Stability Criteria: Routh-Hurwitz

Hurwitz matrix $\Rightarrow$ stable

TF:

Loop TF: $L(s) = C(s)G(s)$

$$T(s) = \frac{L(s)}{1+L(s)} \qquad R(s) = \frac{G(s)}{1+L(s)}$$

stability: roots of $1 + L(s) = 0$

$\rightarrow$ in OLHP

open left half plane.

$1+L(s) = as^2 + bs + c \Rightarrow$ quadratic eqn.

what if

$$1+L(s) = p_0 s^n + p_1 s^{n-1} + p_2 s^{n-2} + \dots$$

PID Controller:

$$1+L(s) = ms^3 + k_d s^2 + k_p s + K_I = 0$$

Goal: bounds on gains that maintain stability.

Necessary: $m>0$, $k_d>0$, $k_p>0$, $k_I>0$

$$(s+\lambda_1)\cdots(s+\lambda_n) = s^n + \bigcirc s^{n-1} \bigcirc - \bigcirc$$

roots negative real parts $\Longrightarrow$ positive coeffs not roots w/ neg. real parts $\Leftarrow$ if not positive coeffs

(proof by contrapositive)

$\sim p$ $\sim q$

ASIDE: $\neg, \sim, !$: "not"

| STATEMENT: EQUIV. | CONTRAPOSITIVE

if p then q $\Longleftrightarrow$ if $\sim q$ then $\sim p$

$p \Rightarrow q$ $\sim q \Rightarrow \sim p$

| INVERSE   EQUIV | CONVERSE

if $\sim p$ then $\sim q$ $\Longleftrightarrow$ if q then p

$\sim p \Rightarrow \sim q$ $q \Rightarrow p$

statement: q is true if p.

inverse: q is true only if p

both: q if and only if p (q iff p)

($q \Leftrightarrow p$

iff: necessary & sufficient conditions

$p \Leftrightarrow q$ necessary: $p \Leftarrow q$

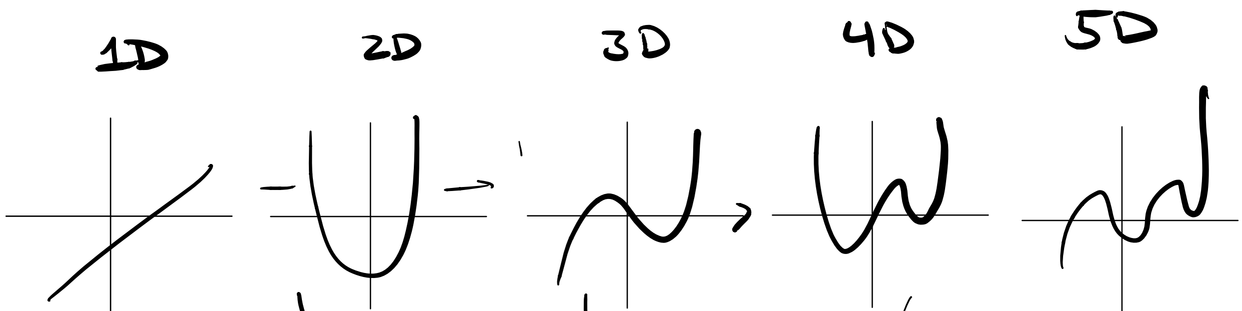
sufficient: $p \Rightarrow q$

$m>0,\ k_d>0,\ k_p>0,\ k_I>0$ not enough to guarantee stability

What is enough to guarantee stability (sufficient)

Polynomials

Ex. poly degree...



$p_0 s + p_1$

$p_0 s^2 + p_1 s + p_2$

$s^2 + p_2 = 0$

$(s - i\sqrt{p_2})(s + i\sqrt{p_2}) = 0$

$p_0 s^3 + p_1 s^2 + p_2 s + p_3$

$p_0 s^4 + p_1 s^3 + p_2 s^2 + p_3 s + p_4 = 0$

$p_0 s^5 + p_1 s^4 + p_2 s^3 + p_3 s^2 + p_4 s + p_5$

Notes: n deg poly.

- n degree polynomial: n roots
- if n is odd → go off to ∞ in different directions
  (↑ max degree)

  if n is even → " " " same direction
  (↑ max degree)

Odd & Even Functions:

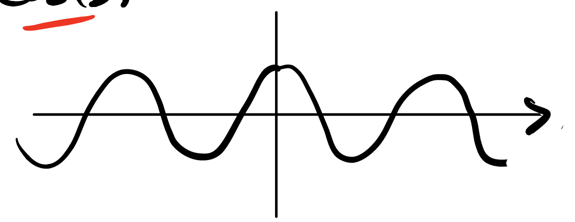
$f(s)$ is even if $\boxed{f(-s) = f(s)}$

$f(s)$ is odd if $\boxed{f(-s) = -f(s)}$

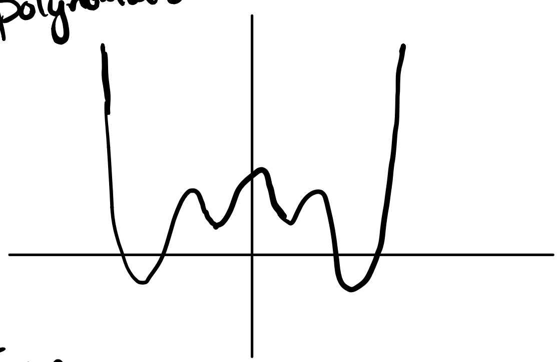
Ex.

Even Functions

cos(s)



polynomials



Form n = even

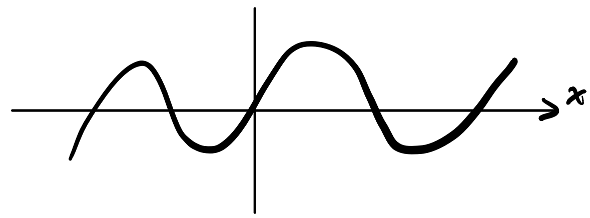
$p_0 s^n + p_2 s^{n-2} + \cdots$

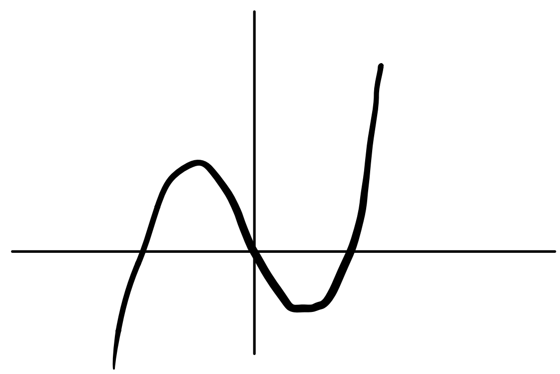
even powers of s only

sign doesn't matter

Odd functions

sin(s)





n = even

$p_0 s^{n+1} + p_2 s^{n-1} + p_4 s^{n-3} + \cdots$

odd powers of s only

sign does matter

Routh-Hurwitz: (PROCEDURE)

Ex. $2s^5 + 8s^4 + 4s^3 + 10s^2 + 6s + 12 = 0$

Table:

max deg odd

	s^5 / s^4	s^3 / s^2	s / s^0		
s^5	2	4	6	0	⇐ odd terms
s^4	8	10	12	0	⇐ even terms
s^3	$\frac{8(4)-2(10)}{8}$ $\boxed{1.5}$	$\frac{8(6)-2(12)}{8}$ $\boxed{3}$	$\boxed{0}$	0	
s^2	$\frac{(1.5)10-8(3)}{1.5}$ $\boxed{-6}$	$\frac{(1.5)(12)-\cancel{0}(8)}{1.5}$ $\boxed{12}$	$\boxed{0}$	0	
s^1	$\frac{-6(3)-18(1.5)}{-6}$ $\boxed{6}$	$\frac{-6(0)-1.5(0)}{-6}$ $\boxed{0}$	0		
s^0	$\frac{6(12)-(-6)(0)}{6}$ $\boxed{12}$	$\boxed{0}$			

⇑ look at first column.

s^5	2
s^4	8
s^3	1.5
s^2	-6
s^1	6
s^0	12

how many sign changes are there in the first column?

1st sign change

2nd sign change

# of sign changes is the # of roots w/ positive real parts.

is this polynomial stable? $\Rightarrow$ $\boxed{\text{NO}}$

In Matlab:

>> roots([2 8 4 10 6 12])

$\rightarrow$ vector repres. of polynomial

roots

-3.79

$\boxed{0.52} \pm i(1.01)$

$-0.62 \pm i(0.91)$

$\rightarrow$ complex eigenvalues in conjugates pairs (polynomials w/ real coeffs)

PID Controller:

$$ms^3 + k_d s^2 + k_p s + K_I = 0$$

Table:

s^3	m	k_p	0
s^2	k_d	k_I	0
s^1	$\dfrac{k_p k_d - m k_I}{k_d}$	0	
s^0	k_I		

Before:

$m>0$, $k_d>0$, $k_p>0$, $k_I>0$

Now

$m>0$, $k_d>0$, $k_I>0$

$$\frac{k_p k_d - m k_I}{k_d} > 0$$

clearly : k_p must be > 0

also need: $k_p k_d > m k_I$

PID Control:

$m > 0 \quad k_d > 0$

$k_p > 0 \quad k_I > 0$

$$\boxed{\underbrace{k_p}_{\text{prop.}} \underbrace{k_d}_{\text{derivative}} > \underbrace{m}_{\text{mass}} \underbrace{k_I}_{\text{integral}}}$$

Intuition

- large mass → requires large k_p and/or k_d
- large $\underbrace{k_I}_{\text{gain on accumulated error}}$ → destabilize system is k_p & k_d aren't big enough
- lowering k_p or k_d "too much" → destabilize...

  relative to m and k_I
- etc.

Ex: $s^3 + 2s^2 + s + 2 = 0$

Table

$$
\begin{array}{c|cc}
s^3 & 1 & 1 \\
s^2 & 2 & 2 \\
\hline
s^1 & \cancel{\frac{2-2}{2}} \quad 0 = \varepsilon > 0 & \boxed{0} \\
s^0 & \frac{2(\varepsilon) - 2(0)}{\varepsilon} &
\end{array}
$$

if 0 in the 1st col...

add $\varepsilon > 0$

$$
\begin{array}{c|cc}
s^3 & 1 & 1 \\
s^2 & 2 & 2 \\
s^1 & \boxed{\varepsilon > 0} & 0 \\
s^0 & 2 &
\end{array}
$$

zero

$\Rightarrow$ no sign change in 1st columns

if zero in 1st col

$\Rightarrow$ root w/ zero real part.

on imag axis

(only marginally stable)

Roots:

$-2, \pm i$