Connections TF and State space

state space:
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\rightarrow \quad sX(s) = AX + BU \quad \rightarrow \quad X = (sI - A)^{-1}BU$$
$$Y = CX + DU \qquad\qquad Y = [C(sI-A)^{-1}B + D]U$$

TF:

$$G(s) = [C(sI-A)^{-1}B + D]$$

TF to state space:

- multiple state space models
- transfer function determines the minimum # of states in the state space, but could be more states

want a state space with the minimum number of states → minimal realization

$$G(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \leftarrow$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Canonical minimal realization

$$C = \begin{bmatrix} \beta_0 & \cdots & \beta_{n-1} \end{bmatrix} \qquad D = \boxed{0}$$

strictly proper transfer function $G(s)$ since the numerator has degree less than the denominator

if we have a proper trans. function (not strictly proper), ie. the degree of the numerator is equal to degree of denominator

$$G(s) = C\underline{(sI - A)}^{-1}B + D$$

constant

$$(sI - A)^{-1} = \frac{1}{\underline{\det(sI - A)}} \left[\mathrm{Adj}(sI - A) \right]$$

$A \in \mathbb{R}^{n \times n}$

n degree polynomial

denominator of $G(s)$

Adjugate matrix

ea. term is a determinant of a submatrix

$$G(s) = \frac{\overbrace{C\,\mathrm{Adj}(sI - A)B}^{n-1 \text{ deg. poly.}} + \overbrace{D\det(sI - A)}^{n \text{ deg poly.}}}{\det(sI - A)}$$

all terms only go up to degree $n-1$

for proper (not strictly proper) TF:

$$G(s) = \frac{\beta_n s^n + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

Minimal Realization:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -\alpha_0 & -\alpha_1 & & -\alpha_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\beta_0 - \alpha_0 \beta_n, \beta_1 - \alpha_1 \beta_n, \ldots, \beta_{n-1} - \alpha_{n-1}\beta_n] \qquad D = \beta_n$$

C here is the previous $C + \det(sI - A)D$

Intuitively:

$$x = \begin{bmatrix} z \\ \dot{z} \\ \ddot{z} \\ \vdots \\ z^{(n-1)} \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \\ \ddot{z} \\ \vdots \end{bmatrix} + Bu$$

Stability: $\text{eig}(A) \in \text{OLHP}$ $\qquad \text{eig}(A)$

if degree of numerator of $G(s)$ greater than denominator → improper TF.

→ no physical system would give this TF

→ violates causality.

Stability Assessment:

so far:

- Routh Hurwitz
- Eigenvalues of sys matrix.
- Nyquist Criteria. } → allow us to model delays

Delay:

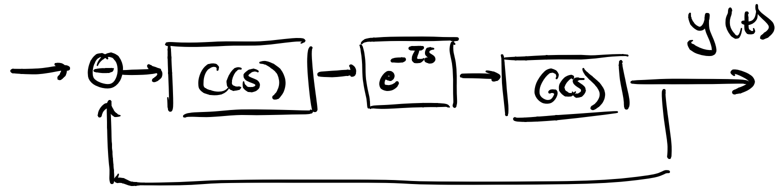
$$u(t) \longrightarrow \boxed{\text{Delay}} \longrightarrow y(t) = u(t-\tau)$$

Laplace transform $\quad Y(s) = e^{-\tau s} U(s)$

$|e^{-\tau i \omega}| = 1$ time delay doesn't change the magnitude

$\angle e^{-\tau i \omega} = -\tau \omega$ shifts the phase of the signal by $-\tau\omega$

Before:



open loop TF $\quad L(s) = e^{-\tau s} C(s) G(s)$

char eqn $\quad 1 + L(s) = 0 \implies 1 + \underbrace{e^{-\tau s} C(s) G(s) = 0}_{\text{roots ?}}$

Ex. PD control of a car.

$$C(s) = k_p + k_d s \qquad G(s) = \frac{1}{ms^2}$$

$$1 + L(s) = 1 + e^{-\tau s}\left(\frac{k_p + k_d s}{ms^2}\right) = 0$$

$$ms^2 + \underbrace{e^{-\tau s}}(k_p + k_d s) = 0$$

infinite degree polynomial

$$e^{-\tau s} = 1 - \tau s + \frac{(-\tau s)^2}{2} + \frac{(-\tau s)^3}{3!} + \cdots$$

Taylor Expansion of $e^{-\tau s}$

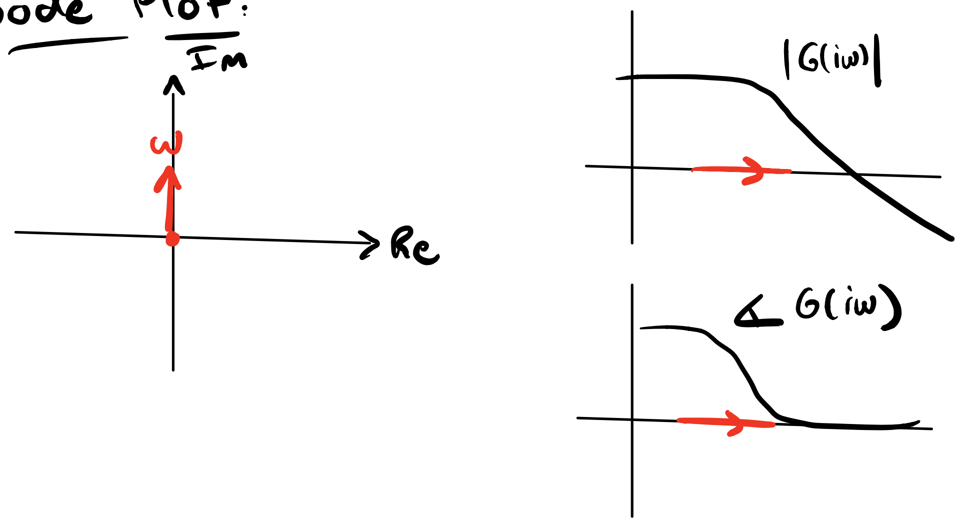
$$ms^2 + e^{-\tau s}\left(k_p + k_d s\right) = 0 \;\} \rightarrow$$ infinite number of roots.

any state space model
$\rightarrow$ infinite # of states.

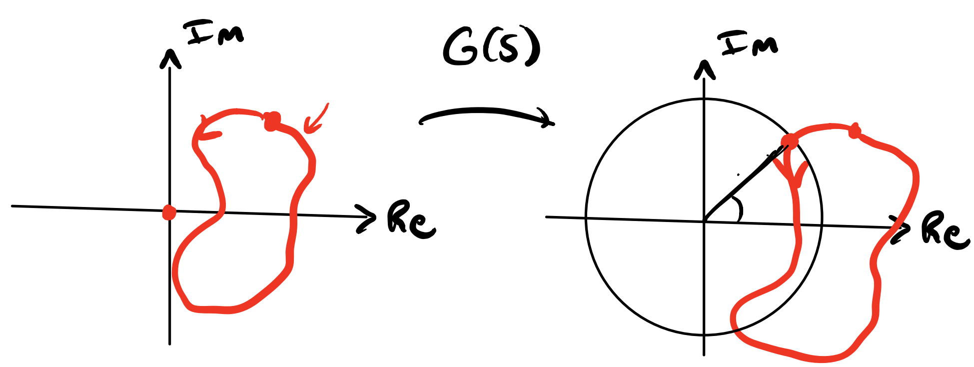
Nyquist Plots

another way to analyze TF

Bode Plot:

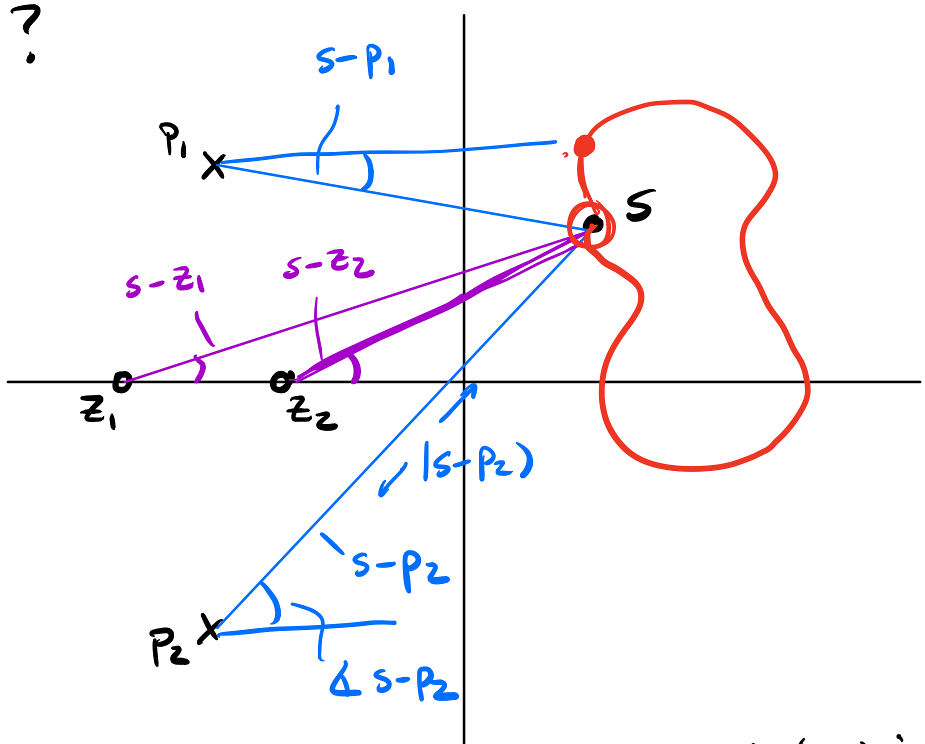


Nyquist Plot



$$G(s) = \frac{(s-z_1)\cdots(s-z_\ell)}{(s-p_1)\cdots(s-p_n)}$$

$G(s) = ?$

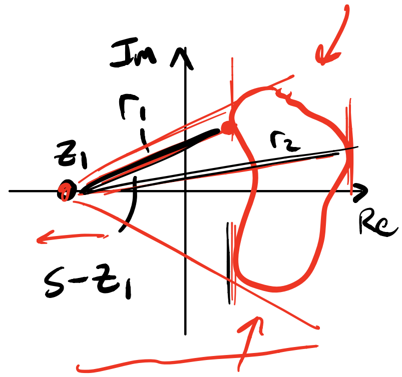


$$G(s) = \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)} = \frac{|s-z_1||s-z_2|}{|s-p_1||s-p_2|} \frac{e^{i\measuredangle(s-z_1)} e^{i\measuredangle(s-z_2)}}{e^{i\measuredangle(s-p_1)} e^{i\measuredangle(s-p_2)}}$$

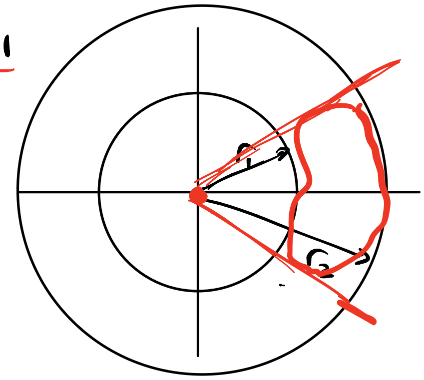
$$= \frac{|s-z_1||s-z_2|}{|s-p_1||s-p_2|} e^{\measuredangle(s-z_1) + \measuredangle(s-z_2) - \measuredangle(s-p_1) - \measuredangle(s-p_2)}$$

in general:

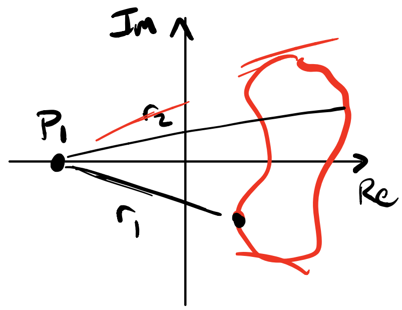
$$G(s) = \frac{\prod_k |s-z_k|}{\prod_{k'} |s-p_{k'}|} e^{i\sum_k \measuredangle(s-z_k) - i\sum_{k'} \measuredangle(s-p_{k'})}$$

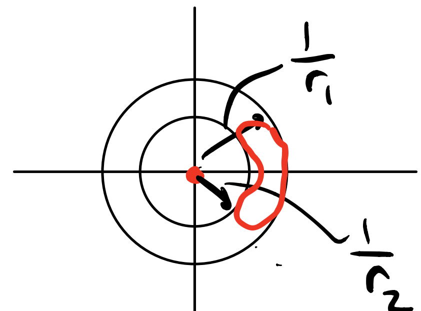


$$G(s) = s - z_1$$



$$G(s) = \frac{1}{s - p_1}$$





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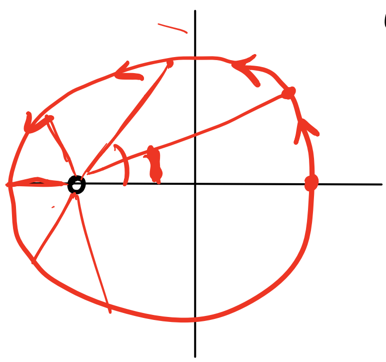

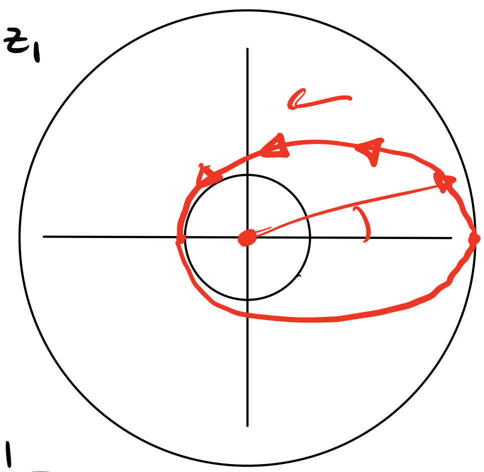

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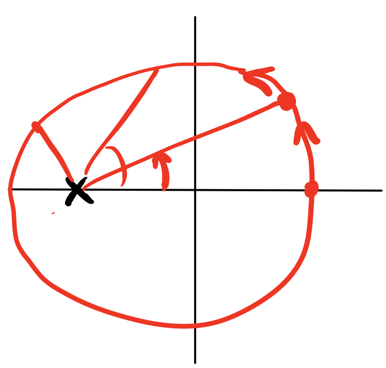
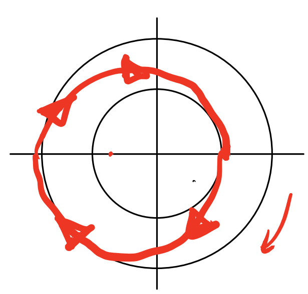
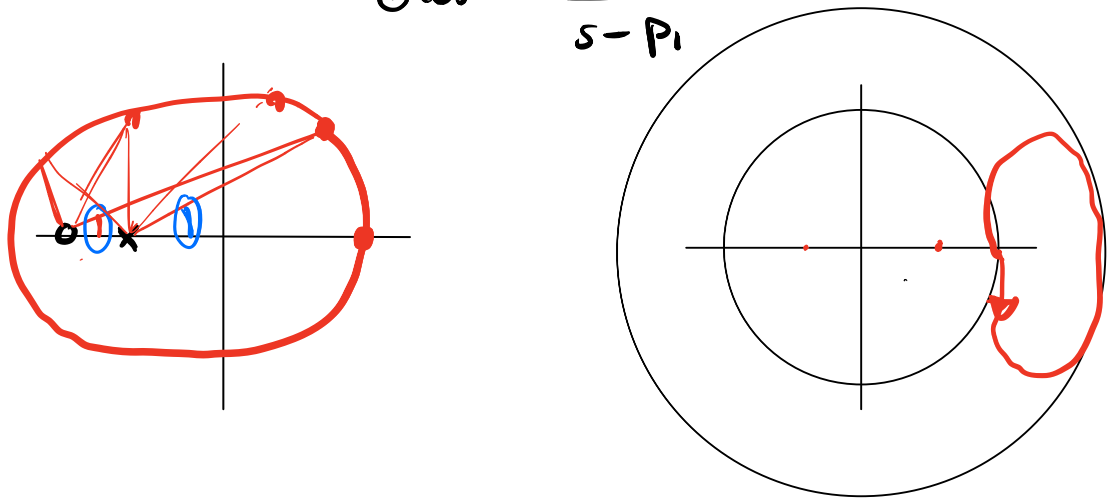
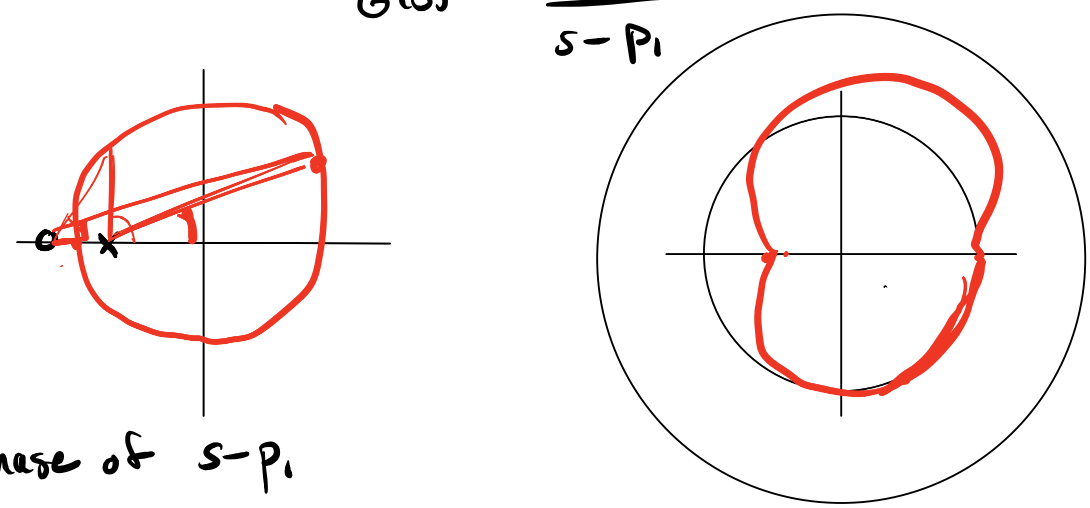




$$G(s) = \frac{s - z_1}{s - p_1}$$



$$G(s) = \frac{s - z_1}{s - p_1}$$



phase of $s - p_1$

0 all the way to 360

phase of $s - z_1$

45° and -45°

Cauchy Argument Principle:

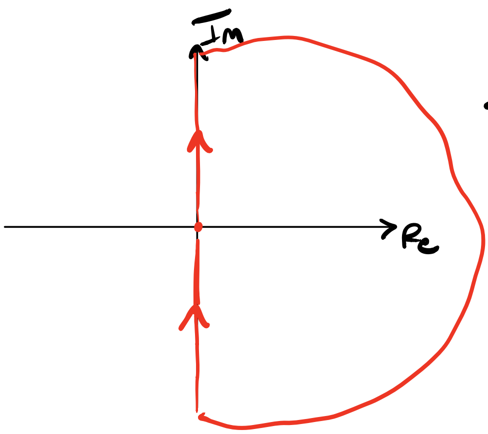
For a complex rational function $G(s)$

zeros inside a contour = # CW encirclements of origin + # of poles inside a contour

or

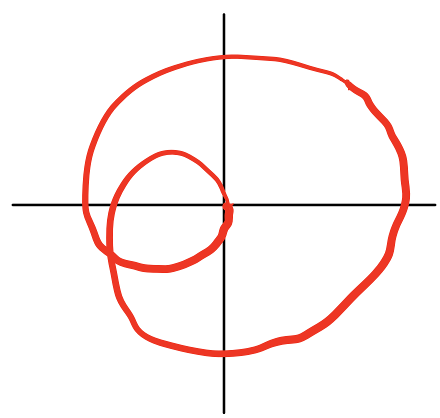
zeros inside a contour − # of poles inside a contour = # CW encirclements of origin

Nyquist Contour → encloses the RHP



$1+L(s)$ →

Nyquist Plot:

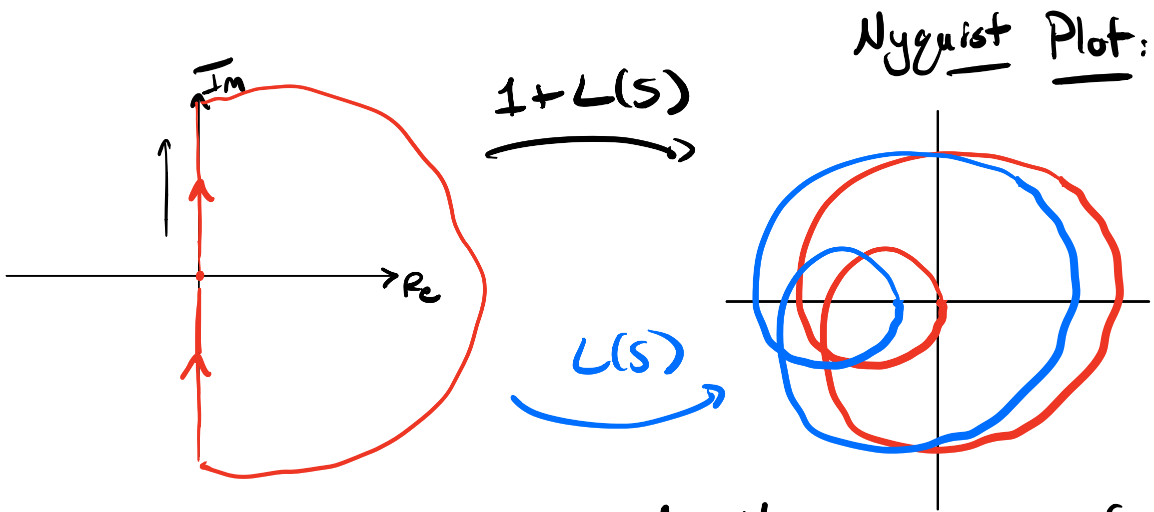


For stability:

interested in zeros $1+L(s)=0$

$L(s) = C(s)\,G(s)$

count encirclements of the origin



we can understand the zeros of $1+L(s)$ by counting the number of times that $L(s)$ encircles -1 in the Nyquist plot.

$$G(s) = \frac{num_G}{den_G} \qquad C(s) = \frac{num_C}{den_C}$$

$$1 + L(s) = 1 + G(s)C(s) = 1 + \frac{num_G}{den_G}\frac{num_C}{den_C}$$

roots are zeros

$$= \frac{den_G\, den_C + num_G\, num_C}{den_G\, den_C}$$

poles

- Draw Nyquist plot of $L(s)$
- count # of CW encirclements of -1

- know

→ $\# \text{RHP zeros} = \# \text{CW encirclements of } -1 + \# \text{RHP poles}$

want none in RHP: roots of $1 + L(s)$

want this to be 0

RHP poles of $L(s)$

roots of $den_G \, den_C$

know this

Want

$-\# \text{CW encirclements of } -1 = \# \text{RHP poles of } L(s)$ stability

$$\boxed{\# \text{CCW encirclements of } -1 = \# \text{RHP poles of } L(s)}$$

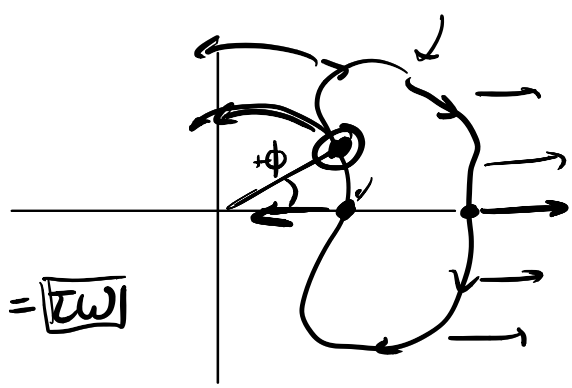
Nyquist stability Criterion

Time delay:

$\underline{L(s)}\,\underline{e^{-\tau s}}$

$e^{-i\phi}$

$\phi = \boxed{\tau\omega}$



$L(s) + \boxed{1}$