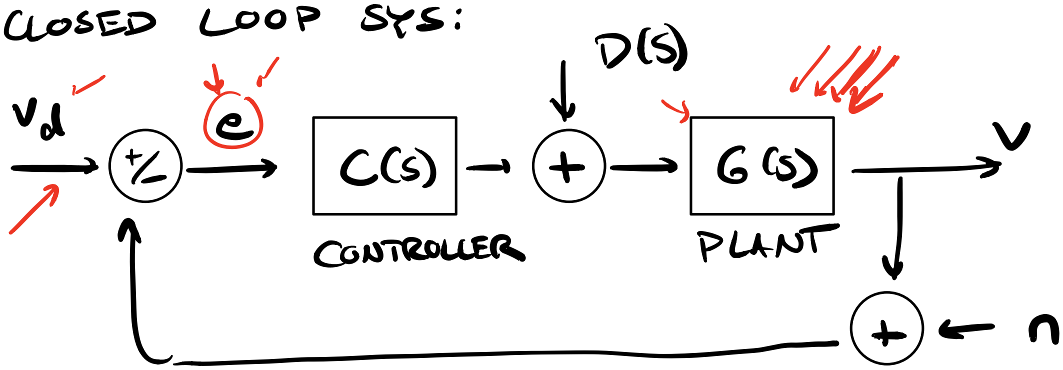
CLOSED LOOP SYS:



$$L(s) = C(s)\,G(s)$$

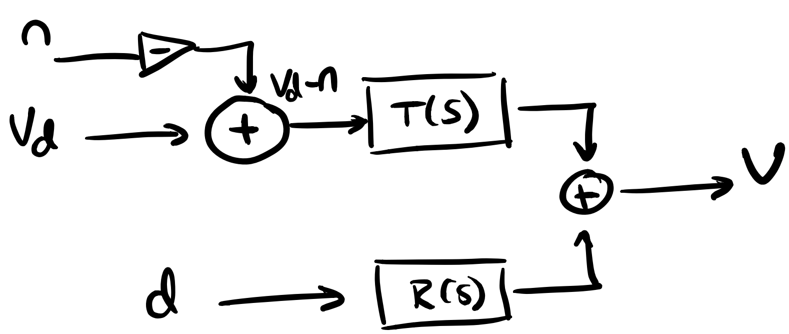
Plant: $G(s) = \frac{1}{ms}$

proportional controller: $C(s) = k_p$

Disturbance: $D(s)$ process noise affects dynamics

MEAS NOISE: $N(s)$

$$V(s) = \underbrace{\frac{C(s)G(s)}{1+C(s)G(s)}}_{\triangleq\, T(s)}\left(V_d(s) - N(s)\right) + \underbrace{\frac{G(s)}{1+C(s)G(s)}}_{\triangleq\, R(s)} D(s)$$



- $T(s)$ and $R(s)$ are closed loop TFs
- Loop TF, $\underset{\text{loop}}{L(s)} \triangleq \underset{\text{controller}}{\boxed{C(s)}}\,\underset{\text{plant}}{G(s)}$

$$T(s) = \frac{L(s)}{1+L(s)} \qquad R(s) = \frac{G(s)}{1+L(s)}$$

$$V(s) = T(s)\left(\underline{V_d(s)} - \underline{N(s)}\right) + R(s)D(s)$$

Ideally $T(s) \approx 1$ and $\underline{R(s) \approx 0}$

good tracking — good disturbance rejection.

Sort of good → problem : no noise filtering...

Fortunately, for real world systems...

$V_d(s)$ & $D(s)$ ⟶ low frequency signals

$N(s)$ ⟶ high frequency signal

			Range	Criteria for $L(s)$
Tracking	$\lvert T(s)\rvert = 1$	$T(s) = \frac{L(s)}{1+L(s)}$	low freq.	$\lvert L(s)\rvert \approx \infty$
Disturbance Rejection	$\lvert R(s)\rvert = 0$	$R(s) = \frac{G(s)}{1+L(s)}$	low freq.	$\lvert L(s)\rvert \approx \infty$
Noise Filtering	$\lvert T(s)\rvert = 0$	$T(s) = \frac{L(s)}{1+L(s)}$	high freq.	$\lvert L(s)\rvert \approx 0$

Summary: Design criteria.

- $L(s) \to \infty$ at low freq.
- $L(s) \to 0$ at high freq.

also TF to be BIBO stable.

Denominators of $T(s)$ & $R(s)$

$1 + L(s) = 0$ roots are called poles.

all poles of $T(s)$ & $R(s)$ to have negative real parts $\to$ BIBO stability

DESIGN CRITERIA:

1. BIBO stability

   solns of $1 + L(s) = 0$ have negative real parts

2. $|L(s)|$ large for low freq.
3. $|L(s)|$ small for high freq.

Car on the road example:

$$\left.\begin{aligned} G(s) &= \frac{1}{ms} \quad \text{plant} \\ C(s) &= k_p \quad \text{control} \end{aligned}\right\} \Rightarrow L(s) = \frac{k_p}{ms}$$

1. Stability: $1 + L(s) = 1 + \frac{k_p}{ms} = 0$

$$\Rightarrow \frac{ms + k_p}{ms} = 0 \Rightarrow \boxed{s = -\frac{k_p}{m}}$$

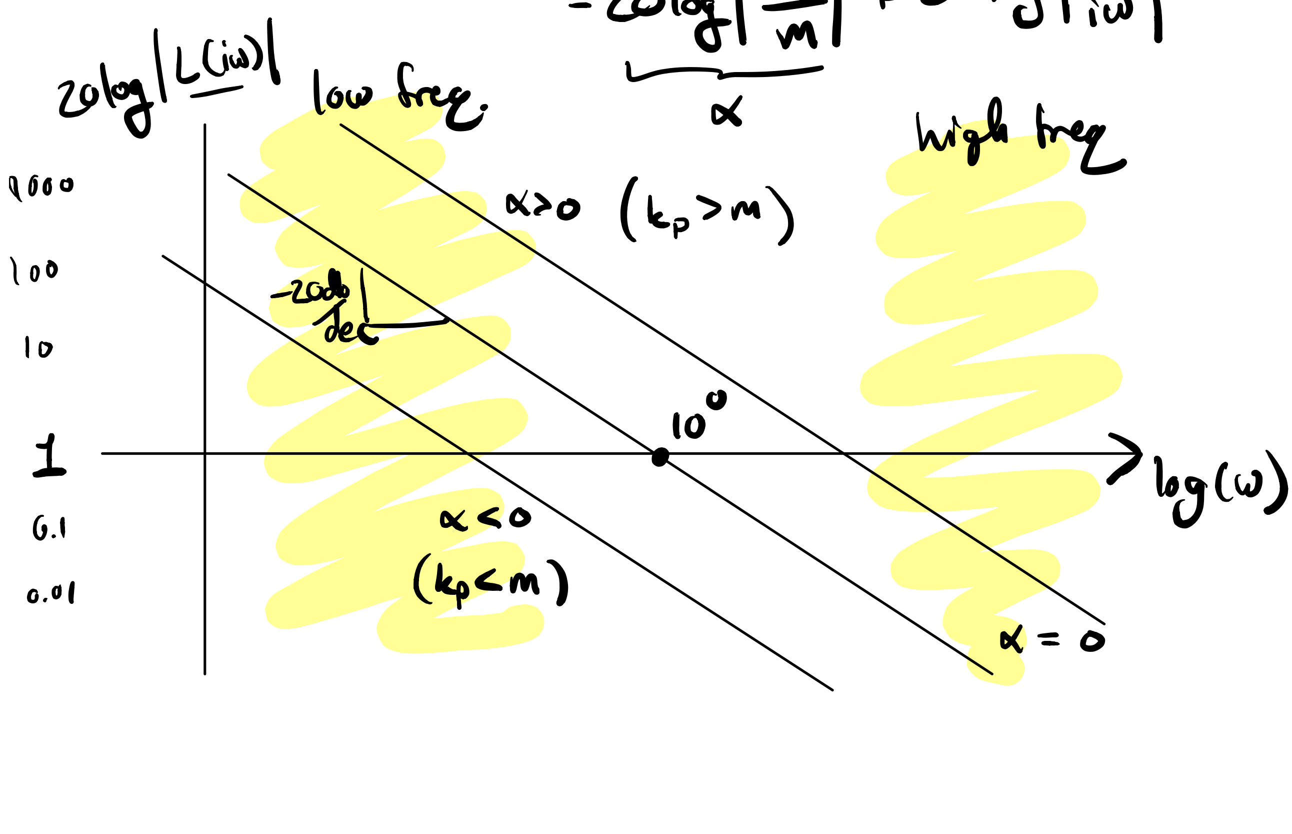
only closed loop pole

$k_p > 0$ guarantees stability.

$L(s) = \frac{k_p}{ms} \qquad k_p > 0$

$$\Rightarrow 20\log|L(i\omega)| = 20\log\left|\frac{k_p}{m}\frac{1}{i\omega}\right|$$

$$= \underbrace{20\log\left|\frac{k_p}{m}\right|}_{\alpha} + 20\log\left|\frac{1}{i\omega}\right|$$



· Suppose constant disturbance d.

$\omega = 0$
$i\omega = i0$.

$D(s) = \frac{a}{s}$

0 frequency

$$R(s) = \frac{G(s)}{1+L(s)} = \frac{\frac{1}{ms}}{\frac{ms+k_p}{ms}} = \frac{1}{ms+k_p}$$

constant input: $\omega = 0$
$u_{step} = e^{i0t} = 1$

assume

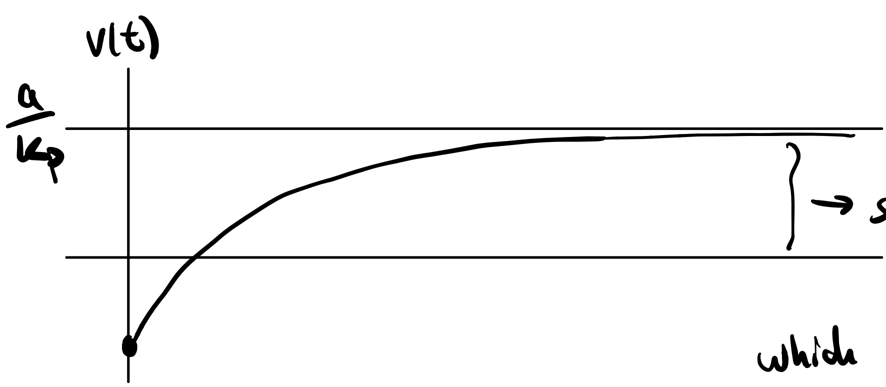
$v_d(t) = 0 \quad n(t) = 0$

$$\Rightarrow \quad u(t) = a\, u_{step}(t) = a e^{i0t} \qquad \underline{i\omega = 0}$$

$$V(s) = R(s)\,U(s)$$

$$v_{ss}(t) = |R(0)|\, e^{i(0+\angle R(i0))}\, a$$

$$v_{ss}(t) = \underline{R(0)}\, a = \frac{a}{k_p}$$



→ steady state tracking error which is not zero!

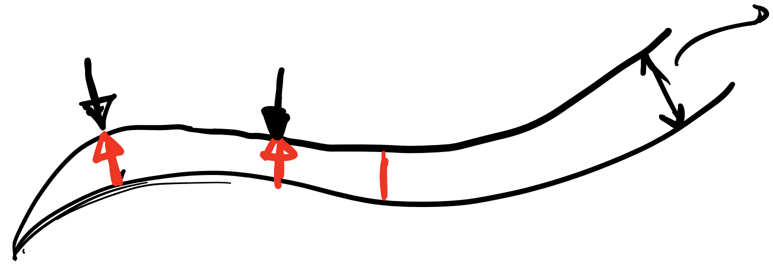
Reminder

$$u \longrightarrow \boxed{H(s)} \longrightarrow y \qquad Y(s) = H(s)\,U(s)$$

assuming BIBO stability

$$y(t) \longrightarrow y_{ss}(t) = |H(i\omega)|\, e^{i(\omega t + \measuredangle H(i\omega))}$$

if $u(t) = e^{i\omega t}$



$\frac{a}{k_p}$

lager control value will make the tracking error smaller

Problems: w large k_p

- actuators might not be able to make k_p large enough
- even for large k_p, tracking error is still > 0.

what can we do?

New controller: Proportional - Integral (PI) Controller

$$C(s) = \underbrace{k_p}_{\text{proportional part}} + \underbrace{\frac{k_I}{s}}_{\text{integral part}}$$

$$L(s) = G(s)C(s) = \frac{1}{ms}\left(k_p + \frac{k_I}{s}\right)$$

$$\Rightarrow L(s) = \frac{k_p s + k_I}{ms^2}$$

1. stability? $1 + L(s) = 0$

$$1 + \frac{k_p s + k_I}{ms^2} = 0$$

$$\Rightarrow ms^2 + k_p s + k_I = 0$$

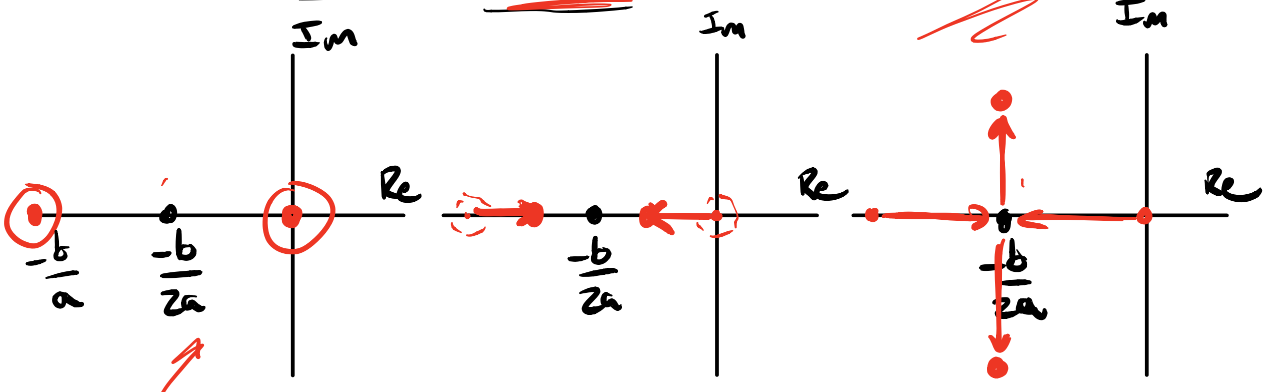
$m>0$, $\underline{k_p>0}$, $\underline{k_I>0}$ } → enough to guarantee stability.

Details:

$as^2 + bs + c = 0 \qquad a, b, c > 0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \underline{-\frac{b}{2a}} \pm \sqrt{\underline{\left(\frac{b}{2a}\right)^2} - \underline{\frac{c}{a}}}$$



$c=0$

$c>0$, $\left(\frac{b}{2a}\right)^2 - \frac{c}{a} > 0$

$c>0$

$\left(\frac{b}{2a}\right)^2 - \frac{c}{a} < 0$

Disturbance Rejection: constant disturbance

$V_{ss}(t) = R(0)\,a \qquad V_d(t) = 0$

$$R(s) = \frac{\frac{1}{ms}}{1 + (k_p s + k_I)/ms^2} = \frac{s}{ms^2 + k_p s + k_I}$$

plug in $i\omega = 0$

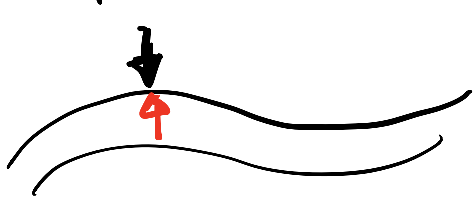
$$R(0) = \frac{0}{0 + 0 + k_I} = 0 \implies V_{ss}(t) = 0 = V_d(t)$$

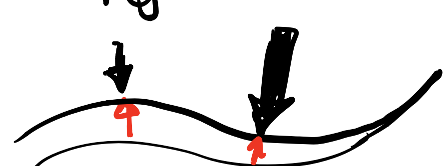
Remark:

car control system. TF $G(s) = \frac{1}{ms}$
(single integrator)

a proportional controller
P-controller : stabilizes the system & can attenuate tracking but it can't do full disturbance rejection.

a prop. integral controller
PI controller : stability & complete disturbance rejection





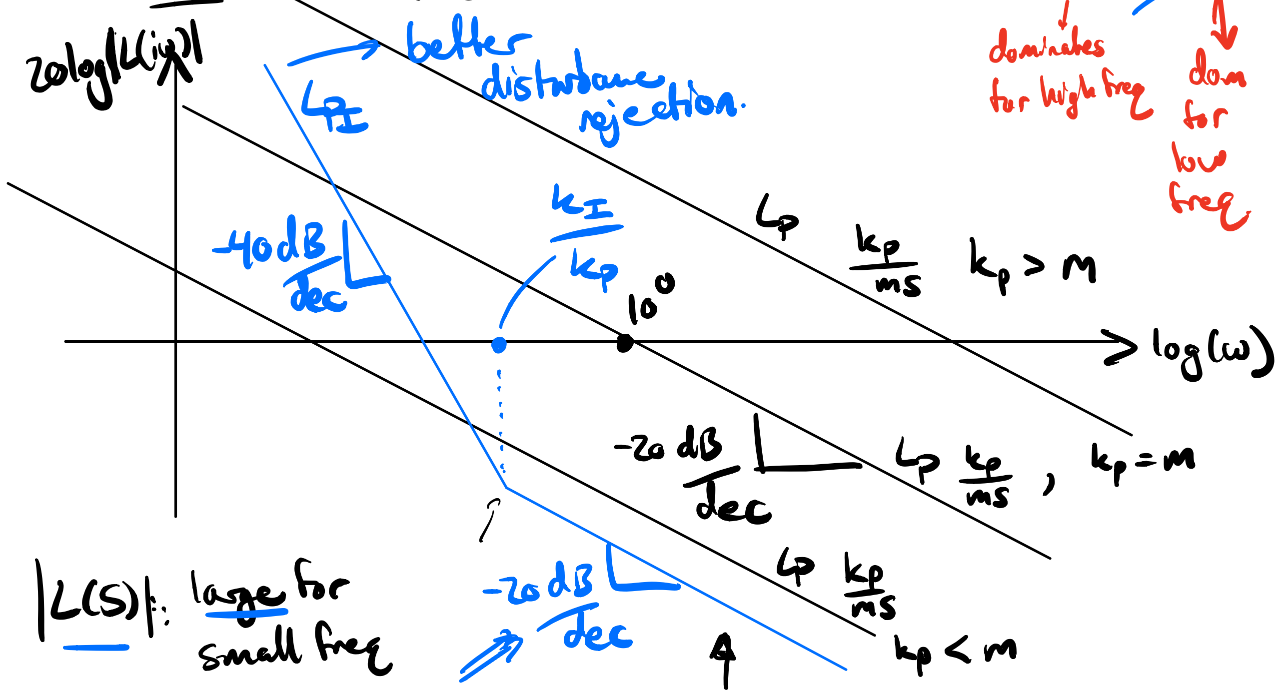
making control input sensitive to accumulated errors

BODE PLOT – PI CONTROLLER

$$L_P(s) = \frac{k_P}{ms} \qquad \boxed{L_{PI}(s)} = \frac{k_P s + k_I}{ms^2} = \boxed{\frac{k_P}{ms}} + \boxed{\frac{k_I}{ms^2}}$$

dominates for high freq

dom for low freq



$|L(s)|$: large for small freq

$|L(s)|$: small for large freq.

$$20\log\left(\frac{1}{s}\right) = 0 - 20\log(s)$$

Cutoff of $\frac{k_P}{k_I}$:

$$20\log\left(\frac{1}{s^2}\right) = -40\log(s)$$

$$\frac{k_P}{ms} = \frac{k_I}{ms^2} \Rightarrow k_P = \frac{k_I}{s} \Rightarrow s = \frac{k_I}{k_P}$$

$$L_{PI}(s) = \frac{k_P s + k_I}{ms^2} = \frac{s + \frac{k_I}{k_P}}{\frac{m}{k_P}s^2} = \frac{1}{\frac{m}{k_P}s^2}\left(s + \frac{k_I}{k_P}\right)$$