Dual Programs: LP

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad Cx \geq d \;\; (\mu)$$

Primal Program

Dual Programs: LP

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Primal Program

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: LP

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad Cx \geq d \quad \mu$$

Primal Program

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: LP

Primal Program

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad Cx \geq d \;\; (\mu)$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x \quad (r^\top + \lambda^\top A + \mu^\top C) x - \lambda^\top b - \mu^\top d$$

Dual Programs: LP

Primal Program

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x \quad \underbrace{(r^\top + \lambda^\top A + \mu^\top C)}_{\substack{\text{must be 0 for inner} \\ \text{problem to be bounded}}} x \; - \lambda^\top b - \mu^\top d$$

Dual Programs: LP

Primal Program

$$\max_{x} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad Cx \geq d \;\; (\mu)$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_{x} \quad \underbrace{(r^\top + \lambda^\top A + \mu^\top C)}_{\text{must be 0 for inner problem to be bounded}} x \; - \lambda^\top b - \mu^\top d$$

Dual Program

$$\min_{\lambda, \mu} \quad -\lambda^\top b - \mu^\top d$$

$$\text{s.t.} \quad r^\top + \lambda^\top A + \mu^\top C = 0 \,, \quad \mu \geq 0$$

Dual Programs: QP

$$\max_x \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Note: $Q = Q^\top \prec 0$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x + r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: QP

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$$\max_{x} \quad \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: QP

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$$\max_{x} \; \min_{\lambda,\mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda,\mu \geq 0} \; \max_{x} \quad \tfrac{1}{2}x^\top Q x \; + \; (r^\top + \lambda^\top A + \mu^\top C) x \; - \lambda^\top b - \mu^\top d$$

Dual Programs: QP

$$\max_x \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\max_x \; \min_{\lambda,\mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\min_{\lambda,\mu \geq 0} \; \max_x \quad \underbrace{\tfrac{1}{2}x^\top Q x \; + \; \big(r^\top + \lambda^\top A + \mu^\top C\big)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Dual Programs: QP

Primal Program

$$\max_x \quad \tfrac{1}{2}x^\top Q x + r^\top x$$

$$\text{s.t.} \quad Ax = b \ \ \lambda \qquad Cx \geq d \ \ \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x \underbrace{\tfrac{1}{2}x^\top Q x + (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C,$

Dual Programs: QP

Primal Program

$$\max_x \quad \tfrac{1}{2}x^\top Q x + r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad Cx \geq d \;\; (\mu)$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x \quad \underbrace{\tfrac{1}{2}x^\top Q x + (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\quad \dfrac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Q x + \xi^\top x\right) = 0$

Dual Programs: QP

$$\max_x \quad \tfrac{1}{2}x^\top Qx \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x,\lambda,\mu) = \tfrac{1}{2}x^\top Qx \; + r^\top x \; + \lambda^\top(Ax-b) + \mu^\top(Cx-d)$$

$$\max_x \quad \min_{\lambda,\mu\geq 0} \quad \tfrac{1}{2}x^\top Qx \; + r^\top x \; + \lambda^\top(Ax-b) + \mu^\top(Cx-d)$$

$$\min_{\lambda,\mu\geq 0} \quad \max_x \quad \underbrace{\tfrac{1}{2}x^\top Qx \; + \; (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} \; -\lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C, \quad \frac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Qx + \xi^\top x\right) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \; \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\max_{x} \; \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\min_{\lambda, \mu \geq 0} \; \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x \; + \; (r^\top + \lambda^\top A + \mu^\top C)\,x}_{\text{maximize explicitly...}} \; -\lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\quad \frac{\partial}{\partial x}\big(\tfrac{1}{2}x^\top Q x + \xi^\top x\big) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Plug in x...

Dual Programs: QP

Primal Program

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x + r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda,\mu \geq 0} \quad \tfrac{1}{2}x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda,\mu \geq 0} \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x + (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\quad \frac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Q x + \xi^\top x\right) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Plug in x...

Dual Program

$$\min_{\xi,\lambda,\mu} \quad -\tfrac{1}{2}\xi^\top Q^{-1}\xi - \lambda^\top b - \mu^\top d$$

$$\text{s.t.} \quad \xi^\top = r^\top + \lambda^\top A + \mu^\top C, \quad \mu \geq 0$$

Note: $-Q^{-1} = -Q^{-\top} \succ 0$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad Cx \geq d \quad \mu \qquad ||E_i x + e_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \quad i \in \mathcal{I} \;\; \theta_i$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

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$$y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$Ax = b$ (λ) $\quad$ $Cx \geq d$ (μ) $\quad$ $||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I}$ (ν_i)

$y_i = E_i x + b_i \quad i \in \mathcal{I}$ (θ_i)

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

Dual Programs: SOCP (lin objective)

Primal Program

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$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \quad \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x}_{\text{must be 0 for inner problem to be bounded}} \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$Ax = b$ $\quad\lambda$

$Cx \geq d$ $\quad\mu$

$||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I}$ $\quad\nu_i$

$y_i = E_i x + b_i \quad i \in \mathcal{I}$ $\quad\theta_i$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \; \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)}_{\text{must be 0 for inner problem to be bounded}} x \quad -\lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

$$\sup_{y_i} - \; \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad Cx \geq d \;\; (\mu) \qquad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; (\nu_i)$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; (\theta_i)$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)}_{\text{must be 0 for inner problem to be bounded}} x - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

$$\sup_{y_i} - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) = \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \quad i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x}_{\text{must be 0 for inner problem to be bounded}} \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

$$\text{Suppose} \quad ||\theta_i||_2 \leq \nu_i \qquad (\nu_i \geq 0)$$

$$-\theta_i^\top y_i \leq \nu_i ||y_i||_2 \qquad \forall y_i$$

$$-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) \leq 0 \qquad \forall y_i$$

$$\sup_{y_i} - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

$$= \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

s.t. $Ax = b$ (λ) $\quad Cx \geq d$ (μ) $\quad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I}$ (ν_i)

$y_i = E_i x + b_i \;\; i \in \mathcal{I}$ (θ_i)

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \; \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x}_{\text{must be 0 for inner problem to be bounded}} \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Suppose $||\theta_i||_2 > \nu_i \qquad (\nu_i \geq 0)$

take $y_i = -s\theta_i \qquad s \in \mathbb{R}_+$

$$-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) = - \; s\left(\nu_i ||\theta_i||_2 - ||\theta_i||_2^2\right) > 0$$

$$\sup_{y_i} - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

$$= \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$$Ax = b \quad \lambda$$

$$Cx \geq d \quad \mu$$

$$||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

$$y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x}_{\text{must be 0 for inner problem to be bounded}} \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right] + \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Dual Program

$$\min_{\substack{\lambda, \mu \\ \nu_i, \theta_i}} \quad -\lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

s.t.

$$r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right] = 0, \quad \mu \geq 0,$$

$$||\theta_i||_2 \leq \nu_i, \;\; \nu_i \geq 0 \quad i \in \mathcal{I}$$

$$\sup_{y_i} - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) = \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (quad objective)

Primal Program

$$\max_{x,y_i} \ \frac{1}{2}x^\top Q x + r^\top x$$

s.t. $Ax = b \quad \lambda$ $\quad Cx \geq d \quad \mu$ $\quad ||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$

$y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \frac{1}{2}x^\top Q x + r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i\left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top(E_i x + e_i - y_i)$$

$$\max_{x,y_i} \min_{\substack{\lambda,\mu\geq 0\\ \nu_i \geq 0, \theta_i}} \frac{1}{2}x^\top Q x + r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i\left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top(E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda,\mu\geq 0\\ \nu_i \geq 0, \theta_i}} \max_{x,y_i} \underbrace{\frac{1}{2}x^\top Q x + \left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right] + \sum_i \underbrace{-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Dual Program

$$\min_{\substack{\lambda,\mu,\xi\\ \nu_i,\theta_i}} -\frac{1}{2}\xi^\top Q^{-1}\xi - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

s.t. $r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right] = \xi^\top, \quad \mu \geq 0,$

$||\theta_i||_2 \leq \nu_i, \quad \nu_i \geq 0 \quad i \in \mathcal{I}$

$$\sup_{y_i} -\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) = \begin{cases} 0 & ; \ ||\theta_i||_2 \leq \nu_i \\ \infty & ; \ \text{otherwise} \end{cases}$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \langle R, X \rangle$$

$$\text{s.t.} \quad \langle A_i, X \rangle = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \langle C_j, X \rangle \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Suppose $X \succeq 0 \quad (X = X^\top)$

Dual Programs: SDP (lin objective)

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

Primal Program

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i \Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j \Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i \Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j \Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists}\\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(X^{\top/2} U^\top X^{1/2})$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}})$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \underset{\text{unique}}{\text{exists}} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}})$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}})$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

$$\mathrm{Tr} = \text{sum eigs} \quad \Rightarrow \quad \iff \mathrm{Tr}(X^{\top/2} U^\top X^{1/2}) \geq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}}) \geq 0$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

$$\mathrm{Tr} = \text{sum eigs} \;\; \Rightarrow \qquad \iff \mathrm{Tr}(X^{\top/2} U^\top X^{1/2}) \geq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Suppose $\quad X \not\succeq 0 \quad (X = X^\top)$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Suppose $X \not\succeq 0 \quad (X = X^\top) \quad \Rightarrow$ exists neg. (real) eval with evec u

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Suppose $\quad X \not\succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad$ exists neg. (real) eval with evec u

take $\quad U = s u u^\top \succeq 0$

$$s \in \mathbb{R}_+$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \ X = X^\top \succeq 0 \\ -\infty & ; \ \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \not\succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \text{exists neg. (real) eval with evec } u$$

$$\text{take} \quad U = suu^\top \succeq 0 \quad \Rightarrow \quad \mathrm{Tr}(U^\top X) = s\mathrm{Tr}(u^\top X u) < 0$$

$$s \in \mathbb{R}_+$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \left(\mathrm{Tr}(A_i^\top X) - b_i\right) + \sum_j \mu_j \left(\mathrm{Tr}(C_j^\top X) - d_j\right) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \left(\mathrm{Tr}(A_i^\top X) - b_i\right) + \sum_j \mu_j \left(\mathrm{Tr}(C_j^\top X) - d_j\right) + \mathrm{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \max_X \mathrm{Tr}\left(\underbrace{\left[R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top\right]}_{\text{must be 0 for inner problem to be bounded}} X \right) - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \max_{X} \mathrm{Tr}\Bigg(\underbrace{\Big[R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top\Big]}_{\text{must be 0 for inner problem to be bounded}} X\Bigg) - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Program

$$\min_{\lambda_i, \mu_j, U} \quad -\sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

$$\text{s.t.} \quad R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top = 0\,, \quad \mu_j \geq 0 \quad j \in \mathcal{J} \quad U \succeq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \;\; \lambda_i \qquad X \geq M \;\; W \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i \Big) + \mathrm{Tr}\big(W^\top (X - M)\big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, W \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i \Big) + \mathrm{Tr}\big(W^\top (X - M)\big) + \mathrm{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i, W \geq 0 \\ U \succeq 0}} \max_{X} \mathrm{Tr}\Big(\underbrace{\Big[R^\top + \sum_i \lambda_i A_i^\top + W^\top + U^\top \Big]}_{\text{must be 0 for inner problem to be bounded}} X \Big) - \mathrm{Tr}\big(W^\top M\big) - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Program

$$\min_{\lambda_i, W, U} \quad -\mathrm{Tr}\big(W^\top M\big) - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

$$\text{s.t.} \quad R^\top + \sum_i \lambda_i A_i^\top + W^\top + U^\top = 0\,, \quad W \geq 0 \quad U \succeq 0$$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

Dual Programs: General Convex Function

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad Cx \geq d \;\; (\mu)$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \; \min_{\lambda, \mu \geq 0} \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\max_x \quad \min_{\lambda,\mu \geq 0} \qquad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda,\mu \geq 0} \quad \max_x \qquad F(x) \; + \left[\lambda^\top A + \mu^\top C\right] x \; - \lambda^\top b - \mu^\top d$$

Dual Programs: General Convex Function

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \; \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \; \max_{x} \quad F(x) + \left[\lambda^\top A + \mu^\top C\right] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \underbrace{\max_x \quad F(x) + \left[\lambda^\top A + \mu^\top C\right] x}\; - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$ $\qquad F^*(\xi) = \sup_x \; F(x) - \xi^\top x$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda,\mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda,\mu \geq 0} \underbrace{\max_x F(x) + \left[\lambda^\top A + \mu^\top C\right] x}\; - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

Legendre Transform
(Fenchel Conjugate)

Dual Programs: General Convex Function

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_{x} \quad F(x) + \left[\lambda^\top A + \mu^\top C\right] x - \lambda^\top b - \mu^\top d$$

$$\text{Define} \quad \xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$$

$$F^*(\xi) = \sup_{x} \; F(x) - \xi^\top x$$

Legendre Transform
(Fenchel Conjugate)

$$\min_{\xi, \lambda, \mu} \quad F^*(\xi) - \lambda^\top b - \mu^\top d$$

$$\text{s.t.} \quad \xi^\top = -\left[\lambda^\top A + \mu^\top C\right], \quad \mu \geq 0$$

Dual Program

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \quad \min_{\lambda, \mu \geq 0} \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_x \quad F(x) \; + \left[\lambda^\top A + \mu^\top C\right] x \; - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$

$$F^*(\xi) = \sup_x \; F(x) - \; \xi^\top x$$

Legendre Transform
(Fenchel Conjugate)

Example: $F(x) = \frac{1}{2} x^\top Q x \; + \; r^\top x$

Dual Programs: General Convex Function

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \quad \min_{\lambda, \mu \geq 0} \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_{x} \quad F(x) \; + \left[\lambda^\top A + \mu^\top C\right] x \; - \lambda^\top b - \mu^\top d$$

$$\text{Define} \;\; \xi^\top = -\left[\lambda^\top A + \mu^\top C\right] \qquad F^*(\xi) = \sup_{x} \; F(x) - \xi^\top x$$

Legendre Transform
(Fenchel Conjugate)

Example: $F(x) = \frac{1}{2} x^\top Q x \; + \; r^\top x$

$$\Rightarrow \quad \frac{d}{dx}(F - \xi^\top x) = 0$$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$
$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \quad \min_{\lambda, \mu \geq 0} \qquad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_x \qquad F(x) \; + \left[\lambda^\top A + \mu^\top C\right] x \; - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$

$$F^*(\xi) = \sup_x \; F(x) - \; \xi^\top x$$

Legendre Transform
(Fenchel Conjugate)

Example: $F(x) = \frac{1}{2} x^\top Q x \; + \; r^\top x$

$$\Rightarrow \quad \frac{d}{dx}(F - \xi^\top x) = 0 \quad \Rightarrow \quad x = Q^{-1}(\xi - r)$$

Dual Programs: General Convex Function

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_{x} \quad F(x) + \left[\lambda^\top A + \mu^\top C\right] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -\left[\lambda^\top A + \mu^\top C\right]$

$$F^*(\xi) = \sup_{x} F(x) - \xi^\top x$$

Legendre Transform
(Fenchel Conjugate)

Example: $F(x) = \frac{1}{2} x^\top Q x + r^\top x$

$$\Rightarrow \frac{d}{dx}(F - \xi^\top x) = 0 \Rightarrow x = Q^{-1}(\xi - r) \Rightarrow F^*(\xi) = -\frac{1}{2}(\xi - r)^\top Q^{-1}(\xi - r)$$

Dual Programs: General Convex Function

Primal Program

$$\max_x \quad F(x)$$
$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu$$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \; \min_{\lambda, \mu \geq 0} \quad F(x) \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \; \max_x \quad F(x) \; + \left[\lambda^\top A + \mu^\top C\right] x \; - \lambda^\top b - \mu^\top d$$

$$\text{Define} \;\; \xi^\top = -\left[\lambda^\top A + \mu^\top C\right] \qquad F^*(\xi) = \sup_x \; F(x) - \; \xi^\top x$$

Legendre Transform (Fenchel Conjugate)

Example: $F(x) = \frac{1}{2} x^\top Q x \; + \; r^\top x$

$$\Rightarrow \; \frac{d}{dx}(F - \xi^\top x) = 0 \;\; \Rightarrow \;\; x = Q^{-1}(\xi - r) \;\; \Rightarrow \;\; F^*(\xi) = -\tfrac{1}{2}(\xi - r)^\top Q^{-1}(\xi - r)$$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad x \in \mathcal{X} \quad y$$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \ \ (\lambda) \qquad x \in \mathcal{X} \ \ (y)$$

generalized convex cone

cone: $x \in \mathcal{X} \quad \Rightarrow \quad \alpha x \in \mathcal{X}, \ \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \quad \Rightarrow \quad x + y \in \mathcal{X}$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_x \quad F(x)$$

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad x \in \mathcal{X} \quad y$$

generalized convex cone

cone: $x \in \mathcal{X} \Rightarrow \alpha x \in \mathcal{X}, \ \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \Rightarrow x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + y^\top x$$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad x \in \mathcal{X} \;\; y$$

generalized convex cone

cone: $x \in \mathcal{X} \Rightarrow \alpha x \in \mathcal{X}, \;\; \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \Rightarrow x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + y^\top x$$

$$\max_{x} \min_{\lambda,\, y \in \mathcal{Y}} F(x) + \lambda^\top (Ax - b) + y^\top x$$

Dual Programs: General Convex Function + General Convex Cones

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad x \in \mathcal{X} \quad y$$

Primal Program

generalized convex cone

cone: $x \in \mathcal{X} \;\Rightarrow\; \alpha x \in \mathcal{X}, \;\; \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \;\Rightarrow\; x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

$$\max_{x} \quad \min_{\lambda,\, y \in \mathcal{Y}} \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

what does $\mathcal{Y}$ *need to be so that*

$\min_{y \in \mathcal{Y}} \; y^\top x$ *forces* $x \in \mathcal{X}$ *?*

Dual Programs: General Convex Function + General Convex Cones

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad x \in \mathcal{X} \;\; y$$

Primal Program

generalized convex cone

cone: $x \in \mathcal{X} \;\; \Rightarrow \;\; \alpha x \in \mathcal{X}, \;\; \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \;\; \Rightarrow \;\; x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

$$\max_{x} \;\; \min_{\lambda,\, y \in \mathcal{Y}} \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

what does $\mathcal{Y}$ *need to be so that*

$\min_{y \in \mathcal{Y}} \; y^\top x$ *forces* $x \in \mathcal{X}$ *?*

$$\mathcal{Y} = \left\{ \; y \;\middle|\; \langle y, x \rangle \geq 0, \;\; \forall x \in \mathcal{X} \; \right\}$$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad x \in \mathcal{X} \;\; y$$

generalized convex cone

cone: $x \in \mathcal{X} \;\; \Rightarrow \;\; \alpha x \in \mathcal{X}, \;\; \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \;\; \Rightarrow \;\; x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

$$\max_{x} \;\; \min_{\lambda,\, y \,\in\, \mathcal{Y}} \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

what does $\mathcal{Y}$ *need to be so that*

$\min_{y \in \mathcal{Y}} \; y^\top x$ *forces* $x \in \mathcal{X}$ *?*

$$\mathcal{Y} = \left\{ \; y \;\middle|\; \langle y, x \rangle \geq 0, \;\; \forall x \in \mathcal{X} \; \right\}$$

$$\inf_{y \in \mathcal{Y}} \langle y, x \rangle \;\; = \begin{cases} 0 & ; \;\; x \in \mathcal{X} \\ -\infty & ; \;\; \text{otherwise} \end{cases}$$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad x \in \mathcal{X} \;\; (y)$$

generalized convex cone

cone: $x \in \mathcal{X} \;\Rightarrow\; \alpha x \in \mathcal{X},\; \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \;\Rightarrow\; x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \;+ \lambda^\top (Ax - b) + \; y^\top x$$

$$\max_{x} \;\; \min_{\lambda,\, y \in \mathcal{Y}} \quad F(x) \;+ \lambda^\top (Ax - b) + \; y^\top x$$

what does $\mathcal{Y}$ *need to be so that*

$$\min_{y \in \mathcal{Y}} \; y^\top x \quad \textit{forces} \quad x \in \mathcal{X} \;\; ?$$

$$\mathcal{Y} = \left\{ \; y \;\middle|\; \langle y, x \rangle \geq 0,\; \forall x \in \mathcal{X} \; \right\}$$

$$\inf_{y \in \mathcal{Y}} \langle y, x \rangle \;\; = \begin{cases} 0 & ; \; x \in \mathcal{X} \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Examples:

$$\mathcal{X} = \{ \; x \mid Ax = 0 \}$$

$$\mathcal{Y} = \{ \; y \mid y = A^\top z \}$$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_x \quad F(x)$$
$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad x \in \mathcal{X} \;\; y$$

generalized convex cone

cone: $x \in \mathcal{X} \;\Rightarrow\; \alpha x \in \mathcal{X}, \;\alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \;\Rightarrow\; x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

$$\max_x \quad \min_{\lambda,\, y \in \mathcal{Y}} \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

what does $\mathcal{Y}$ *need to be so that*

$\min_{y \in \mathcal{Y}} \; y^\top x$ *forces* $x \in \mathcal{X}$ *?*

$$\mathcal{Y} = \left\{ \, y \mid \langle y, x \rangle \geq 0, \;\; \forall x \in \mathcal{X} \, \right\}$$

$$\inf_{y \in \mathcal{Y}} \langle y, x \rangle \;\; = \begin{cases} 0 & ; \;\; x \in \mathcal{X} \\ -\infty & ; \;\; \text{otherwise} \end{cases}$$

Examples:

$\mathcal{X} = \{ \, x \mid Ax = 0 \}$	$\mathcal{Y} = \{ \, y \mid y = A^\top z \}$
$\mathcal{X} = \{ \, x \mid x \in \mathbb{R}^n_+ \}$	$\mathcal{Y} = \{ \, y \mid y \in \mathbb{R}^n_+ \}$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_{x} \quad F(x)$$

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what does $\mathcal{Y}$ *need to be so that*

$\min_{y \in \mathcal{Y}} \; y^\top x$ *forces* $x \in \mathcal{X}$ *?*

$$\mathcal{Y} = \left\{ \; y \mid \langle y, x \rangle \geq 0, \;\; \forall x \in \mathcal{X} \; \right\}$$

$$\inf_{y \in \mathcal{Y}} \langle y, x \rangle \;\; = \begin{cases} 0 & ; \;\; x \in \mathcal{X} \\ -\infty & ; \;\; \text{otherwise} \end{cases}$$

Examples:

$\mathcal{X} = \{ \; x \mid Ax = 0 \}$	$\mathcal{Y} = \{ \; y \mid y = A^\top z \}$
$\mathcal{X} = \{ \; x \mid x \in \mathbb{R}^n_+ \}$	$\mathcal{Y} = \{ \; y \mid y \in \mathbb{R}^n_+ \}$
$\mathcal{X} = \{ \; X \mid X \succeq 0 \}$	$\mathcal{Y} = \{ \; Y \mid Y \succeq 0 \}$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad x \in \mathcal{X} \;\; y$$

generalized convex cone

cone: $x \in \mathcal{X} \;\Rightarrow\; \alpha x \in \mathcal{X},\; \alpha \in \mathbb{R}_+$

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what does $\mathcal{Y}$ *need to be so that*

$\min_{y \in \mathcal{Y}} \; y^\top x$ *forces* $x \in \mathcal{X}$ *?*

$$\mathcal{Y} = \left\{ \; y \; \middle| \; \langle y, x \rangle \geq 0, \;\; \forall x \in \mathcal{X} \; \right\}$$

$$\inf_{y \in \mathcal{Y}} \langle y, x \rangle \;\; = \begin{cases} 0 & ; \;\; x \in \mathcal{X} \\ -\infty & ; \;\; \text{otherwise} \end{cases}$$

Examples:

$\mathcal{X} = \{ \; x \mid Ax = 0 \}$	$\mathcal{Y} = \{ \; y \mid y = A^\top z \}$
$\mathcal{X} = \{ \; x \mid x \in \mathbb{R}^n_+ \}$	$\mathcal{Y} = \{ \; y \mid y \in \mathbb{R}^n_+ \}$
$\mathcal{X} = \{ \; X \mid X \succeq 0 \}$	$\mathcal{Y} = \{ \; Y \mid Y \succeq 0 \}$
$\mathcal{X} = \{ (x, r) \mid \lVert x \rVert \leq r \}$	$\mathcal{Y} = \{ (y, s) \mid \lVert y \rVert_* \leq s \}$

Dual Programs: General Convex Function + General Convex Cones

Primal Program

$$\max_{x} \quad F(x)$$

$$\text{s.t.} \quad Ax = b \;\; (\lambda) \qquad x \in \mathcal{X} \;\; (y)$$

generalized convex cone

cone: $x \in \mathcal{X} \;\Rightarrow\; \alpha x \in \mathcal{X},\; \alpha \in \mathbb{R}_+$

convex: $x, y \in \mathcal{X} \;\Rightarrow\; x + y \in \mathcal{X}$

$$\mathcal{L}(x, \lambda, \mu) = \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

$$\max_{x} \;\; \min_{\lambda,\, y \in \mathcal{Y}} \quad F(x) \; + \lambda^\top (Ax - b) + \; y^\top x$$

Dual Program

$$\min_{\xi, \lambda, y} \quad F^*(\xi) \; - \lambda^\top b$$

$$\text{s.t.} \quad \xi^\top = -\left[\lambda^\top A + y^\top\right]$$

$$y \in \mathcal{Y}$$

Examples:

$\mathcal{X} = \{\; x \mid Ax = 0\}$	$\mathcal{Y} = \{\; y \mid y = A^\top z\}$
$\mathcal{X} = \{\; x \mid x \in \mathbb{R}^n_+\}$	$\mathcal{Y} = \{\; y \mid y \in \mathbb{R}^n_+\}$
$\mathcal{X} = \{\; X \mid X \succeq 0\}$	$\mathcal{Y} = \{\; Y \mid Y \succeq 0\}$
$\mathcal{X} = \{(x, r) \mid \|\|x\|\| \leq r\}$	$\mathcal{Y} = \{(y, s) \mid \|\|y\|\|_* \leq s\}$