PID CONTROLLERS:

last Proportional Controllers P- controllers

Prop. Integral Controllers PI- controllers

For velocity control:

P- controllers : enough to stabilize

no complete disturbance rejection

PI -controller : stabilize

disturbance rejection.

CLOSED LOOP TF

PLANT: $G(s) = \frac{1}{ms}$

P- CONTROL: $\boxed{C_p(s) = k_p}$

$$T_p(s) = \frac{C_p(s)G(s)}{1+C_p(s)G(s)} = \frac{k_p/ms}{1+k_p/ms} = \boxed{\frac{k_p}{ms+k_p}}$$

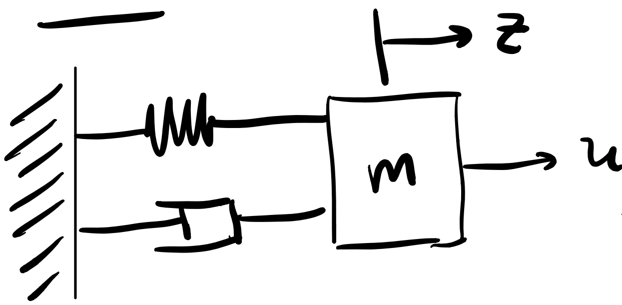
$$R_p(s) = \frac{G(s)}{1+C_p(s)G(s)} = \frac{1/ms}{1+k_p/ms} = \boxed{\frac{1}{ms+k_p}}$$

PI- CONTROL $\boxed{C_{PI}(s) = k_p + \frac{k_I}{s}}$

$$T_{PI}(s) = \frac{C_{PI}(s)G(s)}{1+C_{PI}(s)G(s)} = \frac{\left(k_p+\frac{k_I}{s}\right)\frac{1}{ms}}{1+\left(k_p+\frac{k_I}{s}\right)\frac{1}{ms}} = \boxed{\frac{k_p s + k_I}{ms^2 + k_p s + k_I}}$$

$$R_{PI}(s) = \frac{G(s)}{1 + C_{PI}(s)G(s)} = \frac{\frac{1}{ms}}{1 + \left(k_p + \frac{k_I}{s}\right)\frac{1}{ms}} = \boxed{\frac{s}{ms^2 + k_p s + k_I}}$$

Recall



Newton's Law:

$$\rightarrow m\ddot{z} + c\dot{z} + kz = u$$

$$\rightarrow (ms^2 + cs + k)z = u$$

$$y = b_1 z + b_2 \dot{z} \Longrightarrow Y(s) = (b_1 + b_2 s)z$$

position ↑ $b_1 z$, velocity ↑ $b_2\dot{z}$

$$P(s) = \frac{Y(s)}{u(s)} = \frac{(b_1 + b_2 s)\cancel{z}}{(ms^2 + cs + k)\cancel{z}}$$

freq dependent "ratio" between inputs & outputs

Then $T_{PI}(s)$ is exactly like $P(s)$ with

c (damper) $\rightarrow k_p$ $\quad b_1 = k_I$

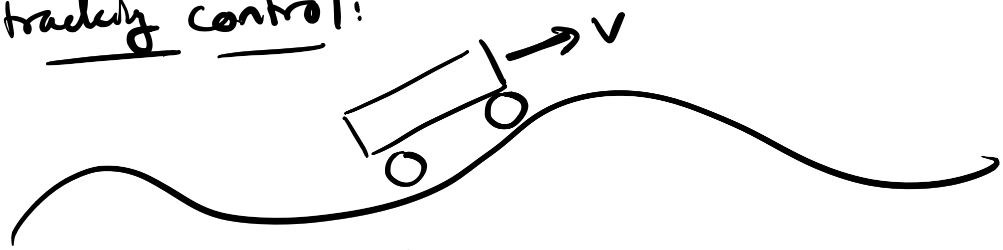
k (spring) $\rightarrow k_I$ $\quad b_2 = k_p$

R_{PI}

c (damper) $\rightarrow k_p$ $\quad b_1 = 0$

k (spring) $\rightarrow k_I$ $\quad b_2 = 1$ $\Big\} \rightarrow y(t) = \dot{z}$

Car tracking control:



Previously, we wanted v to track some desired velocity v_d

Next want to track position

$r \longrightarrow r_d$

r ↓ position

Model: ODE (state space model)

NEWTONS 2ND LAW:

$$\ddot{r} = \frac{1}{m}(f + d) \implies$$

$$y = r + n$$

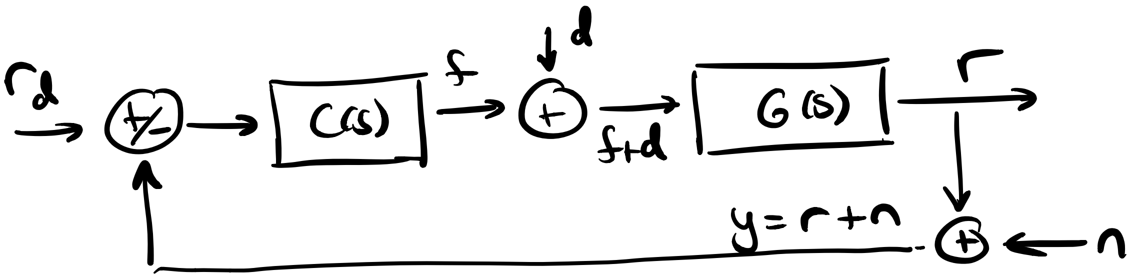
State space model: $\dot{x} = Ax + Bu$

$x \in \mathbb{R}^2$ $\quad x = \begin{bmatrix} r \\ v \end{bmatrix}$ $\quad$ 2ND ORDER SYS

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & \boxed{1} \\ 0 & \cancel{\ast} \end{bmatrix} \begin{bmatrix} r \\ \boxed{v} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f}_{\text{control input}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} d}_{\text{disturbance}}$$

spring $\quad$ damper

air resistance $= \ast\,-|v|v$



Assuming 0 init conds:

For velocity plant.

$$G(s) = \frac{1}{ms}$$

scaled by 1/m

"large m means need more force"

integrate force to get velocity

Position Model

$$G(s) = \frac{1}{ms^2} \rightarrow \text{double integral}$$

$$s^2 R(s) = \frac{1}{m}\left(F(s) + D(s)\right)$$

$$G(s) = \frac{R(s)}{F(s)+D(s)} = \frac{1}{ms^2}$$

$$L(s) = C(s)G(s):$$

How do we design $C(s)$ for BIBO stability & freq domain conditions on transfer functions

New controller:

Proportional Derivative controller PD-controller

$$C_{PD}(s) = K_P + K_D s$$

BIBO stability

$$T(s) = \frac{L(s)}{1+L(s)} \qquad R(s) = \frac{G(s)}{1+L(s)}$$

$1 + L(s) = 0$ } → solutions of this eqn determine stability.

$$1 + L(s) = 1 + C_{PD}(s)G(s)$$

$$= 1 + (k_p + k_D s)\frac{1}{ms^2} = \frac{ms^2 + k_d s + k_p}{ms^2} = 0$$

$$\Rightarrow \boxed{ms^2 + \underline{k_d} s + \underline{k_p} = 0} \leftarrow$$

Conditions for stability:

$k_p > 0$, $k_d > 0$ ⇒ poles have negative real parts

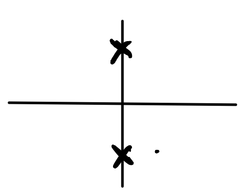
what happens if $k_d = 0$

in position control: k_p ← acting like a spring

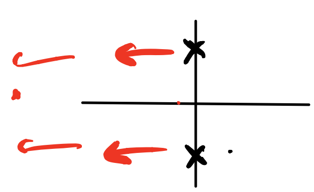
k_d ← acting like a damper

intuitively: not losing energy.

$$ms^2 + k_p = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{\frac{k_p}{m}}$$



as $\boxed{k_d} > 0$:



Freq. Response Closed Loop TF:

$$T_{PD}(s) = \frac{L_{PD}(s)}{1+L_{PD}(s)} = \frac{(k_p + k_d s)\frac{1}{ms^2}}{\frac{ms^2 + k_d s + k_p}{ms^2}}$$

$$= \frac{k_p + k_d s}{ms^2 + k_d s + k_p}$$

$$R_{PD}(s) = \frac{G(s)}{1+L_{PD}(s)} = \frac{1}{ms^2 + k_d s + k_p}$$

$R_{PD}(0) \neq 0 \implies$ constant disturbances are not rejected perfectly.

They're attenuated by $1/k_p$

$T_{PD}(0) = \frac{k_p}{k_p} = 1 \rightarrow$ allows us to track low frequencies signals

$$L_{PD}(s) = (k_p + k_d s)\frac{1}{ms^2}$$

New controller

Proportional Integral Derivative Controller

PID - Controller

$$C_{PID}(s) = \underset{(P)}{k_p} + \underset{(D)}{k_d s} + \underset{(I)}{\frac{k_I}{s}}$$

$$L_{PID}(s) = \left(k_p + k_d s + \frac{k_I}{s}\right)\left(\frac{1}{ms^2}\right)$$

$$= \frac{k_d s^2 + k_p s + k_I}{ms^3}$$

$$T_{PID}(s) = \frac{k_d s^2 + k_p s + k_I}{ms^3 + k_d s^2 + k_p s + k_I}$$

$$R_{PID}(s) = \frac{s}{ms^3 + k_d s^2 + k_p s + k_I}$$

<u>BIBO</u> <u>stability</u>:

roots of $ms^3 + k_d s^2 + k_p s + k_I$

have negative real parts

cubic polynomial (as opposed to quadratic)

more complicated conditions for stability:

<u>Routh</u> - <u>Harwitz</u> Test : Next week.

$m > 0$

necessary cond for stability

$$\underbrace{k_d > 0,\ k_p > 0,\ k_I > 0}_{\text{always be true for stable TF}}$$

not enough to guarantee stability

if stable: $(s+\lambda_1)(s+\lambda_2)(s+\lambda_3) = 0$

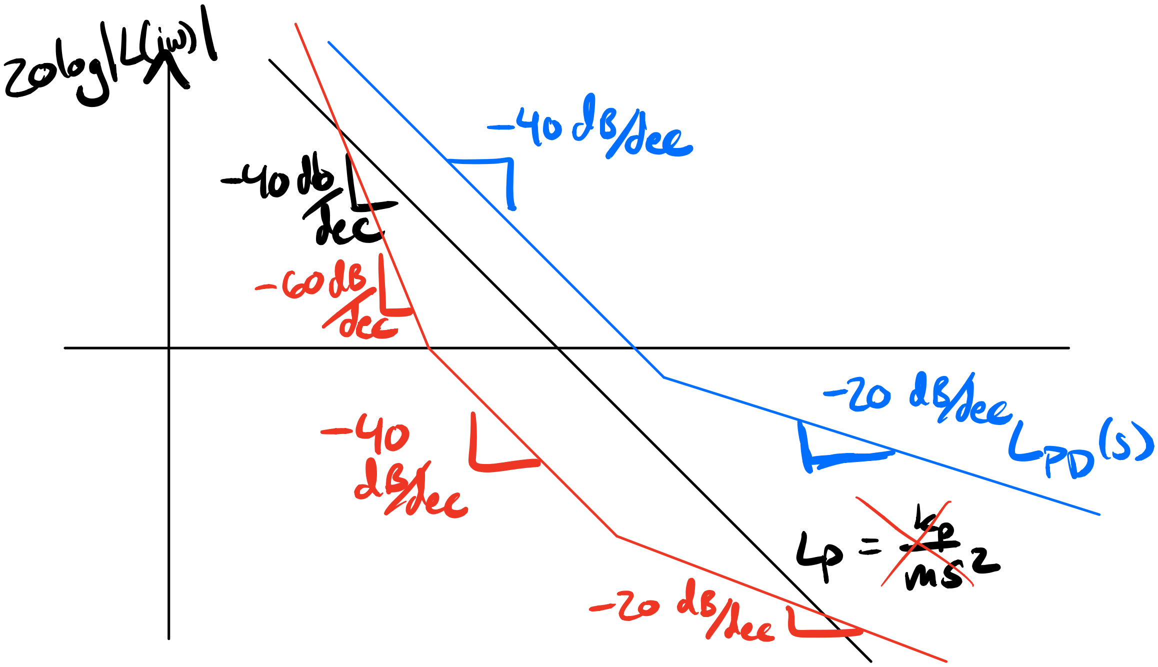
FREQ DOMAIN PERFORMANCE:

$$T_{PID}(\cancel{s}\,0) = \frac{k_d \cancel{s^2} + k_p \cancel{s} + K_I}{m\cancel{s^3} + k_d \cancel{s^2} + k_p \cancel{s} + K_I} = 1$$

$$R_{PID}(\cancel{s}\,0) = \frac{\cancel{s}\,0}{m\cancel{s^3} + k_d \cancel{s^2} + k_p \cancel{s} + \underline{K_I}} = 0$$

GOOD TRACKING & GOOD DISTURBANCE REJECTION

$$L_p(s) = \frac{k_p}{ms^2} \quad L_{PD}(s) = \frac{k_d s + k_p}{ms^2} \quad L_{PID}(s) = \frac{k_d s^2 + k_p s + k_I}{ms^3}$$



PID Controller in State Space:

$$x = \begin{bmatrix} r \\ v \end{bmatrix} \in \mathbb{R}^2$$

$$\begin{bmatrix} \dot r \\ \dot v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \underset{\text{spring}}{*} & \underset{\text{damper}}{*} \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} d$$

$$y = r + n$$

$$F(s) = C(s)R(s) = \left(k_p + k_d s + k_I \frac{1}{s}\right) R(s)$$

$$f(t) = k_p r(t) + k_d \dot r(t) + \underbrace{k_I \int r(t)}_{z}$$

$$= k_p r(t) + k_d v(t) + \ ?$$

Augment the state vector x . . .

before $x \in \mathbb{R}^2$ $x = \begin{bmatrix} r \\ v \end{bmatrix}$

now... $x \in \mathbb{R}^3$ $x = \begin{bmatrix} z \\ r \\ v \end{bmatrix}$

r: position meter

v: velocity meter/second

z: integral meters·sec

Controller:

$$f(t) = k_p r + k_d v + k_I z \qquad z = \int r + z(0)$$

$$= \underbrace{\begin{bmatrix} k_I & k_p & k_d \end{bmatrix}}_{K} \begin{bmatrix} z \\ r \\ v \end{bmatrix}$$

$$f(t) = \underline{K} x(t)$$

new dynamics: $\dot{z} = r$

$$\begin{bmatrix} \dot{z} \\ \dot{r} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & * & * \end{bmatrix}}_{A} \begin{bmatrix} z \\ r \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix}}_{B_1} f + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix}}_{B_2} d$$

spring damper

plug in controller in the time domain.

$$f(t) = K x(t) \qquad K \in \mathbb{R}^{1\times 3}$$

$$\begin{aligned} \dot{x} &= Ax + B_1 f + B_2 d \\ &= Ax + B_1 K x + B_2 d \\ &= \underbrace{(A + B_1 K)}_{\text{eigenvalues of this matrix determine stability}} x + B_2 d \end{aligned}$$

characteristic polynomial of $\underline{A+B_1K}$...

$$ms^3 + k_d s^2 + k_p s + K_I$$

$$A + B_1 K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & * & * \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/m \end{bmatrix} \begin{bmatrix} K_I & K_p & K_D \end{bmatrix}}_{\text{rank 1 } 3\times 3 \text{ matrix}}$$

dyad

outer product