DESIGNING CONTROLLERS TO ELIMINATE STEADY STATE ERRORS

Tracking Error

$$E(s) = \frac{1}{1+L(s)}\left[G(s)D(s) - V_d(s) - L(s)N(s)\right]$$

$$L(s) = C(s)\,G(s)$$

Plant

$$G(s) = \frac{num_G(s)}{den_G(s)} = \frac{n_G(s)}{d_G(s)}$$

$\rightarrow \frac{1}{ms}, \rightarrow \frac{1}{ms^2}, \ldots$

Controller

$$C(s) = \frac{num(s)}{den(s)} = \frac{n(s)}{d(s)}$$

PID

$$C(s) = k_d s + k_p + \frac{k_I}{s}$$

$$\rightarrow = \frac{k_d s^2 + k_p s + k_I}{s}$$

Design Goals:

1. make $e(t) \rightarrow 0$ in steady state
2. maintain stability.

$$E(s) = \frac{1}{1+L(s)}\left[G(s)\underbrace{D(s)}_{\text{disturbance}} - \underbrace{V_d(s)}_{\text{reference signal}} - L(s)\underbrace{N(s)}_{\text{noise}}\right]$$

Reference

$$\frac{-1}{1+L(s)} V_d(s)$$

$$\underbrace{\qquad}$$

$$\frac{-1}{1+\frac{num_G}{den_G}\frac{num}{den}}$$

Disturbanc

$$\frac{G(s)}{1+L(s)} D(s)$$

$$\frac{\frac{num_G}{den_G}}{1+\frac{num_G}{den_G}\frac{num}{den}}$$

NOISE

$$\frac{L(s)}{1+L(s)} N(s)$$

$$\frac{\frac{num_G}{den_G}\frac{num}{den}}{1+\frac{num_G}{den_G}\frac{num}{den}}$$

$$\boxed{\frac{-den_G\, den}{den_G\, den + num_G\, num} \qquad \frac{num_G(s)\, den(s)}{den_G(s)\, den(s) + num_G(s)\, num(s)} \qquad \frac{num_G(s)\, num(s)}{den_G(s)\, den(s) + num_G(s)\, num(s)}}$$

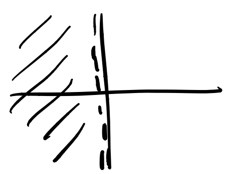
1. want ea. term to go to 0 in steady state

2. $1 + L(s) = 0 \Rightarrow$ roots in OLHP

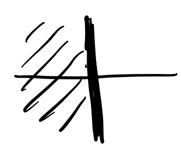
$$\boxed{den_G(s)\, den(s) + num_G(s)\, num(s) = 0 \Rightarrow \text{roots in OLHP}}$$

Note: complex plane

$Re(z) < 0$ OLHP



$Re(z) \leq 0$ CLHP



FINAL VALUE THM: $f(t)$ w/ Laplace $F(s)$

if $F(s)$ has poles in OLHP or at the origin (with only 1 pole at origin)

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Proof: $\mathcal{L}[\dot{f}] = \int_0^\infty \dot{f} e^{-st} dt = sF(s) - f(0)$

take $\lim_{s\to 0}$ $= \int_0^\infty \dot{f}\, 1\, dt = \lim_{s\to 0} sF(s) - f(0)$

$= \lim_{t\to\infty} f - f(0)$

$\Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} s F(s)$

Sanity check: $F(s) = \underbrace{G(s)}_{\text{sys}} \underbrace{\frac{1}{s}}_{\text{step input}}$

freq. response of $G(s)$ at ω is just $G(i\omega)$ $\qquad \underline{\text{step input}}$ $\omega = 0$

$\omega = 0$: $G(0) = \lim_{s\to 0} \cancel{s}\, G(s) \frac{1}{\cancel{s}} = G(0)$ ✓

How to use ↙

$\rightarrow \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{\beta_k s^k + \beta_{k-1} s^{k-1} \cdots \beta_1 s + \cancel{\beta_0}}{\alpha_n s^n + \alpha_{n-1} s^{n-1} \cdots \alpha_1 s + \cancel{\alpha_0}}$

what is lim? $\frac{\beta_0}{\alpha_0}$, $\frac{\beta_1}{\alpha_1}$ if $\beta_0 = 0$, $\alpha_0 = 0$

really a statement about minimum degree terms of the rational expression

- min deg term of top & bot have <u>same</u> degree $\implies$ soln. ratio of coeffs.
- min deg term of top is greater than min deg term of bot. $\implies$ soln 0

$$\frac{\beta_k s^k + \cdots + \beta_1 s}{\alpha_n s^n + \cdots + \alpha_1 s + \alpha_0} \implies \frac{s}{\alpha_0} = 0$$

- min deg term of bot greater than top $\implies$ soln $= \infty$

<u>Reference</u>	<u>Disturbance</u>	NOISE
$\underbrace{\frac{-1}{1+L(s)}}\,\underline{V_d(s)}$	$\frac{G(s)}{1+L(s)}\,\underline{D(s)}$	$\frac{L(s)}{1+L(s)}\,\underline{N(s)}$

$$\boxed{\frac{-\mathrm{den}_G\,\mathrm{den}}{\mathrm{den}_G\,\mathrm{den} + \mathrm{num}_G\,\mathrm{num}} \qquad \frac{\mathrm{num}_G(s)\,\mathrm{den}(s)}{\mathrm{den}_G(s)\,\mathrm{den}(s) + \mathrm{num}_G(s)\,\mathrm{num}(s)} \qquad \frac{\mathrm{num}_G(s)\,\mathrm{num}(s)}{\mathrm{den}_G(s)\,\mathrm{den}(s) + \mathrm{num}_G(s)\,\mathrm{num}(s)}}$$

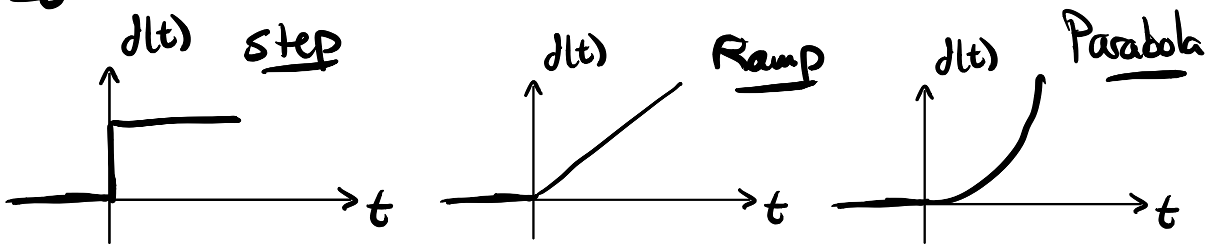
Ref. tracking

$$\lim_{s \to 0} \frac{-\,den_G(s)\,den(s)}{den_G(s)\,den(s) + num_G(s)\,num(s)}\, s\,(\text{input})$$

Disturbance

$$\lim_{s \to 0} \frac{num_G(s)\,den(s)}{den_G(s)\,den(s) + num_G(s)\,num(s)}\, s\,(\text{input})$$

Types of Inputs ⟸



$1/s$: "steady push"

$\frac{1}{s^2}$ "increasing push"

$\frac{1}{s^3}$ "accelerating push"

Plant $G(s) = \dfrac{num_G(s)}{den_G(s)}$

$\text{degree}(num_G) \leq \text{degree}(den_G)$

$G(s)$ proper TF from causality

Ex.

single integrator $G(s) = \boxed{\dfrac{1}{ms}}$ ←

double integrator $G(s) = \dfrac{1}{ms^2}$

Controllers

(PI) $C(s) = \dfrac{k_p s + k_I}{s}$

(PD) $C(s) = k_d s + k_p$

(PID) $C(s) = \dfrac{k_d s^2 + k_p s + k_I}{s}$ ←

PIID

$$C(s) = k_d s + k_p + \frac{k_I}{s} + \frac{k_{II}}{s^2}$$

$$= \frac{k_d s^3 + k_p s^2 + k_I s + k_{II}}{s^2} \leftarrow$$

Assume: input $\frac{1}{s^k}$

Reference

$$\lim_{s\to 0} \frac{\boxed{den_G}\,\boxed{den}}{den_G\, den + num_G\, num}\left(\frac{s}{s^\ell}\right)$$

Disturbance

$$\lim_{s\to 0} \frac{\boxed{num_G}\,\boxed{den}}{den_G\, den + num_G\, num}\left(\frac{s}{s^\ell}\right)$$

Want: min deg term in the top to be a higher power of s than the min deg term in the bottom

Design: $C(s) = \frac{num}{den}$

pick den to cancel out $\left(\frac{s}{s^\ell}\right)$

Try: $C(s) = \frac{k}{s^\ell}$

for plant $G(s) = \frac{1}{ms}$

$$\lim_{s\to 0} \frac{\overset{ms}{\cancel{den_G}}\,\overset{s^\ell}{\cancel{den}}}{\underset{ms}{\cancel{den_G}}\,\underset{s^\ell}{\cancel{den}} + \underset{1}{\cancel{num_G}}\,\underset{k}{\cancel{num}}}\left(\frac{s}{s^\ell}\right)$$

$$\lim_{s\to 0} \frac{\overset{1}{\cancel{num_G}}\,\overset{s^\ell}{\cancel{den}}}{\underset{ms}{\cancel{den_G}}\,\underset{s^\ell}{\cancel{den}} + \underset{1}{\cancel{num_G}}\,\underset{k}{\cancel{num}}}\left(\frac{s}{s^\ell}\right)$$

$$\lim_{s\to 0} \frac{ms^2}{ms^{\ell+1}+k} \to 0 \checkmark$$

$$\lim_{s\to 0} \frac{s}{ms^{\ell+1}+k} \to 0 \checkmark$$

$$C(s) = \frac{k}{s^{l-1}} ?$$

$$\lim_{s \to 0} \frac{ms}{ms^l + k} = 0 \qquad \lim_{s \to 0} \frac{1\,\cancel{s^{l-1}}}{\underbrace{ms^l + k}} \cancel{\left(\frac{s}{s^l}\right)}$$

$$\lim_{s \to 0} \frac{1}{ms^l + k} = \frac{1}{k}$$

needed a $\frac{k}{s^l}$ term in controller to reject disturbance.

PROBLEM: stability.

$$\text{den}_G\,\text{den} + \text{num}_G \boxed{\text{num}} = 0$$

$$ms\,s^l + 1\,k$$

$$ms^{l+1} + k = 0$$

necessary cond. for stability: all coeffs > 0

NOT STABLE

Now we can pick num for stability.

$$ms^{l+1} + 1(\text{num})$$

$$\to \beta_l s^l + \beta_{l-1} \underbrace{s^{l-1}}_{} + \cdots + \beta_1 s + \beta_0$$

determine coeffs based on RH

need all terms for stability.

if l is large... BIG RH table

$\Downarrow$

Complicated nonlinear conditions on gains for stability.

$$\cancel{C(s) = \frac{k}{s^l}}$$

FIXED:

$$C(s) = \frac{\beta_l s^l + \cdots + \beta_1 s + \beta_0}{s^l}$$

k

$$= \underbrace{\beta_l}_{\text{prop.}} + \underbrace{\frac{\beta_{l-1}}{s}}_{\text{integral}} + \cdots + \underbrace{\frac{\beta_0}{s^l}}_{\text{higher order integral terms}}$$

Another input:

sinusoid of freq ω $\quad D(s) = \dfrac{\omega}{s^2 + \omega^2}$

Reference

$$\lim_{s\to 0} \frac{den_G\, den}{den_G\, den + num_G\, num} \frac{s\,\omega}{s^2+\omega^2}$$

Disturbance

$$\lim_{s\to 0} \frac{num_G\, den}{den_G\, den + num_G\, num} \frac{s\,\omega}{s^2+\omega^2}$$

$$C(s) = \frac{k}{s^2+\omega^2} \; ?$$

Designed to reject sinusoids

$$G(s) = \frac{1}{ms}$$

$$\lim_{s\to 0} \frac{ms(s\omega)}{ms(s^2+\omega^2)+k} = 0 \qquad \lim_{s\to 0} \frac{s\omega}{ms(s^2+\omega^2)+k} = 0$$

with this controller what happens if we apply a step, ramp, etc disturbance?

$$\lim_{s\to 0} \frac{ms(s^2+\omega^2)}{ms(s^2+\omega^2)+k}\frac{s}{s^l} \qquad \lim_{s\to 0} \frac{s(s^2+\omega^2)}{ms(s^2+\omega^2)+k}\frac{s}{s^l}$$

$$\frac{ms^3+m\omega^2 s}{(ms(s^2+\omega^2)+k)s^{l-1}}$$

much higher degree

similar

DOESN'T WORK...

if $l=2$

$$\frac{ms^3+\boxed{m\omega^2} s}{(ms(s^2+\omega^2)\boxed{+k})s} = \frac{m\omega^2}{k} \neq 0$$

Try $C(s) = \dfrac{k}{s^l(s^2+\omega^2)}$

$$\lim_{s\to 0} \frac{m s\, s^l \cancel{(s^2+\omega^2)}}{m s^{l+1}(s^2+\omega^2)+k}\left(\frac{s\omega}{\cancel{s^2+\omega^2}}\right) = 0$$ sinusoid disturbance

$$\lim_{s\to 0} \frac{m s\, \cancel{s^l}(s^2+\omega^2)}{m s^{l+1}(s^2+\omega^2)+k}\left(\frac{s}{\cancel{s^l}}\right) = 0$$ step, ramp disturbance

---

Again stability is a problem...

modify $C(s) = \dfrac{num(s)}{s^l(s^2+\omega^2)}$

$$ms(s^l(s^2+\omega^2)) + \underbrace{num(s)}_{\text{pick this for stability}} = 0$$

$m s^{l+3} + m\omega^2 s^{l+1}$

num(s) needs to have at least terms of the form

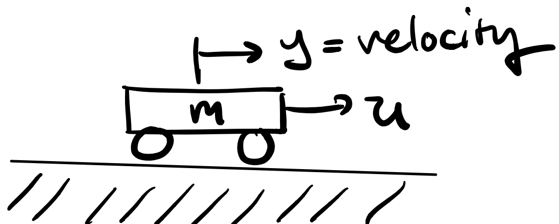
$s^{l+2}, s^l, s^{l-1}, s^{l-2}, \ldots, s, s^0$

use RH table $\Longrightarrow$ for stability criteria final controller:

wasn't needed

$$C(s) = \frac{\beta_{l+2} s^{l+2} + \boxed{\beta_{l+1} s^{l+1} +} \beta_l s^l + \cdots + \beta_1 s + \beta_0}{s^l (s^2 + \omega^2)}$$

EXAMPLE:

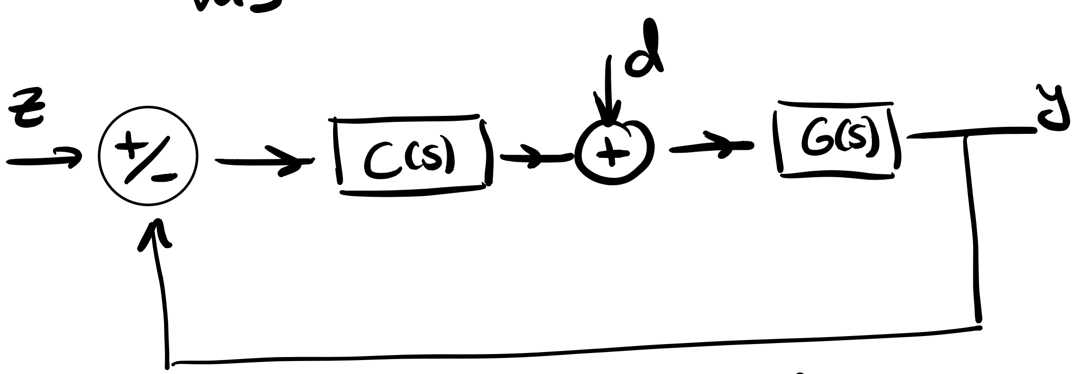


$$\dot{y} = \frac{u+d}{m}$$

u: control

d: disturbance – want to reject
- constant – WIND
- sinusoidal of freq ω_0

$$\Rightarrow G = \frac{1}{ms}$$



$$C(s) = \underbrace{k_p}_{\text{stability}} + \underbrace{\frac{k_I}{s}}_{\text{+ const. rejection}} + \underbrace{\frac{k_N}{s^2 + \omega_0^2}}_{\text{sinusoidal rejection}}$$

$$= \frac{k_p s(s^2+\omega_0^2) + k_I(s^2+\omega_0^2) + k_N s}{s(s^2+\omega_0^2)}$$

check stability: $G(s) = \frac{1}{ms}$

$$1+L(s) = \underset{den_G}{ms}\underset{den}{(s)}\cancel{(s^2+\omega_0^2)} + k_p s\cancel{(s^2+\omega_0^2)} + k_I\cancel{(s^2+\omega_0^2)} + k_N \cancel{s}$$

$$= ms^4 + k_p s^3 + (m\omega_0^2 + k_I)s^2 + (k_p\omega_0^2 + k_N)s + k_I\omega_0^2$$

Necessary: $m > 0$ ✓ $\quad m\omega_0^2 + k_I > 0 \quad k_I\omega_0^2 > 0$

$\boxed{k_p > 0} \quad k_p\omega_0^2 + k_N > 0$

RH TABLE:

s^4	m	$m\omega_0^2 + k_I$	$k_I\omega_0^2$
s^3	k_p	$k_p\omega_0^2 + k_N$	0
s^2	$(m\omega_0^2+k_I) - \frac{m}{k_p}(k_p\omega_0^2+k_N)$	$\boxed{k_I\omega_0^2}\cancel{(k_p)} - \cancel{m(0)}/\cancel{k_p}$	
s^1	$\boxed{*}$	0	
s^0	$k_I\omega_0^2$		

$$m > 0 \quad , \quad k_p > 0 \quad , \quad \underbrace{k_I w_0^2 > 0}_{k_I > 0}$$

$$(m w_0^2 + k_I) - \frac{m}{k_p}\left(k_p w_0^2 + k_N\right) > 0$$

$$* = k_p w_0^2 + k_N - \frac{k_p k_I w_0^2}{(m w_0^2 + k_I) - \frac{m}{k_p}\left(k_p w_0^2 + k_N\right)} > 0$$

Manipulate to get conditions for stability...