ROUTH HURWITZ

0's in first column:

↳ poles on $i\omega$ axis

↳ add $\varepsilon > 0$

Ex. $s^3 + 2s^2 + s + 2 = 0$

$$\begin{array}{c|cc} s^3 & \boxed{1} & 1 \\ s^2 & \boxed{2} & 2 \\ s^1 & 2-\cancel{2}\,\cancel{0} \; \boxed{2\ \varepsilon>0} & \boxed{0} \\ s^0 & 2 & \end{array}$$

FINAL

$$\begin{array}{c|c} s^3 & 1 \\ s^2 & 2 \\ s^1 & \varepsilon > 0 \\ s^0 & 2 \end{array}$$

no sign change

$\Rightarrow$ no RHP poles

right half plane (positive real parts)

Roots: -2, $\boxed{\pm i}$

Ex. $s^3 + 0s^2 + 2s + 1 = 0$

$$\begin{array}{cccc} s^3 & +1 & 2 \\ s^2 & +0 & 1 \\ & \varepsilon > 0 & \\ s^1 & \frac{2\varepsilon - 1}{\varepsilon} = \boxed{2 - \frac{1}{\varepsilon}} & 0 \\ s^0 & +1 & \end{array}$$

$2 - \frac{1}{\varepsilon} < 0$ for small $\varepsilon > 0$

2 sign changes $\Rightarrow$

- 2 roots w/ positive real parts.
- 0 in 1st column $\Rightarrow$ pole on $i\omega$ axis

Roots: $0.23 \pm i\,1.47$, -0.45

Rules of Thumb Degree 2, 3, 4 ...

Degree 2: $as^2 + bs + c = 0$

$$\begin{array}{c|cc} s^2 & a & c \\ s^1 & b & 0 \\ s^0 & c & \end{array} \qquad \left.\begin{array}{l} a > 0 \\ b > 0 \\ c > 0 \end{array}\right\} \text{ iff stability.}$$

Degree 3: $\boxed{a}s^3 + \boxed{b}s^2 + \boxed{c}s + \boxed{d} = 0$

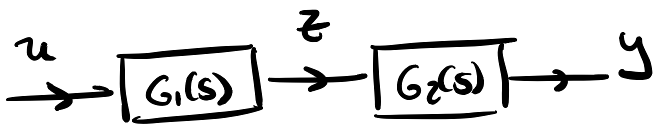
$$\begin{array}{c|cc} s^3 & a & c \\ s^2 & b & d \\ s^1 & \frac{bc-ad}{b} & 0 \\ s^0 & d & \end{array} \qquad \left.\begin{array}{l} a > 0 \\ b > 0 \\ \boxed{bc} > \boxed{ad} \\ d > 0 \\ (c > 0) \end{array}\right\} \text{ iff stability.}$$

Degree 4: $as^4 + bs^3 + cs^2 + ds + f = 0$

$$\begin{array}{c|ccc} s^4 & a & c & f \\ s^3 & b & d & 0 \\ s^2 & \frac{bc-ad}{b} & f & \\ s^1 & d - bf\left(\frac{b}{bc-ad}\right) & & \\ s^0 & f & & \end{array} \qquad \begin{array}{l} a > 0 \\ b > 0 \\ bc > ad \\ d > bf\left(\frac{b}{bc-ad}\right) \Leftarrow \\ f > 0 \\ (d > 0) \\ (c > 0) \end{array} \quad \text{iff } \underline{\text{stable}}$$

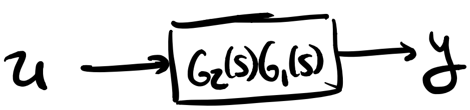
Block Diagrams

- SERIES TF



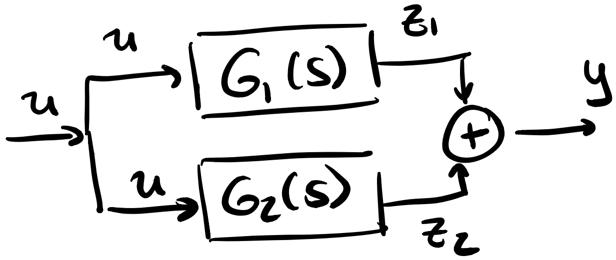
$$\frac{Y(s)}{U(s)} = G(s) = ?$$

$$Z(s) = G_1(s)U(s)$$
$$Y(s) = G_2(s)Z(s)$$
$$\Rightarrow Y(s) = G_2(s)G_1(s)U(s)$$



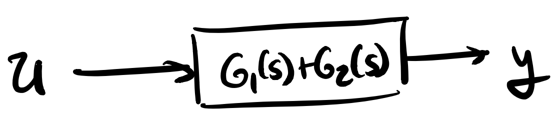
$$\Rightarrow \boxed{G(s) = G_2(s)G_1(s)}$$

- PARALLEL TF:

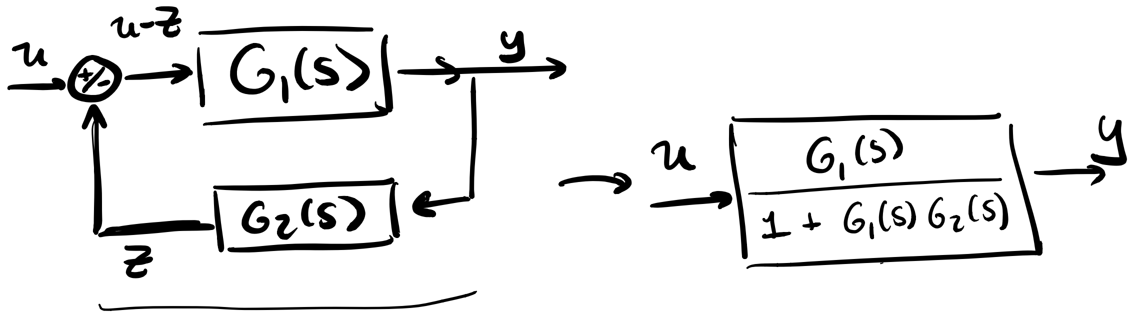


$$Y(s) = [G_1(s)+G_2(s)]U(s)$$

$$\left.\begin{aligned} Y(s) &= Z_1(s) + Z_2(s) \\ Z_1(s) &= G_1(s)U(s) \\ Z_2(s) &= G_2(s)U(s) \end{aligned}\right\} \rightarrow \boxed{G(s) = G_1(s) + G_2(s)}$$



FEEDBACK CONNECTION:



$$G(s) = \frac{Y(s)}{U(s)} = \,?$$

$$Z(s) = G_2(s)Y(s)$$

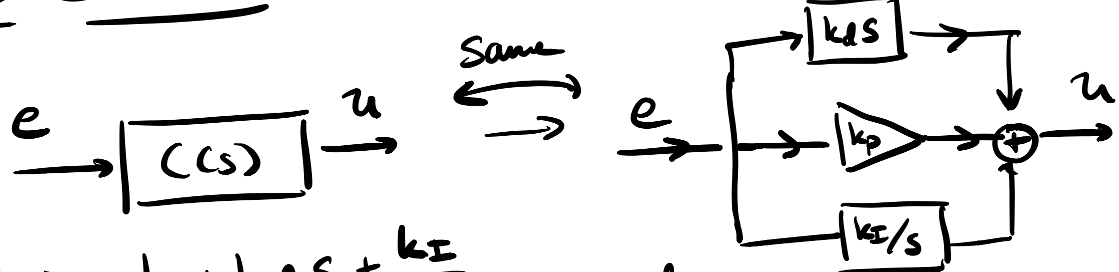
$$Y(s) = G_1(s)\left[U(s) - Z(s)\right]$$

$$= G_1(s)U(s) - G_1(s)G_2(s)Y(s)$$

$$\Rightarrow \left(1 + G_1(s)G_2(s)\right)Y(s) = G_1(s)U(s)$$

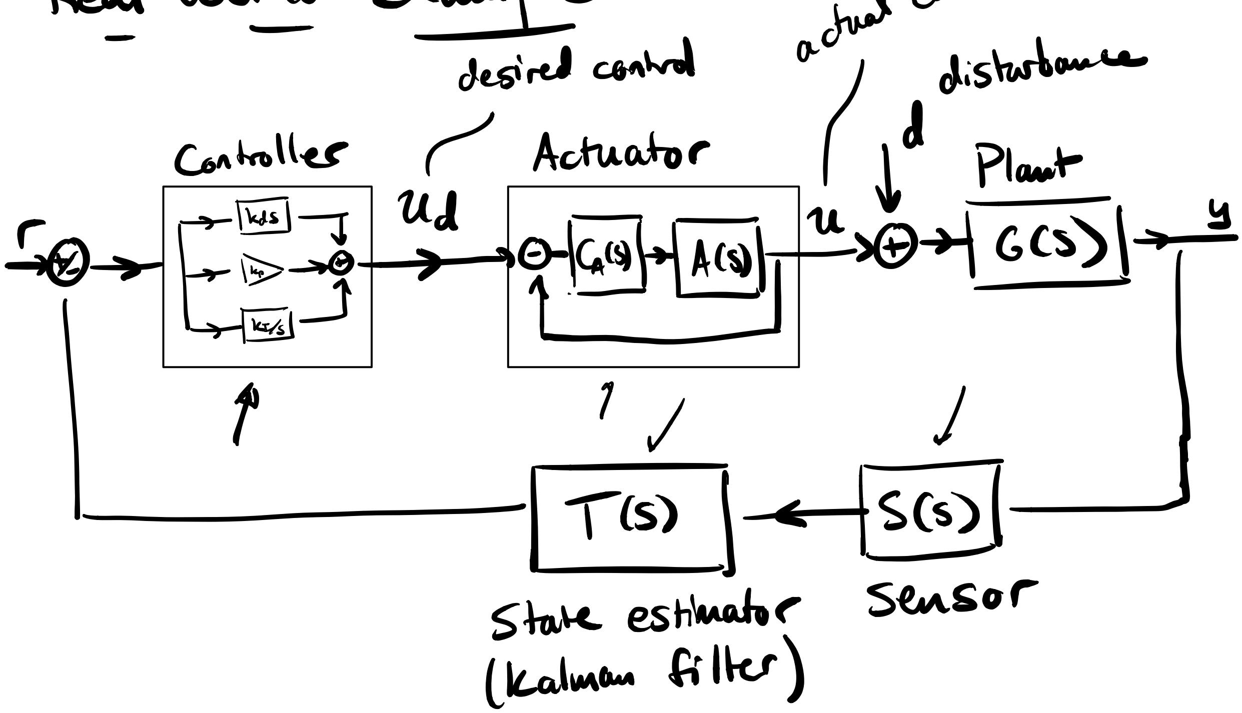
$$\Rightarrow \quad G(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

PID Controller:

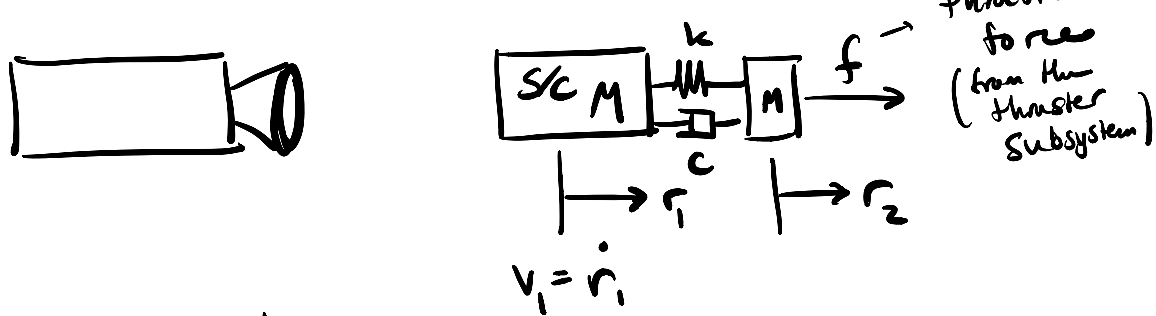


$$C(s) = k_p + k_d s + \frac{k_I}{s}$$ → applied in parallel

Real World example:



ACTUATOR MODELS:



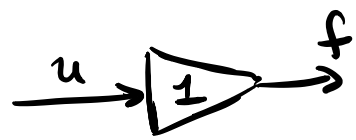
Propulsion Subsystem

Ideal Thruster

u : desired thrust

f : actual

$f = u$

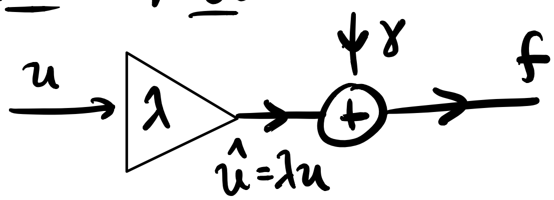


Models of non-ideal thrusters

$f(t) = \lambda u(t) + \gamma$

uncertainty: $\lambda \neq 1$ and $\gamma \neq 0$

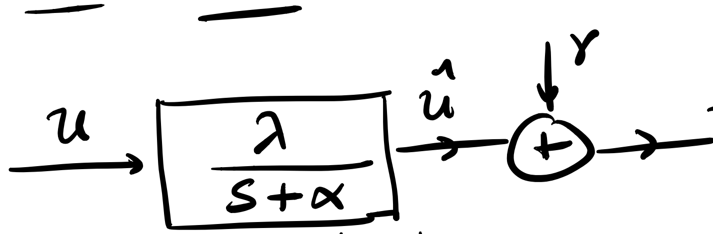
static model...



$f = \hat{u} + \gamma = \lambda u + \gamma$

Time Domain: $\hat{u} = \lambda \boxed{u}$

Dynamic Model:



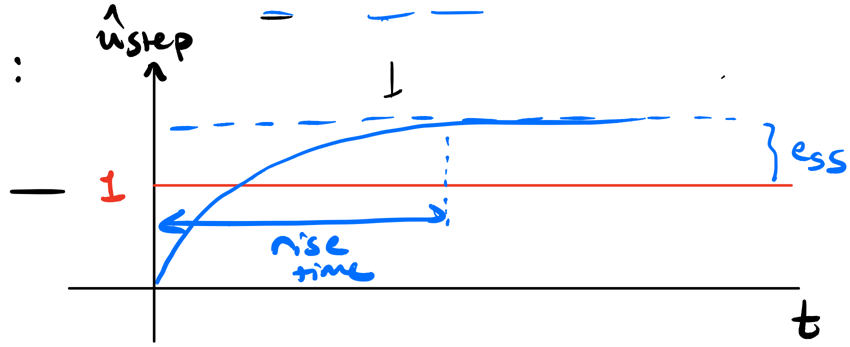
$f = \hat{u} + \gamma$

Dynamic block

Time Domain representation:

$$\Gamma(s) = \frac{\lambda}{s+\alpha} \Rightarrow \begin{aligned} \dot{x}_a &= -\alpha x_a + \boxed{u} \\ \hat{u} &= \lambda x_a \end{aligned}$$

Step response:



• is $\Gamma(s)$ BIBO stable? Yes if $\alpha > 0$

$$e^{i\omega t} \longrightarrow \boxed{\Gamma(s)} \longrightarrow \text{steady state response}$$

$$\underbrace{|\Gamma(i\omega)|}_{\text{amplification / attenuation}}\, e^{i(\omega t + \underbrace{\measuredangle \Gamma(i\omega)}_{\text{phase shift}})}$$

for a step function:

$\omega = 0 \;\rightarrow$ steady state

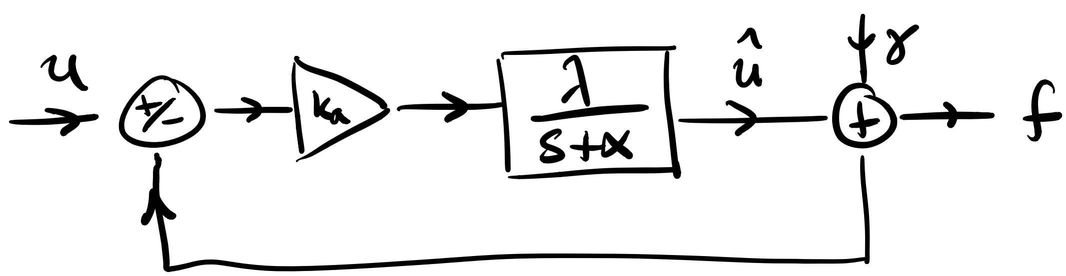
$$\Gamma(0)\, u_{step}(t)$$

$$\Gamma(0) = \frac{\lambda}{\alpha} \implies \text{steady state error} \quad \text{error} = \frac{\lambda}{\alpha} - 1$$

Note: $\alpha > 0$ larger $\implies$ fast converges to steady state

$\frac{\lambda}{\alpha} \neq 1 \implies$ there is some non zero ss. error

Even more complicated dynamic model:



Closed loop feedback system for the thruster $\bar{w}$ a proportional control

- without feedback, we get steady state errors assume $\gamma = 0$: $\Longrightarrow$ $f = \hat{u}$

$$\frac{F(s)}{u(s)} = \frac{\frac{k_a \lambda}{s+\alpha}}{1 + \frac{k_a \lambda}{s+\alpha}} = \underbrace{\frac{\lambda k_a}{s + (\alpha + \lambda k_a)}}_{\text{closed-loop TF for the thruster subsystem}}$$

BIBO stable:

Assuming $\lambda > 0$, $\alpha > 0$

$\Rightarrow$ need $k_a > 0$
(stability)

Note: want $f = u$ ideally!

To make this happen, pick $k_a > 0$ as large as possible

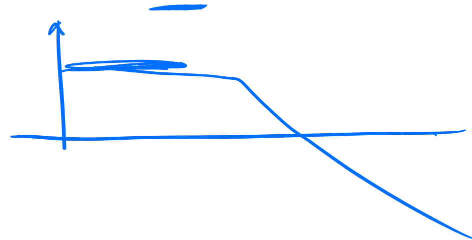
In practice some upper bound on k_a...

if k_a large $\Longrightarrow$ $\frac{F(s)}{u(s)} \approx 1$

$$\frac{F(s)}{u(s)} = \frac{\lambda k_a}{\cancel{s}^{\,i\omega} + (\underbrace{\alpha}_{\text{negligible}} + \lambda k_a)} \approx 1.$$

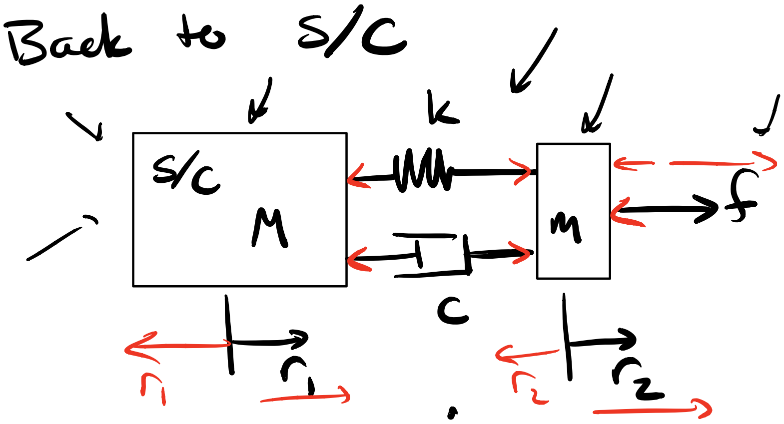
0.01

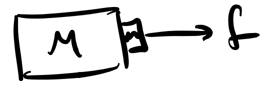
for large k_a



Note: we do not need to know exact α or λ

Back to S/C





$(M+m)\ddot{r} = f$

want to control $\dot{r}_1 = v_1$

S/C : $M\ddot{r}_1 = k(r_2 - r_1) + c(v_2 - v_1)$

thruster: $m\ddot{r}_2 = k(r_1 - r_2) + c(v_1 - v_2) + f$

output: $y = v_1$

want TF: $G(s) = \dfrac{Y(s)}{F(s)}$

S/C: $Ms^2 R_1(s) = k\left(R_2(s) - R_1(s)\right) + cs\left(R_2(s) - R_1(s)\right)$

$(Ms^2 + cs + k)R_1(s) = (k + cs)R_2(s)$

$$R_1(s) = \frac{k+cs}{Ms^2+cs+k} R_2(s) \quad \Leftarrow$$

Thrust

$(ms^2 + cs + k)R_2(s) = (k + cs)R_1(s) + F(s)$

$$\left[(ms^2+cs+k)\frac{(Ms^2+cs+k)}{k+cs} - (k+cs)\right] R_1(s) = F(s)$$

$$R_1(s) = \frac{k+cs}{(ms^2+cs+k)(Ms^2+cs+k)-(k+cs)^2}\,F(s)$$

$$Y(s) = V_1(s) = s\,R_1(s)$$

$$= \frac{s\,(k+cs)}{(ms^2+cs+k)(Ms^2+cs+k)-(k+cs)^2}\,F(s)$$

$$G(s) = \frac{Y(s)}{F(s)} = \frac{s\,(k+cs)}{(ms^2+cs+k)(Ms^2+cs+k)-(k+cs)^2}$$

$$\frac{\cancel{s}\,(k+cs)}{Mms^4 + (M+m)s^{\cancel{2}}(cs+k) + \cancel{(cs+k)^2} - \cancel{(k+cs)^2}}$$

$$\Rightarrow \boxed{G(s) = \frac{Y(s)}{F(s)} = \frac{(cs+k)}{s\left[Mms^2 + (M+m)(cs+k)\right]}}$$

rigid body limit: $c=0$, $k=\infty$ ↑

$$\boxed{G(s) = \frac{Y(s)}{F(s)} = \frac{(\overset{0}{\cancel{cs}}+k)}{s\left[\underset{k=\infty}{\overset{0}{\cancel{Mms^2}}} + \underset{(M+m)\,k}{(M+m)(\overset{0}{\cancel{c}}s+k)}\right]}} \quad \cancel{k} \quad \cancel{1/k} \quad 1/k$$

$$G_{rigid}(s) = \frac{1}{(M+m)s}$$

$1/s$: integration

$$Y(s) = G_{rigid}(s)\, F(s)$$

$$Y(s) = \frac{1}{(M+m)s} F(s)$$

$$Y(s) = V_1(s) = sR_1(s)$$

$$\underbrace{(M+m)}_{\text{total mass}}\ \underbrace{s\,V_1(s)}_{\text{acceler.}} = \underbrace{F(s)}_{\text{force}} \Leftarrow$$