Perturbation Analysis: KKT Conditions

Original
Stationarity
Condition

$$\frac{\partial f}{\partial x}\Big(x\Big) + \lambda^\top \frac{\partial g}{\partial x}\Big(x, \theta\Big) = 0$$

Feasibility Condition
(original constraint)

$$g(\underbrace{x}_{\text{variables}}, \underbrace{\theta}_{\text{parameters}}) = 0$$

variables parameters

(defines shape
of constraints)

Perturbation Analysis: KKT Conditions

Original **Stationarity Condition**

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0$$

Perturb Shape of Constraints

$$\theta \ \rightarrow \theta + \Delta\theta$$

Question:
How does this affect other problem variables at optimum?

Feasibility Condition
(original constraint)

$$g(\underbrace{x}_{\text{variables}}, \underbrace{\theta}_{\text{parameters}}) = 0$$

(defines shape of constraints)

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Perturb Shape of Constraints

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Question:
How does this affect other problem variables at optimum?

Perturbation Analysis

Perturbed **Feasibility Conditions**

$$g\big(x + \Delta x, \theta + \Delta\theta\big) = 0$$

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Perturbation Analysis

Perturbed **Feasibility Conditions**

$$g\big(x + \Delta x, \theta + \Delta\theta\big) = 0 \qquad \approx \qquad \boxed{g(x, \theta) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0}$$

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$$g\big(\underset{\text{variables}}{x}, \underset{\text{parameters}}{\theta}\big) = 0$$

(defines shape of constraints)

Perturb Shape of Constraints

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Perturbation Analysis

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Perturbed **Stationarity Conditions**

$$\frac{\partial f}{\partial x}\Big(x + \Delta x\Big) + \big(\lambda + \Delta\lambda\big)^\top \frac{\partial g}{\partial x}\Big(x + \Delta x, \theta + \Delta\theta\Big) = 0$$

Perturbation Analysis: KKT Conditions

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(original constraint)

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(defines shape of constraints)

Perturb Shape of Constraints

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Perturbation Analysis

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Perturbed **Stationarity Conditions**

$$\frac{\partial f}{\partial x}\Big(x + \Delta x\Big) + \big(\lambda + \Delta\lambda\big)^\top \frac{\partial g}{\partial x}\Big(x + \Delta x, \theta + \Delta\theta\Big) = 0$$

$$\boxed{\frac{\partial f}{\partial x}(x) \;+\; \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] \;= 0}$$

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Feasibility Condition
(original constraint)

$$g\big(x, \theta\big) = 0$$

variables: x — parameters: θ (defines shape of constraints)

Perturb Shape of Constraints

$$\theta \;\; \rightarrow \theta + \Delta\theta$$

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Perturbation Analysis

Perturbed **Feasibility Conditions**

$$g\big(x + \Delta x, \theta + \Delta\theta\big) = 0 \qquad \approx \qquad \boxed{g\big(x, \theta\big) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0}$$

Perturbed **Stationarity Conditions**

$$\frac{\partial f}{\partial x}\Big(x + \Delta x\Big) + \big(\lambda + \Delta\lambda\big)^\top \frac{\partial g}{\partial x}\Big(x + \Delta x, \theta + \Delta\theta\Big) = 0$$

$$\boxed{\frac{\partial f}{\partial x}(x) \; + \; \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + \; (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] \; = 0}$$

Procedure: 1. Cancel terms… 2. Solve for $\Delta x, \Delta\lambda$ as functions of $\Delta\theta.$

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Original Conditions

$$\frac{\partial f}{\partial x}\Big(x\Big) + \lambda^\top \frac{\partial g}{\partial x}\Big(x, \theta\Big) = 0 \qquad g\big(x, \theta\big) = 0$$

Perturbed Conditions

$$\frac{\partial f}{\partial x}(x) \; + \; \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] \; + \; (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right]$$

$$+ \sum_i (\lambda_i + \Delta\lambda_i) \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] \; + \; \sum_i (\lambda_i + \Delta\lambda_i) \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] \quad = 0$$

$$g\big(x, \theta\big) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

Perturbation Analysis: KKT Conditions

Original Conditions

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0 \qquad g\left(x, \theta\right) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}}(x) + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + \cancel{(\lambda} + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right]$$

$$+ \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\cancel{g(x, \theta)} + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

1. Cancel zero-order terms…

Perturbation Analysis: KKT Conditions

Original Conditions

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x,\theta\right) = 0 \qquad g(x,\theta) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}(x)} + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + (\cancel{\lambda} + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\cancel{g(x,\theta)} + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

1. Cancel zero-order terms… **2. Cancel second-order terms…**

Perturbation Analysis: KKT Conditions

Original Conditions

$$\frac{\partial f}{\partial x}\Big(x\Big) + \lambda^\top \frac{\partial g}{\partial x}\Big(x, \theta\Big) = 0 \qquad g\big(x, \theta\big) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}(x)} + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + (\cancel{\lambda} + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i}) \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i}) \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\cancel{g(x, \theta)} + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

1. Cancel zero-order terms… **2. Cancel second-order terms…**

$$\Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + \Delta\lambda^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i \lambda_i \, \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i \lambda_i \, \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

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$$\frac{\partial f}{\partial x}\Big(x\Big) + \lambda^\top \frac{\partial g}{\partial x}\Big(x,\theta\Big) = 0 \qquad g\big(x,\theta\big) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}(x)} \;+\; \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; (\cancel{\lambda} + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right]$$

$$+ \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] \;+\; \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] \;=0$$

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1. Cancel zero-order terms… **2. Cancel second-order terms…**

$$\Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; \Delta\lambda^\top \left[\frac{\partial g}{\partial x}\right] \;+\; \sum_i \lambda_i\, \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] \;+\; \sum_i \lambda_i\, \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

organize into a system of equations…

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x \partial\theta} \\ \frac{\partial g}{\partial\theta} \end{bmatrix}\Delta\theta$$

$$Q = \frac{\partial^2 f}{\partial x^2} + \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x^2}$$

$$A = \frac{\partial g}{\partial x}$$

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Original Conditions

$$\frac{\partial f}{\partial x}\big(x\big) + \lambda^\top \frac{\partial g}{\partial x}\big(x, \theta\big) = 0 \qquad g\big(x, \theta\big) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}(x)} + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + (\cancel{\lambda} + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

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1. Cancel zero-order terms… **2. Cancel second-order terms…**

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$$\frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

organize into a system of equations…

$$\begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x \partial\theta} \\ \frac{\partial g}{\partial\theta} \end{bmatrix} \Delta\theta$$

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organize into a system of equations…

$$\begin{bmatrix}\Delta x \\ \Delta\lambda\end{bmatrix} = \begin{bmatrix} Q & A^\top \\ A & 0\end{bmatrix}^{-1} \begin{bmatrix}\sum_i \lambda_i \frac{\partial^2 g_i}{\partial x\partial\theta} \\ \frac{\partial g}{\partial\theta}\end{bmatrix}\Delta\theta$$

$$Q = \frac{\partial^2 f}{\partial x^2} + \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x^2}$$

$$A = \frac{\partial g}{\partial x}$$

$$= \begin{bmatrix} Q^{-1} - Q^{-1}A^\top\left(AQ^{-1}A^\top\right)^{-1}AQ^{-1} & Q^{-1}A^\top\left(AQ^{-1}A^\top\right)^{-1} \\ \left(AQ^{-1}A^\top\right)^{-1}AQ^{-1} & -\left(AQ^{-1}A^\top\right)^{-1}\end{bmatrix} \begin{bmatrix}\sum_i \lambda_i \frac{\partial^2 g_i}{\partial x\partial\theta} \\ \frac{\partial g}{\partial\theta}\end{bmatrix}\Delta\theta$$