FROM LAST TIME:

S/C:

More complicated Model: | Simpler Model (Rigid Body) $k \to \infty$ $c \to 0$

PLANT $G(s) = \dfrac{(cs+k)}{s\left[Mms^2+(M+m)(cs+k)\right]}$ $G_r(s) = \dfrac{1}{(M+m)s}$

$k_a \to \infty$

ACTUATOR $\dfrac{\lambda k_a}{s+(\alpha+\lambda k_a)}$ 1 (idealized accuator)

Controller $C(s) = \dfrac{k_p s + k_I}{s}$

ANALYZE

DESIGN

ANALYZE

DESIGN HEIRARCHY:

1 SIMPLE MODEL	DESIGN	Ex. RIGID BODY
2. MORE COMPLICATED	ANALYSIS	NOT RIGID DYNAMICS
3. VERY HIGH FIDELITY	SIMULATE (CHECK BEHAVIOR)	FEA/CFD/ STRESS MECHS

FINAL VALUE THM: (looking at steady state behavior)

Statement:

$f : [0, \infty) \longrightarrow \mathbb{R}$ w̄ Laplace Transform $F(s)$

if $F(s)$ has poles w̄ negative real parts or at the origin (at zero) but with at most one pole at the origin. then

$$\lim_{t \to \infty} f(t) = L = \lim_{s \to 0} sF(s)$$

assuming limit exists

time domain

freq. domain

Ex. $f(t) = \hat{u}_{step}(t) \Rightarrow F(s) = \frac{1}{s} \Rightarrow$ Pole $= \{0\}$

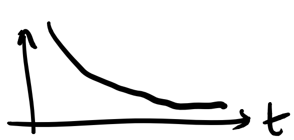
$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\frac{1}{s} = 1$$

Ex. $f(t) = t \Rightarrow F(s) = \frac{1}{s^2} \Rightarrow$ Poles $= \{0, 0\}$

can't apply FVT

Ex. $f(t) = e^{-2t} \Rightarrow F(s) = \frac{1}{s+2} \Rightarrow$ Poles $= \{-2\}$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s}{s+2} = \underline{0}$$



Ex. $f(t) = te^{-2t}$

$f(t) = t^n e^{-at}$ where n integer

$a > 0$

show:

$$\lim_{t \to \infty} t^n e^{-at} = 0$$

Extend Version

$$\lim_{s \to 0^+} sF(s) = L = \lim_{t \to \infty} f(t)$$

$$\lim_{s \to 0^+} sF(s) = +\infty = \lim_{t \to \infty} f(t)$$

$$\lim_{s \to 0^+} sF(s) = -\infty = \lim_{t \to \infty} f(t)$$

BACK TO RIGID BODY S/C

$$C(s) = \frac{k_p s + k_I}{s} \qquad G_r(s) = \frac{1}{\tilde{m} s} \qquad \tilde{m} = M + m$$

$$\tilde{L}(s) = C(s) G_r(s) = \frac{k_p s + k_I}{s} \frac{1}{\tilde{m} s}$$

analyze steady state behavior of error

tracking, disturbances, & noise.

$$E(s) = \tilde{Y}(s) - V_d(s)$$

$$= \tilde{T}(s)\left[V_d(s) - N(s)\right] + \tilde{R}(s)D(s) - V_d(s)$$

$$= \left(\underline{\tilde{T}(s) - 1}\right)V_d(s) - \tilde{T}(s)N(s) + \tilde{R}(s)D(s)$$

$$\tilde{T}(s) = \frac{\tilde{L}(s)}{1+\tilde{L}(s)} \qquad \tilde{R}(s) = \frac{G_r(s)}{1+\tilde{L}(s)}$$

$$\tilde{T}(s) - 1 = \frac{\tilde{L}(s)}{1+\tilde{L}(s)} - 1\left(\frac{1+\tilde{L}(s)}{1+\tilde{L}(s)}\right) = -\frac{1}{1+\tilde{L}(s)}$$

$$\Rightarrow E(s) = \frac{1}{1+\tilde{L}(s)}\left[G_r(s)\underline{D(s)} - \underline{V_d(s)} - \tilde{L}(s)\underline{N(s)}\right]$$

Assume $N(s) = 0$, $D(s) = 0$

$$E(s) = -\frac{1}{\underbrace{1+\tilde{L}(s)}_{\text{BIBO stable}}} V_d(s)$$

Ex.

$V_d(s) = \frac{1}{s}$ constant reference signal $\quad V_d(t) = u_{step}(t)$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{-1}{\underbrace{1+\tilde{L}(s)}_{\text{poles} \in \text{OLHP}}}\underbrace{\left(\frac{1}{s}\right)}_{\text{pole}=0}$$

poles? $\quad$ poles $= \{\underline{0}, \_\_\_\_ \}$ stable in OLHP

$$\lim_{s\to 0} \cancel{s} \frac{-1}{1+\tilde{L}(s)} \frac{1}{\cancel{s}} = \lim_{s\to 0} \frac{-1}{1+\tilde{L}(s)}$$

$$= \underbrace{\frac{-1}{1+\lim_{s\to 0}\tilde{L}(s)}}_{\text{want this to go to } 0}$$

<u>need</u>

$$\lim_{s\to 0} \tilde{L}(s) = \pm\infty$$

we can track a constant reference signal

Now assume $n=0,\ v_d=0$

$$E(s) = \frac{G_r(s)}{1+\tilde{L}(s)} D(s)$$

$$D(s) = \frac{1}{s} \qquad (d(t) = u_{step}(t))$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{G_r(s)}{1+\tilde{L}(s)}\frac{1}{s}$$

$$\Rightarrow \lim_{s\to 0}\frac{G_r(s)}{1+\tilde{L}(s)} = 0$$ } want this for disturbance rejection

$$G_r(s) = \frac{1}{\tilde{m}s}$$

$$C(s) = k_p + \frac{k_I}{s}$$ ⟸ facilitated constant disturbance rejection

$$\tilde{L}(s) = C(s)G_r(s) = \frac{k_p s + k_I}{s}\frac{1}{\tilde{m}s}$$

- if just proportional, $k_I = 0$

$$\Rightarrow \tilde{L}(s) = \frac{k_p}{\tilde{m}s}$$

$$\Rightarrow \lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{G_r(s)}{1+\tilde{L}(s)} \frac{1}{s}$$

$$= \lim_{s\to 0} \frac{1/\tilde{m}s}{1+\frac{k_p}{\tilde{m}s}}$$

$$= \lim_{s\to 0} \frac{1}{\tilde{m}s + k_p} = \frac{1}{k_p} \neq 0$$

with $k_I = 0$, can't do full disturbance rejection.

can pick k_p large to minimize error.

if $k_I \neq 0$:

$$\lim_{s\to 0} \frac{1/\tilde{m}s}{1+(k_p s + k_I)/\tilde{m}s^2} = \lim_{s\to 0} \frac{s}{\tilde{m}s^2 + k_p s + k_I} = 0$$

Integral control: typically used to ensure constant disturbance rejection.

if $d = 0$, $v_d = 0$:

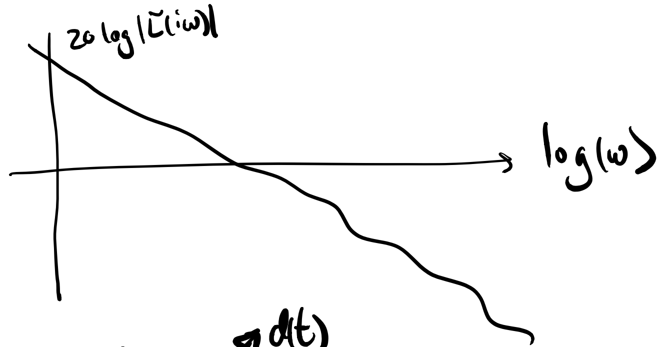
$$E(s) = -\frac{\tilde{L}(s)}{1+\tilde{L}(s)} N(s)$$

$N(s) = \frac{1}{s} \Leftarrow$ constant noise (should probably fix by recalibrating sensor)

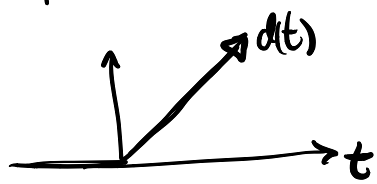
$$\lim_{t\to\infty} e(t) = -\lim_{s\to 0} \cancel{s} \frac{\tilde{L}(s)}{1+\tilde{L}(s)} \frac{1}{\cancel{s}} = \lim_{s\to 0} \frac{-\tilde{L}(s)}{1+\tilde{L}(s)} = -\frac{1}{}$$

from before: wanted $\lim_{s \to 0} \tilde{L}(s) = \pm \infty$

$\tilde{L}(0) \to \pm \infty$



Now: $d(t) = t$



$D(s) = \boxed{\frac{1}{s^2}} \Leftarrow$

disturbance increasing.

$G_r(s) = \frac{1}{\tilde{m} s}$

$\tilde{L}(s) = \frac{1}{\tilde{m} s} C(s)$, $\quad C(s) = \frac{num(s)}{den(s)}$

$$E(s) = \frac{G_r(s)}{1 + \tilde{L}(s)} D(s) = \frac{1/\tilde{m}s}{1 + \frac{1}{\tilde{m}s} \frac{num(s)}{den(s)}} \frac{1}{s^2}$$

$$E(s) = \frac{den(s)}{(\tilde{m} s\, den(s) + num(s))} \frac{1}{s^2} \Leftarrow$$

Goals BIBO stability, reject ramp & constant disturbance

want to use FVT

looks initially like we have 2 roots @ origin need to check for a particular controller

$$\underline{C(s)} = \frac{k_p s + k_I}{s} \begin{array}{l} \leftarrow num(s) \\ \leftarrow den(s) \end{array}$$

for constant was $\frac{1}{s}$

for ramp $\frac{1}{s^2}$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)$$

$$= \lim_{s\to 0} \cancel{s} \frac{den(s)}{(\tilde{m}\, s\, den(s) + num(s))} \frac{1}{s^{\cancel{2}}}$$

$$= \lim_{s\to 0} \frac{\cancel{s}\; 1}{\tilde{m} s^2 + k_p s + k_I} \left(\frac{1}{\cancel{s}}\right) = \boxed{\frac{1}{k_I}}$$

integral controller is not enough to reject ramp disturbances.

$$\lim_{s\to 0} \frac{s\; den(s)}{\underbrace{\tilde{m}\, s\, den(s) + num(s)}_{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}} \frac{1}{s^2} \overset{want}{=} 0$$

try choosing $den(s) = s^2$ $\qquad \frac{k_p s + k_I}{s^2}$

$$\tilde{m}\, s (s^2) + num(s) = \tilde{m} s^3 + num(s)$$

$$= \underbrace{\tilde{m} s^3 + k_p s + k_I}_{\text{not stable}}$$

from RH criteria

not stable because no s^2 term.

fix controller: $C(s) = \frac{as^2+bs+c}{s^2}$

RH criteria on a, b, c for stability.

$a>0,\ b>0,\ c>0,\ ab>\tilde{m}c \Leftarrow$ stability

$$\lim_{s\to 0} s\,\frac{\mathrm{den}(s)}{\tilde{m}s\,\mathrm{den}(s)+\mathrm{num}(s)}\,\frac{1}{s^2}$$

$$\lim_{s\to 0} s\,\frac{s^2}{\tilde{m}s^3+as^2+bs+c}\,\frac{1}{s^2} =$$

$$\lim_{s\to 0} \frac{s}{\tilde{m}s^3+as^2+bs+c} = 0 \Leftarrow$$

$C(s) = \frac{as^2+bs+c}{s^2}$ is able to reject ramp disturbances.

$$= \underbrace{a}_{\text{P-control}} + \underbrace{\frac{b}{s}}_{\text{I-control}} + \boxed{\frac{c}{s^2}}$$

allowed us to reject $D(s) = \frac{1}{s^2}$

II-controller

P-control: enough for BIBO stability

I-control: constant tracking & constant disturbance rejection

II-controller: for ramp disturbance rejection

Internal Model Principle

Roughly "to reject disturbances, you have to embed their models in your controller"

$C(s)$:

$$C(s) = \underbrace{k_p + \frac{k_I}{s}}_{\text{constant d rejection}} \qquad C(s) = \underbrace{a + \frac{b}{s} + \frac{c}{s^2}}_{\text{ramp rejection}}$$

if we remove b:

$$1 + \tilde{L}(s) = \tilde{m}\, s\, \text{den}(s) + \text{num}(s)$$
$$= \tilde{m} s^3 + a s^2 + b s + c$$

from RH if we remove $b \Rightarrow$ unstable

can't use a controller $\quad C(s) = \underbrace{a + \frac{c}{s^2}}_{\text{unstable}}$

if want to use II control term

$\Rightarrow$ we also need an I-control term

Now suppose $d(t) = \sin(\omega t)$

$$D(s) = \frac{\omega}{s^2 + \omega^2}$$

$$E(s) = \frac{G_r(s)}{1 + \tilde{L}(s)} \frac{\omega}{s^2 + \omega^2}$$

$$= \frac{1/\tilde{m}s}{1 + \frac{1}{\tilde{m}s}\frac{num(s)}{den(s)}} \frac{\omega}{\underbrace{s^2 + \omega^2}}$$

roots are $\pm i\omega$ on imag axis not in OLHP

<u>Suppose:</u>

$$den(s) = s^2 + \omega^2$$

$$= \frac{den(s)}{\tilde{m}s\, den(s) + num(s)} \frac{\omega}{(s^2 + \omega^2)}$$

$$\Rightarrow \frac{(s^2 + \omega^2)\ \omega}{[\tilde{m}s(s^2 + \omega^2) + \underbrace{num(s)}]\,(s^2 + \omega^2)}$$

$\tilde{m}s(s^2 + \omega^2) + num(s) \Rightarrow$ poles need to be in OLHP

$\sin(\omega t)$ disturbance

$$\tilde{m}s^3 + \tilde{m}\omega^2 s + \underbrace{num(s)}_{1}$$

needs to contain s^2, s^0

if only use s^2, s^0 terms
run into trouble rejecting constant $\frac{1}{s}$
ramp $\frac{1}{s^2}$

$\Longrightarrow \text{den}(s) = s^2 + \omega^2$

works sinusoidal disturbances
not constant, ramp disturbances...

$G_r(s) = \frac{1}{\underbrace{(M+m)}_{\tilde{m}} s}$

__Suppose__ $\text{den}(s) = s^2(s^2+\omega^2)$

$$E(s) = \frac{G_r(s) D(s)}{1+\tilde{L}(s)} = \frac{G_r(s) D(s)}{1+G_r(s) C(s)} = \frac{s^2(s^2+\omega^2)}{\tilde{m} s (s^2+\omega^2) s^2 + \text{num}(s)} \underbrace{D(s)}$$

For stability:

$$\frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^2+\omega^2}$$

$$\tilde{m} s^5 + \tilde{m} \omega^2 s^3 + \underbrace{\text{num}(s)}$$

needs terms $\alpha_4 s^4 + \underbrace{\alpha_3 s^3}_{\text{optional.}} + \alpha_2 s^2 + \alpha_1 s + \alpha_0$

CHAR EQN:

$$\tilde{m} s^5 + \alpha_4 s^4 + (\tilde{m}\omega^2 + \alpha_3) s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

use RH $\nearrow$ find conditions on coeffs.

$$C(s) = \frac{\alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}{s^2 (s^2 + \omega^2)}$$

partial fraction expansion...

$$C(s) = (\quad)\frac{1}{s} + (\quad)\frac{1}{s^2} + (\quad)\underbrace{\frac{1}{s^2+\omega^2}}_{\text{notch filter.}} + (\quad)$$

I control   II control   notch filter.   P_cont