Farka’s Lemma

Convex Optimization

Dan Calderone

Farka's Lemma Let $H \in \mathbb{R}^{m \times n}$ and $r \in \mathbb{R}^n$.

Then exactly one of the following two assertions is true.

1. There exists $v \in \mathbb{R}^m$ such that $v^\top H = r^\top$ and $v \geq 0$
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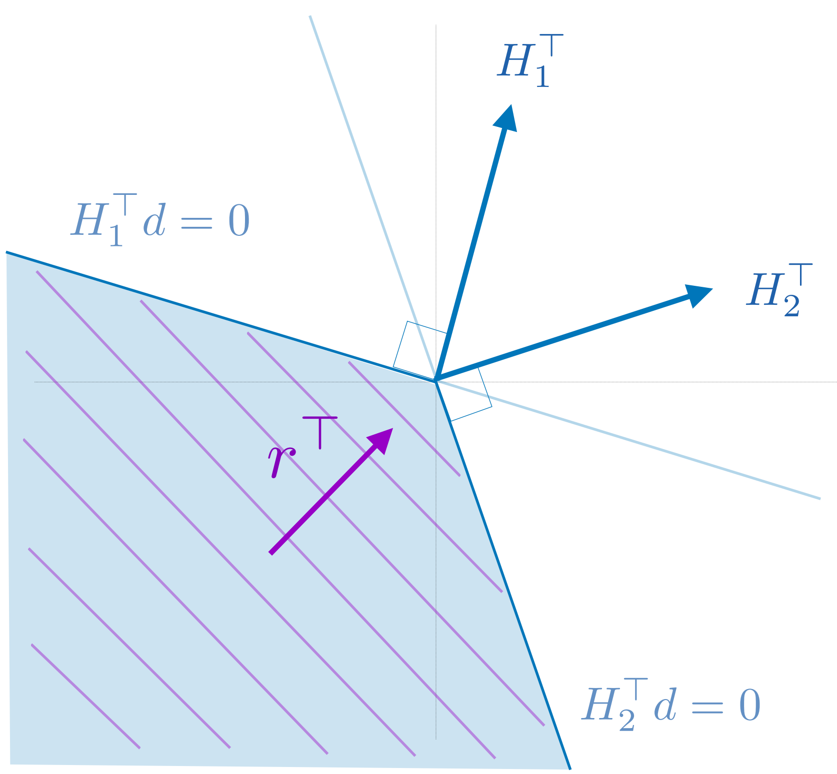
in English…

at a vertex (edge, face, etc) of a linear program, either…

1. *The gradient r points directly into the vertex (edge, face, etc.)*

OR

2. *There is a direction d off of the vertex into the feasible set that improves the objective.*



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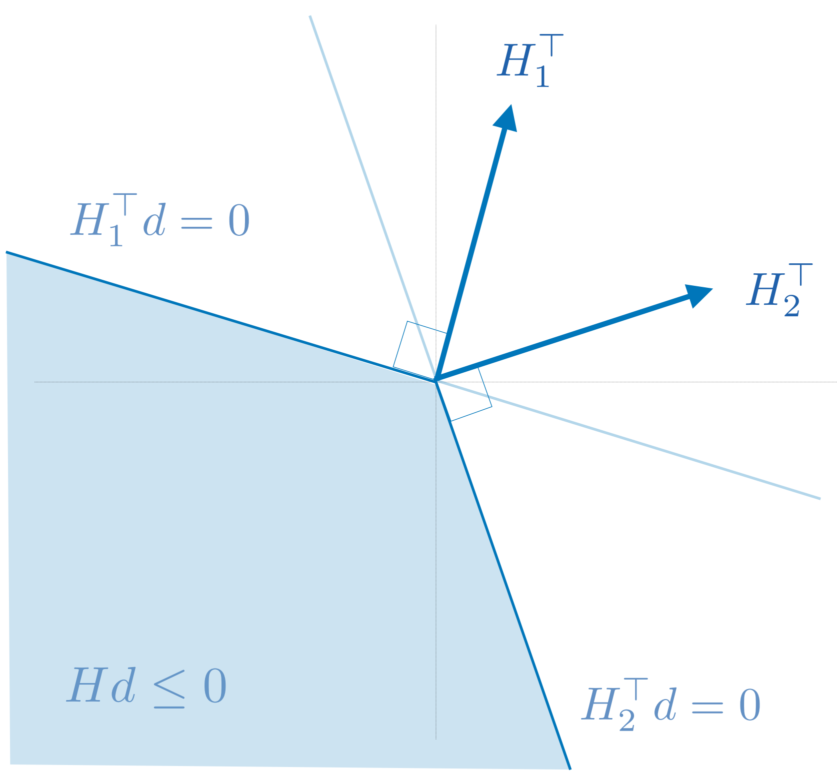
Core Intuition: Square Case

"each row of H is orthogonal to the (other) columns of H inverse"

$$HH^{-1} = \begin{bmatrix} - & H_1^\top & - \\ - & H_2^\top & - \end{bmatrix} \begin{bmatrix} | & | \\ G_1 & G_2 \\ | & | \end{bmatrix} = \begin{bmatrix} H_1^\top G_1 & H_1^\top G_2 \\ H_2^\top G_1 & H_2^\top G_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_i^\top G_j = 0 \quad i \neq j$$



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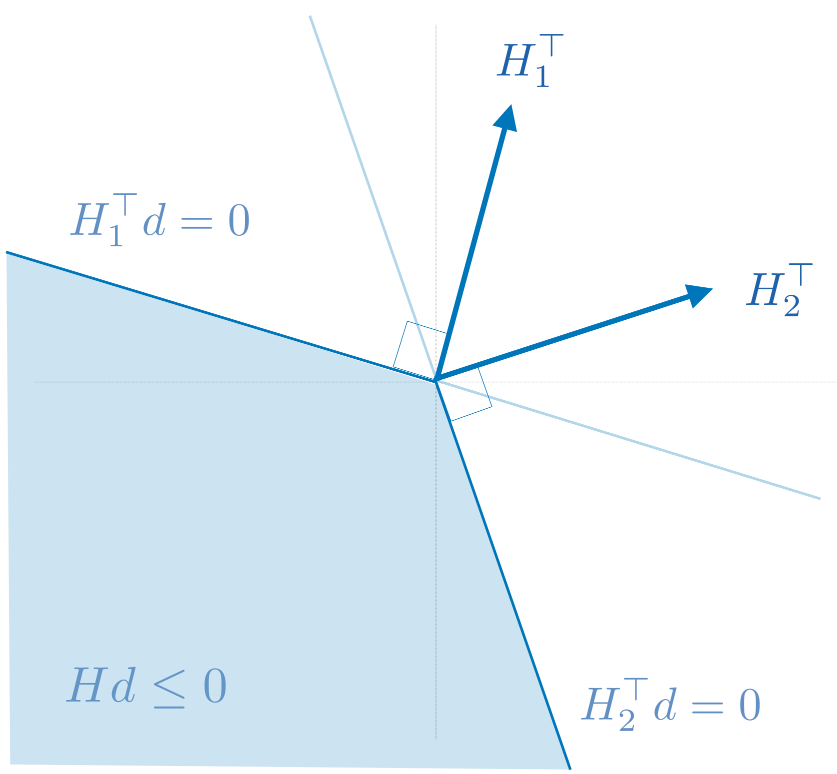

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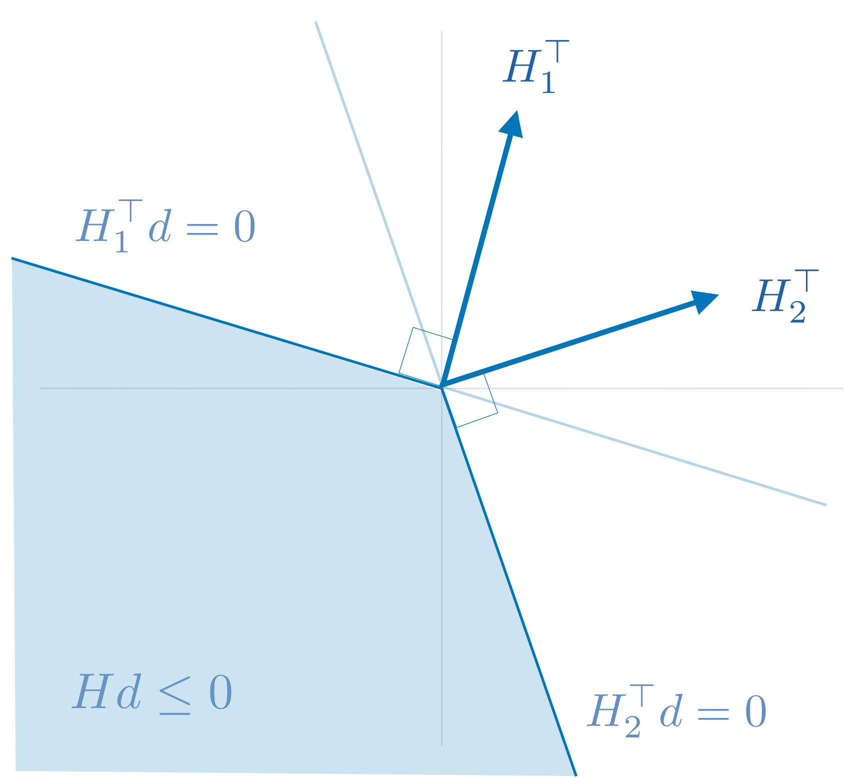

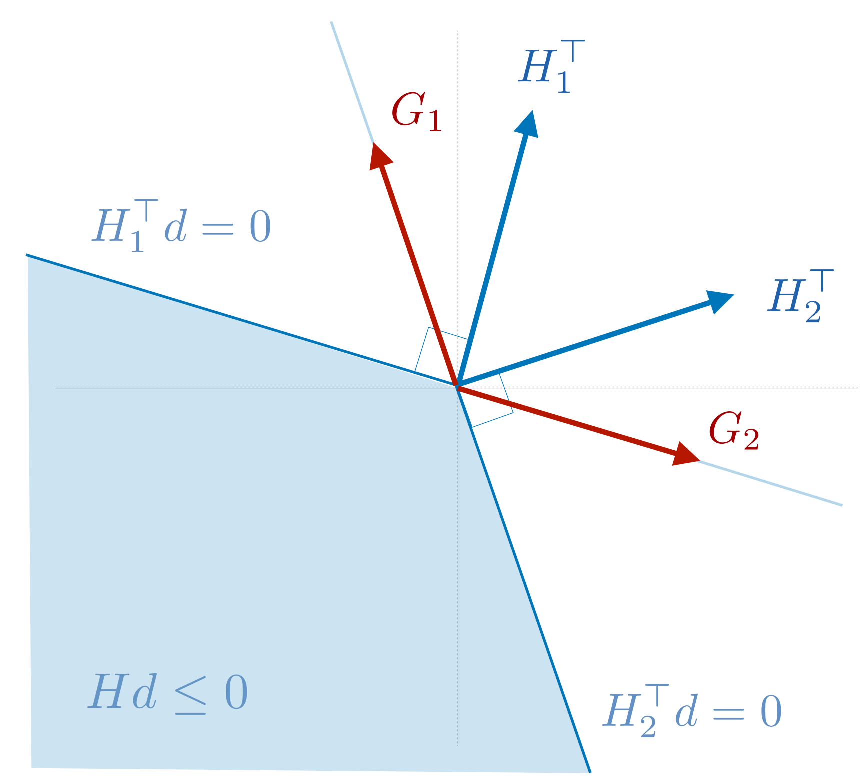

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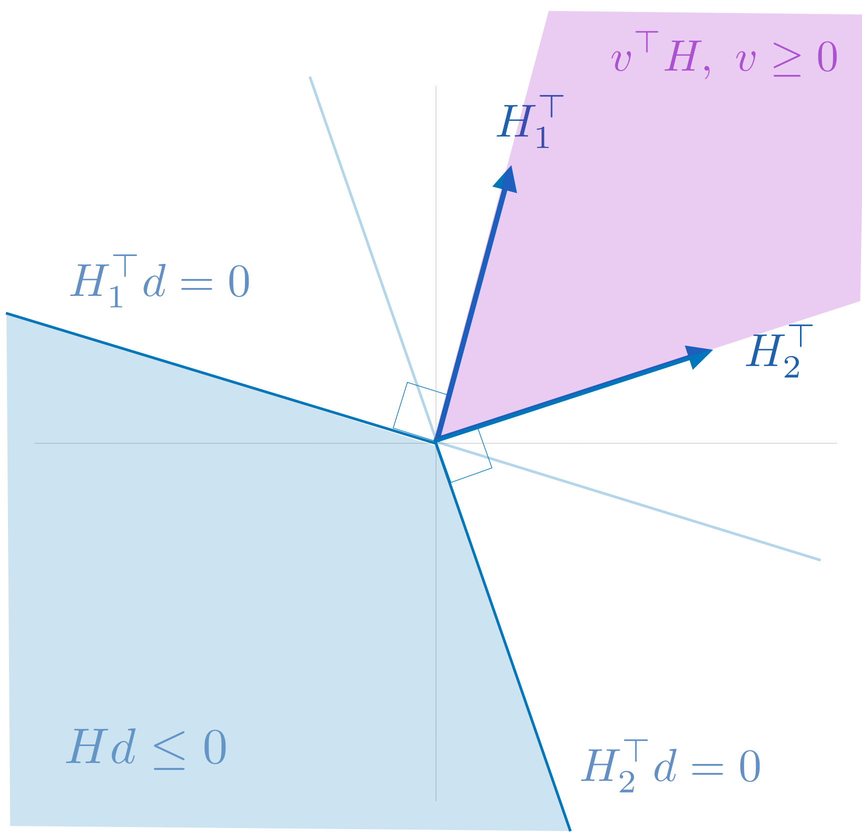

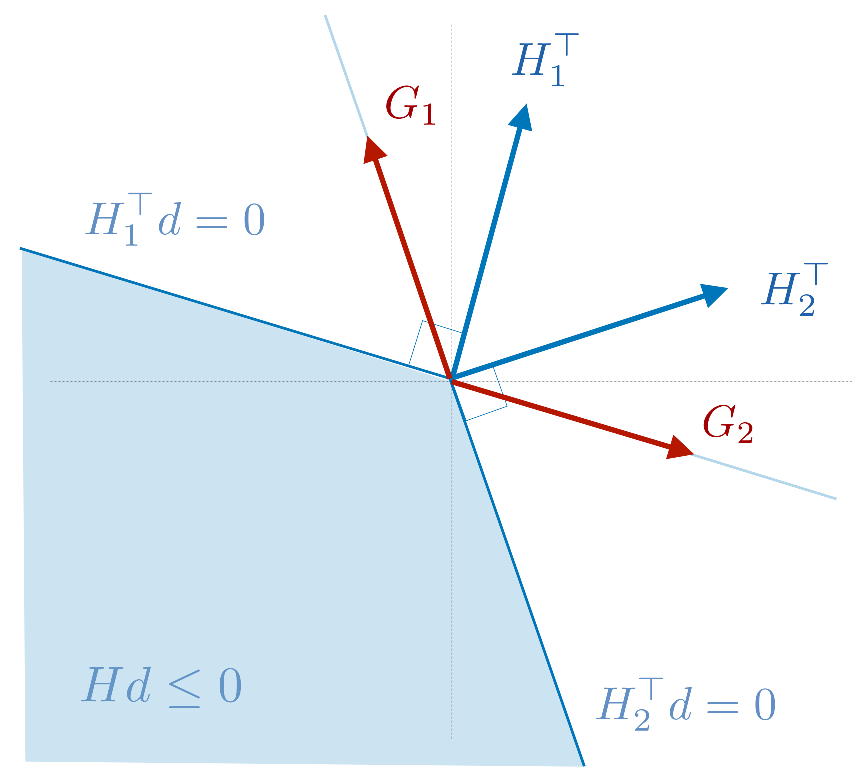

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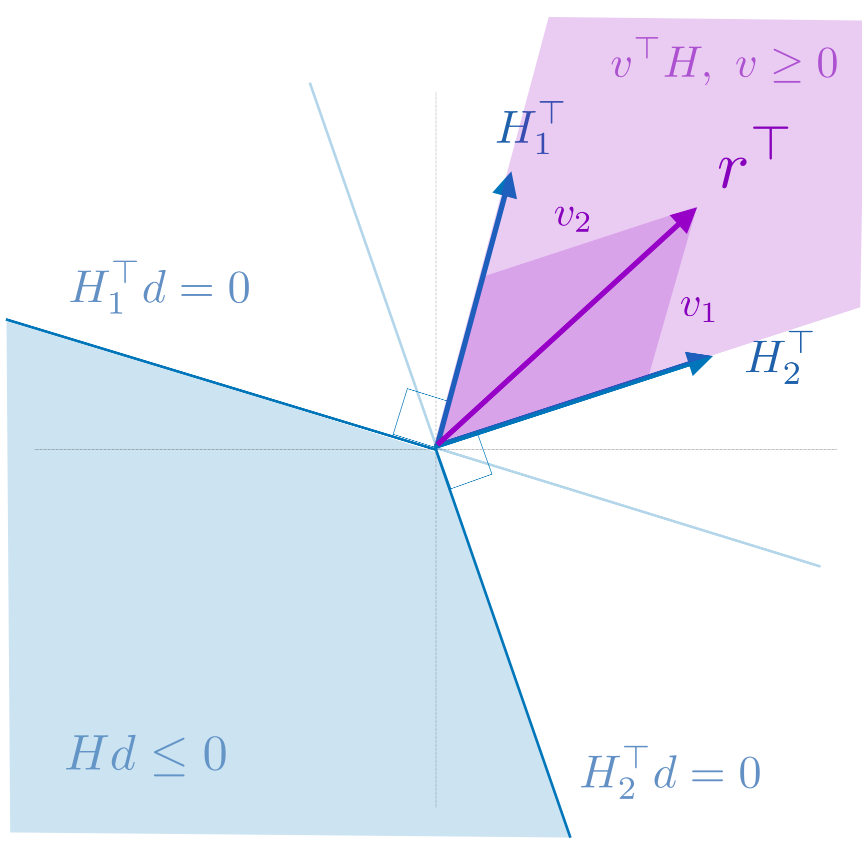

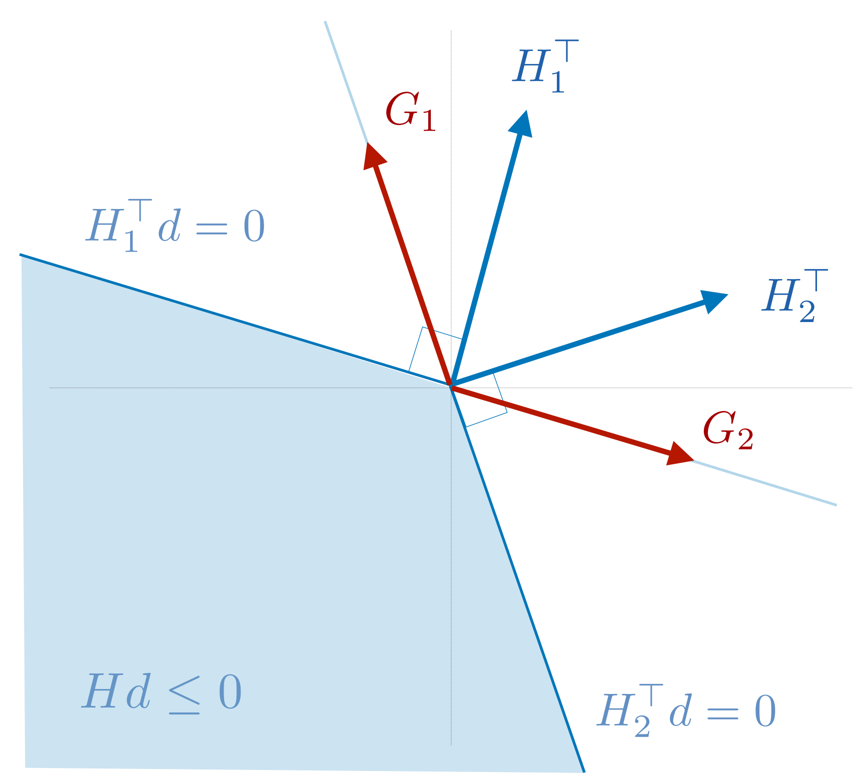

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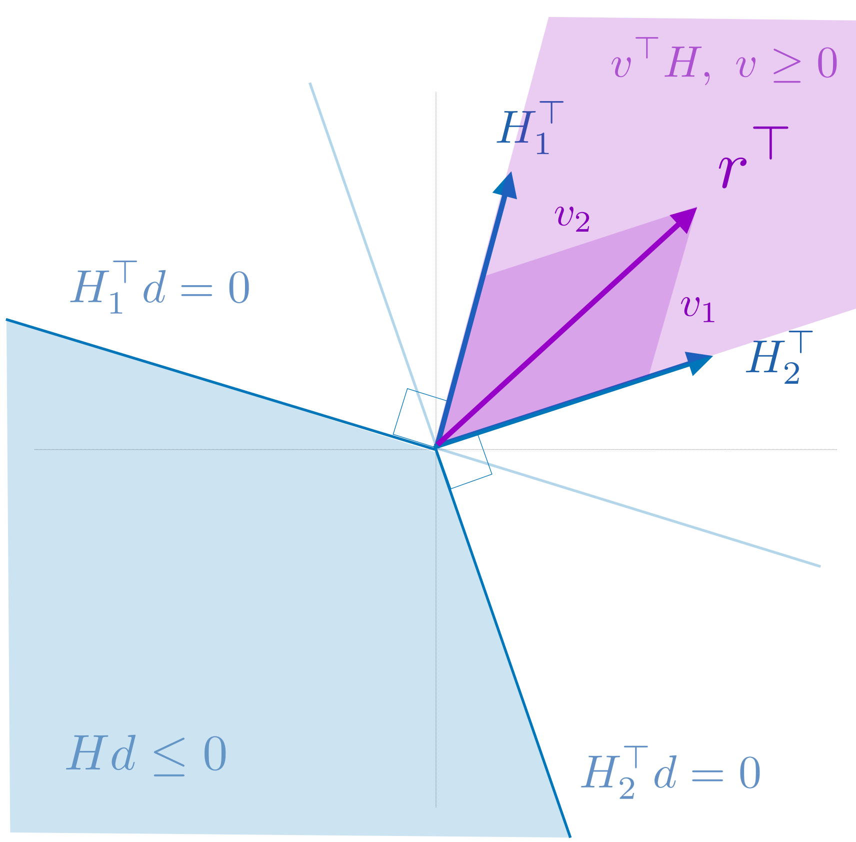
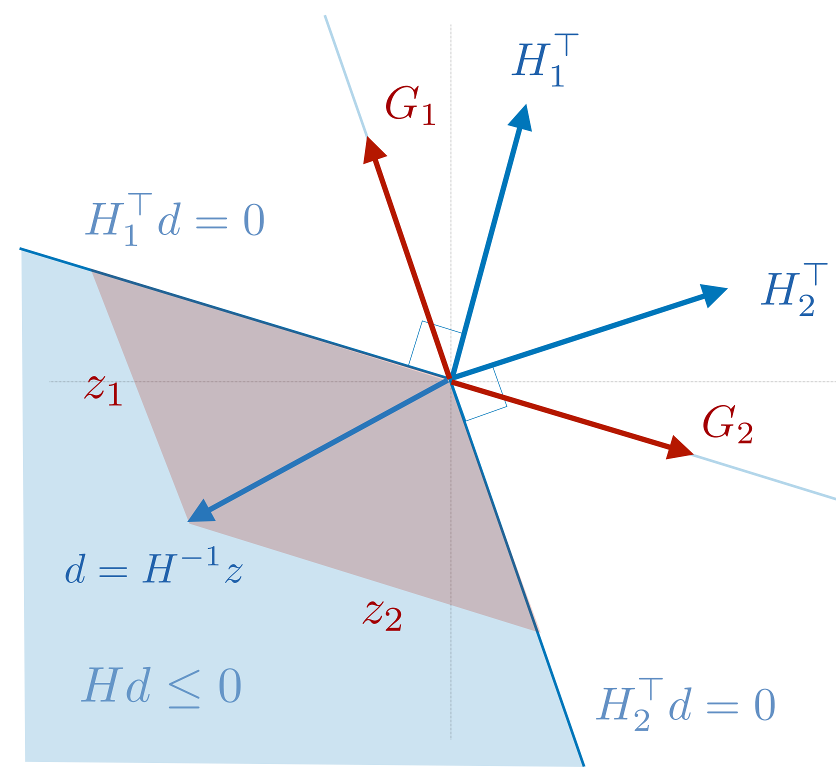
For square, invertible H $\quad \exists\, d \in \mathbb{R}^n$ s.t. $Hd \leq 0 \iff d = H^{-1}z$ for $z \leq 0$





Case 1.

True

False

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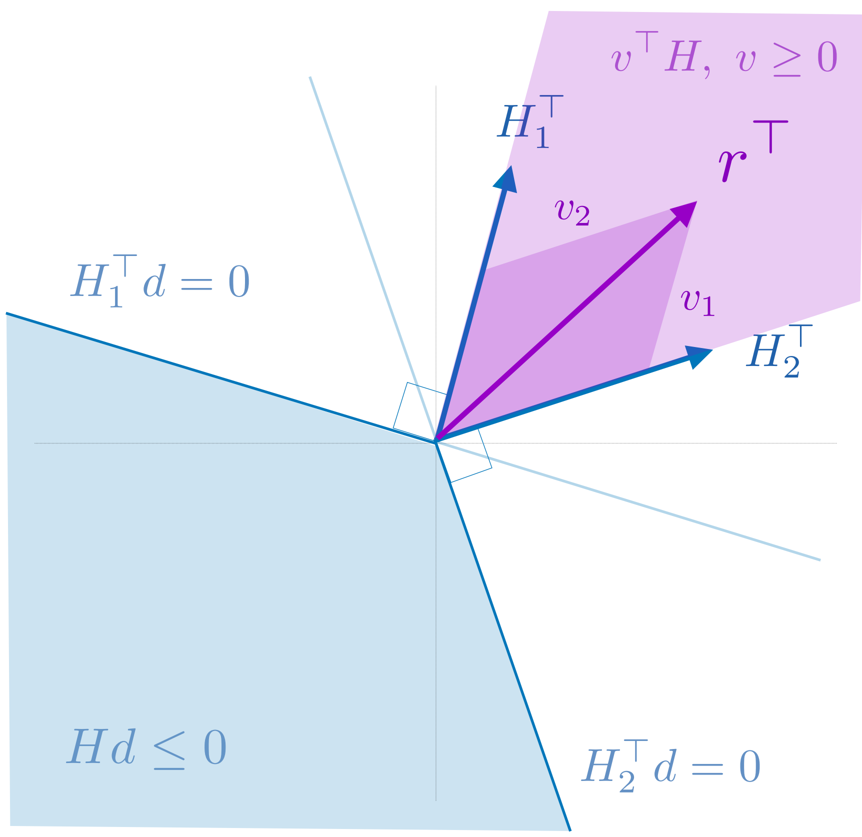
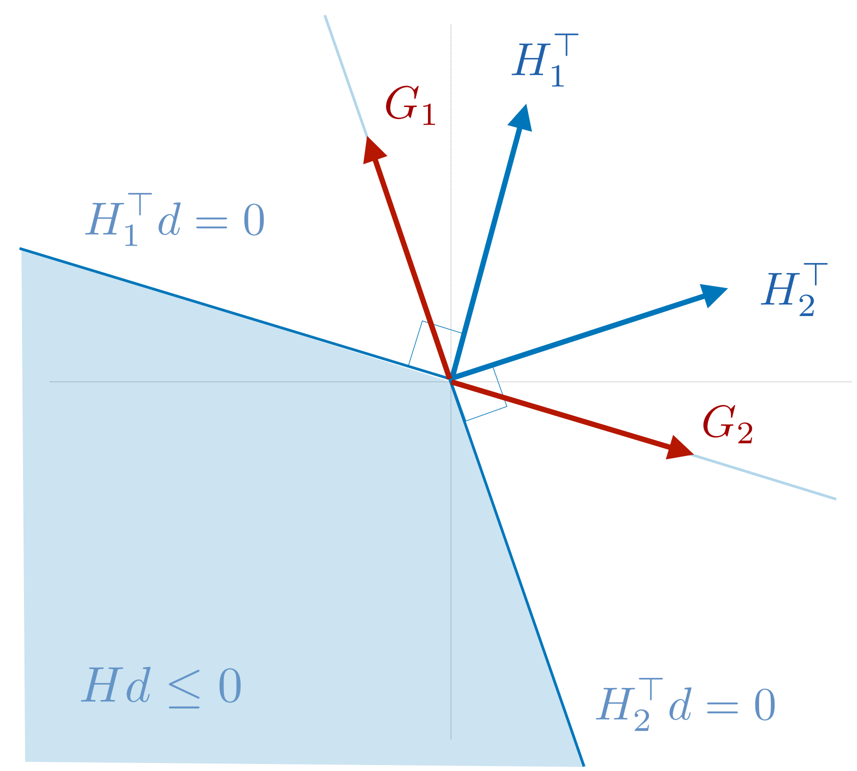




Case 1.

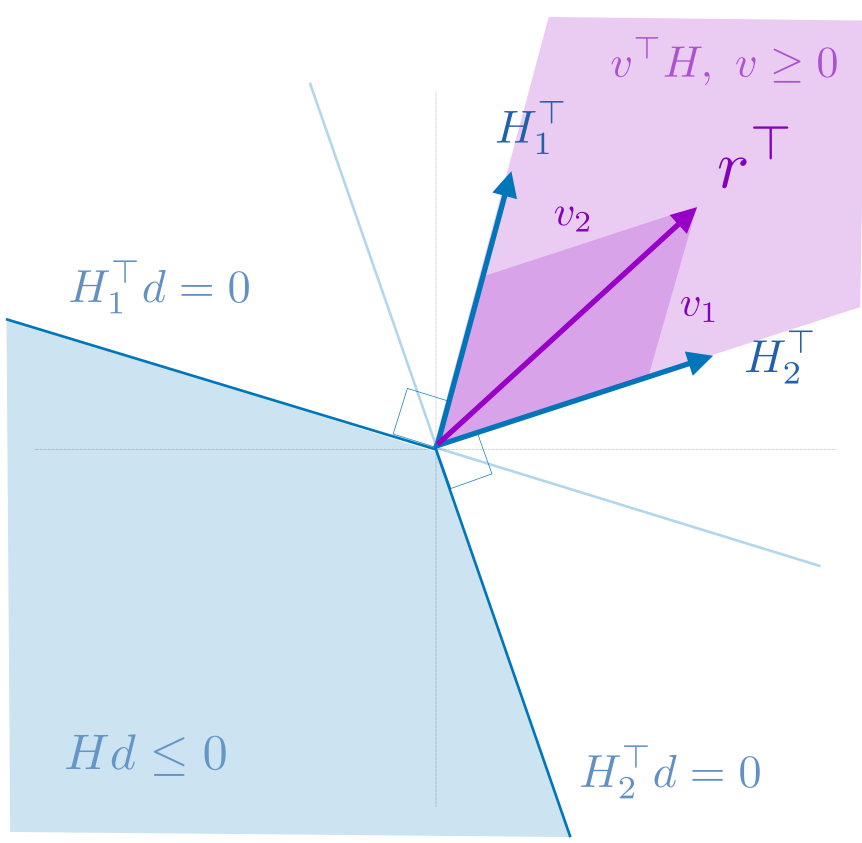
Farka's Lemma Let $H \in \mathbb{R}^{m \times n}$ and $r \in \mathbb{R}^n$.

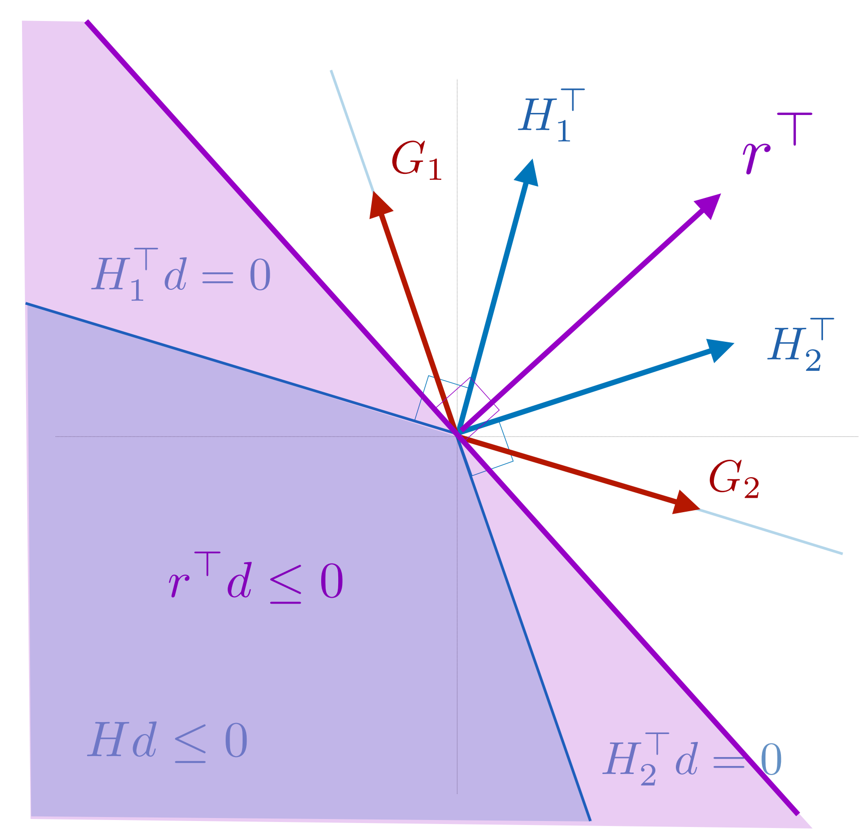
Then exactly one of the following two assertions is true.

True	1. There exists $v \in \mathbb{R}^m$ such that $v^\top H = r^\top$ and $v \geq 0$
False	2. There exists $d \in \mathbb{R}^n$ such that $Hd \leq 0$ and $r^\top d > 0$

not $\Rightarrow$

For all $d \in \mathbb{R}^n$ s.t. $Hd \leq 0$, $r^\top d \leq 0$





Case 2.

False

True

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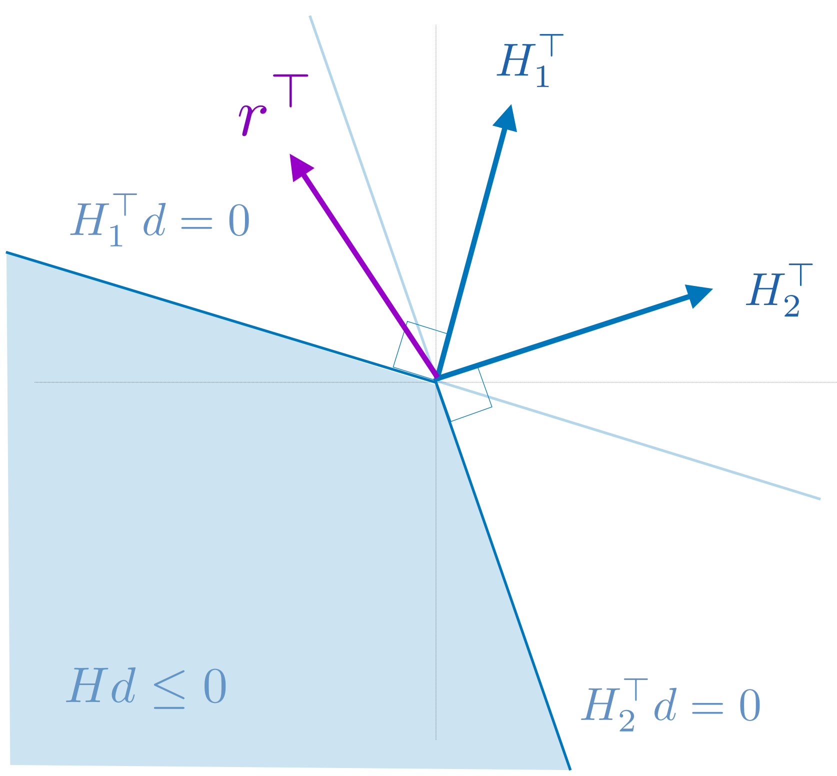

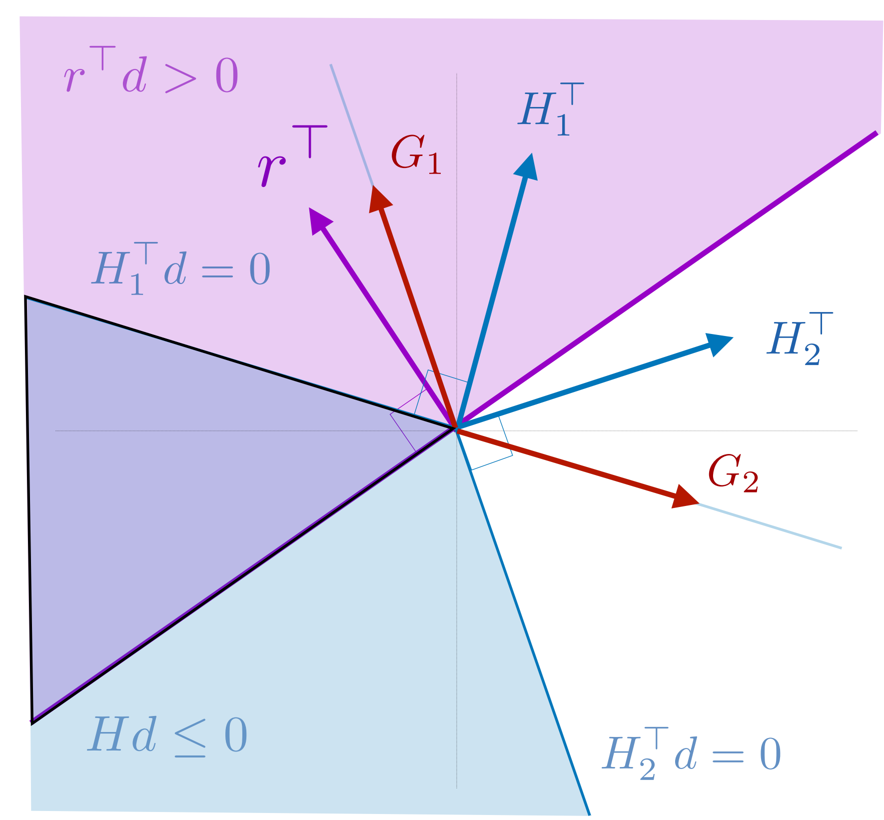

Case 2.

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Then exactly one of the following two assertions is true.

False — 1. There exists $v \in \mathbb{R}^m$ such that $v^\top H = r^\top$ and $v \geq 0$

True — 2. There exists $d \in \mathbb{R}^n$ such that $Hd \leq 0$ and $r^\top d > 0$

not $\Rightarrow$

For all $v \in \mathbb{R}^m$ s.t. $v^\top H = r^\top, v \not\geq 0$