n}_{\text{newer disturbance}}$$

Choosing a large α_p amplifies the effect of noise...



Proportional Control



$f = -k_p(v_d - y)$

Want to write transfer functions to v for all external inputs.

from d to v, v_d to v, n to v

Dynamics $\quad m\dot{v} = f_{net} = f + d$

$$msV(s) = F(s) + D(s) = F_{net}(s)$$

$$F(s) = C(s)\left(V_d(s) - Y(s)\right)$$

$$Y(s) = V(s) + N(s)$$

$$V(s) = G(s)F_{net}(s)$$

$$= G(s)\left[C(s)\left(V_d(s) - Y(s)\right) + D(s)\right]$$

$$= G(s)\left[C(s)\left(V_d(s) - V(s) - N(s)\right) + D(s)\right]$$

$$\left(1+G(s)C(s)\right)V(s) = G(s)C(s)\left[V_d(s) - N(s)\right] + G(s)D(s)$$

$$\boxed{\underbrace{V(s)}_{\text{output}} = \frac{G(s)C(s)}{1+G(s)C(s)}\left[\underbrace{V_d(s)}_{\text{input}} - \underbrace{N(s)}_{\text{input}}\right] + \frac{G(s)}{1+G(s)C(s)}\underbrace{D(s)}_{\text{input}}}$$

↳ Algebraic relationships in frequency domain between the inputs v_d, n, d and the output v, given in terms of the system $G(s)$, $C(s)$

$$T(s) \triangleq \frac{G(s)C(s)}{1+G(s)C(s)} \qquad R(s) = \frac{G(s)}{1+G(s)C(s)}$$

$L(s) \triangleq G(s)C(s)$: loop transfer function

$$T(s) = \frac{L(s)}{1+L(s)} \qquad R(s) = \frac{G(s)}{1+L(s)}$$

Want to design $C(s)$ so that $L(s)$ has desirable properties.

$$V(s) = T(s)\left(V_d(s) - N(s)\right) + R(s)D(s)$$

what do we want $T(s)$ & $R(s)$ to look like?

Performance Criteria:

- tracking (track V_d closely): $T(s) \approx 1$ | ✓ ✗
- noise filtering: $T(s) \approx 0$ | ✗
- disturbance rejection: $R(s) \approx 0$ |

---

frequency dependent relationships..

design $T(s)$, $R(s)$ to have different properties at different frequencies.

- $V_d(s)$: low frequency (desired velocity doesn't change too fast)
- $N(s)$: high frequency
- $D(s)$: low frequency (some external force pushing on system)

Design criteria:

$$\left.\begin{aligned} T(s) &\approx 1 \\ R(s) &\approx 0 \end{aligned}\right\} \rightarrow \text{low freq.}$$ good tracking & disturbance rejection

$$\left.\overline{T(s)} \approx 0\right\} \rightarrow \text{high freq.}$$ noise filtering.

$$T(s) = \frac{L(s)}{1+L(s)} \qquad R(s) = \frac{G(s)}{1+L(s)}$$

How should we shape $L(s)$?

want $L(s)$ large at low frequencies...

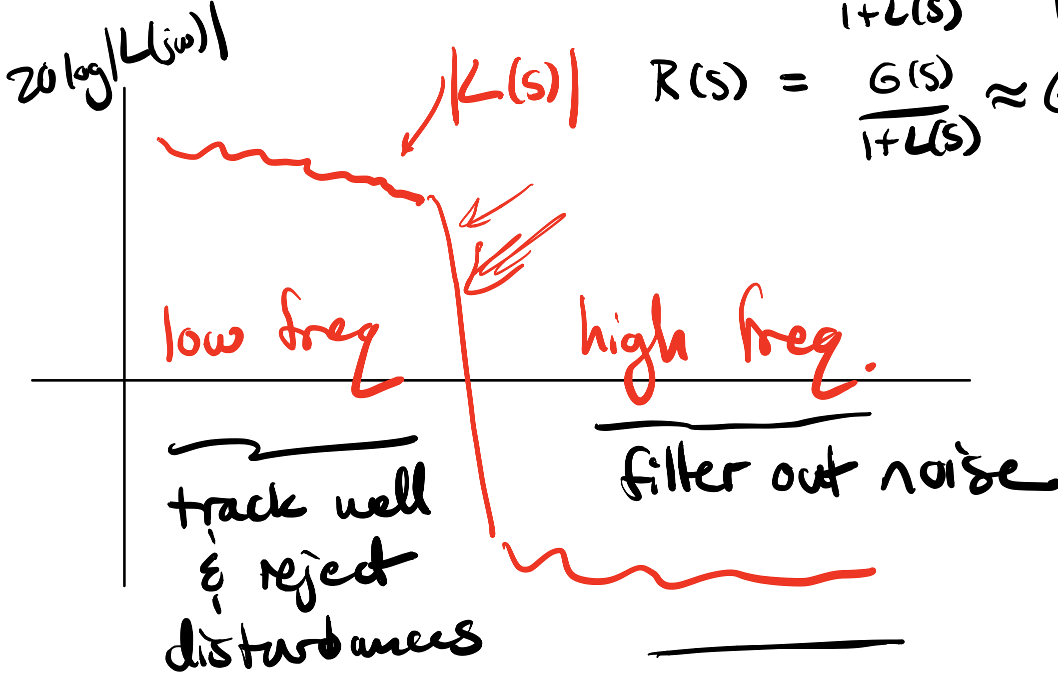
small at high frequencies..

low freq: $L(s)$ large $\quad T(s) = \dfrac{L(s)}{\cancel{1}+L(s)} \approx \boxed{1}$

$$R(s) = \frac{G(s)}{1+L(s)} \approx \boxed{0}$$

high freq: $L(s)$ small $\quad T(s) = \dfrac{L(s)}{1+L(s)} \approx \dfrac{L(s)}{1} \approx \boxed{0}$

$$R(s) = \frac{G(s)}{1+L(s)} \approx G(s)$$



Loop Shaping: designing $L(s) = G(s)\boxed{C(s)}$