KKT Matrix Conditioning

Convex Optimization

Major sources:
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Quadratic Optimization & KKT System

$$\min_x \quad \frac{1}{2}x^\top Q x + c^\top x$$
$$\text{s.t.} \quad Ax = b$$

Primal Variables: $x \in \mathbb{R}^n$

Dual Variables: $v \in \mathbb{R}^m$

$Q \in \mathbb{R}^{n \times n}$ $\quad Q = Q^\top \succ 0$ $\quad c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$ $\quad b \in \mathbb{R}^m$

fat $(m < n)$,

full row rank (rank $= m$)

Lagrangian: $\mathcal{L}(x, v) = \frac{1}{2}x^\top Q x + c^\top x + v^\top \big(Ax - b\big)$

KKT Conditions:

1. Stationarity: $x^\top Q + c^\top + v^\top A = 0$
2. Feasibility: $Ax - b = 0$

KKT System:

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

KKT Matrix

Solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} -c \\ b \end{bmatrix}$$

KKT Solutions

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KKT Matrix

Solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q^{-1} - Q^{-1}A^\top \big(AQ^{-1}A^\top\big)^{-1} AQ^{-1} & Q^{-1}A^\top \big(AQ^{-1}A^\top\big)^{-1} \\ \big(AQ^{-1}A^\top\big)^{-1} AQ^{-1} & -\big(AQ^{-1}A^\top\big)^{-1} \end{bmatrix} \begin{bmatrix} -c \\ b \end{bmatrix}$$

*…using **block matrix inversion** (or just directly verify).*

KKT Solutions

$$\min_x \quad \frac{1}{2}x^\top Qx + c^\top x$$
$$\text{s.t.} \quad Ax = b$$

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KKT Matrix

Solution:

…projection matrix

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q^{-1}\left[I - A^\top\left(AQ^{-1}A^\top\right)^{-1}AQ^{-1}\right] & Q^{-1}A^\top\left(AQ^{-1}A^\top\right)^{-1} \\ \left(AQ^{-1}A^\top\right)^{-1}AQ^{-1} & -\left(AQ^{-1}A^\top\right)^{-1} \end{bmatrix} \begin{bmatrix} -c \\ b \end{bmatrix}$$

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KKT Matrix

Solution:

$$x = Q^{-1}A^\top\big(AQ^{-1}A^\top\big)^{-1}\big(AQ^{-1}c + b\big) - Q^{-1}c$$
$$v = -\big(AQ^{-1}A^\top\big)^{-1}\big(AQ^{-1}c + b\big)$$

KKT Matrix Properties

$$\min_x \quad \frac{1}{2}x^\top Q x + c^\top x$$
$$\text{s.t.} \quad Ax = b$$

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KKT Matrix

Matrix Properties:

$$M = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}$$

M indefinite

$$\text{spec}(M) = \rho_+ \sqcup \rho_-$$

Positive: $\rho_+ = \big\{\lambda_1, \ldots, \lambda_n\big\} > 0$

Negative: $\rho_- = \big\{\lambda_{n+1}, \ldots, \lambda_{n+m}\big\} < 0$

Cardinality

$|\rho_+| = n$

$|\rho_-| = m$

Conditioned KKT Matrix: Option 1*

*many conditionings are possible

Coordinate Transforms

Primal: $x = Px' \in \mathbb{R}^n$

Dual: $v = Wv' \in \mathbb{R}^m$

$$\min_x \quad \tfrac{1}{2}x'^\top P^\top Q P x' + c^\top P x'$$
$$\text{s.t.} \quad W^\top A P x' = W^\top b$$

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Lagrangian: $\mathcal{L}(x', v') = \tfrac{1}{2}x'^\top P^\top Q P x' + c^\top P x' + v'^\top W^\top (APx' - b)$

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1. Stationarity: $x'^\top P^\top Q P + c^\top P + v'^\top W^\top A P = 0$
2. Feasibility: $W^\top A P x' - W^\top b = 0$

KKT System:

$$\begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} \begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} -P^\top c \\ W^\top b \end{bmatrix}$$

Conditioning:

$$M \Rightarrow \begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} = \begin{bmatrix} P^\top & 0 \\ 0 & W^\top \end{bmatrix} \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & W \end{bmatrix}$$

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Conditioning: $AQ^{-\frac{1}{2}} = U\left[\Sigma \ 0\right]V^\top$ **(SVD)** $\Rightarrow$ Take $P = Q^{-\frac{1}{2}}V \quad W = U$

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$$M \Rightarrow \begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & \Sigma \\ 0 & I & 0 \\ \Sigma & 0 & 0 \end{bmatrix} \overset{\textbf{Permute...}}{\Longrightarrow} \begin{bmatrix} I & \Sigma & 0 \\ \Sigma & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

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Subblocks:

$$\begin{bmatrix} 1 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \qquad s(s-1) - \sigma_j^2 = 0$$
$$\lambda_{j,j+1} = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + \sigma_j^2}$$

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$s(s-1) - \sigma_j^2 = 0 \qquad \lambda_{j,j+1} = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + \sigma_j^2}$

$$\text{spec} = \Big\{\underbrace{\overset{\times(n-m)}{1}, \overset{\times m}{\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \sigma_j^2}}}_{+}, \underbrace{\overset{\times m}{\tfrac{1}{2} - \sqrt{\tfrac{1}{4} + \sigma_j^2}}}_{-}\Big\}$$

Conditioned KKT Matrix: Option 2*

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$$M \Rightarrow \begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Subblocks: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$s(s-1) - 1 = 0 \quad \lambda_{j,j+1} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1}$

$$\text{spec} = \Big\{ \underbrace{\overset{\times(n-m)}{1} \;,\; \overset{\times m}{\tfrac{1}{2} + \sqrt{\tfrac{1}{4}+1}}}_{+} \;,\; \underbrace{\overset{\times m}{\tfrac{1}{2} - \sqrt{\tfrac{1}{4}+1}}}_{-} \Big\}$$

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$$\text{spec} = \Big\{ \underbrace{\underset{\times(n-m)}{1}, \underset{\times m}{\tfrac{1+\sqrt{5}}{2}}}_{+}, \underbrace{\underset{\times m}{\tfrac{1-\sqrt{5}}{2}}}_{-} \Big\}$$