Hungarian (Munkres) Algorithm - Convex Relaxation

N workers

N tasks

$$\begin{bmatrix} 9 & 1 & 9 & 9 \\ 2 & 9 & 9 & 9 \\ 9 & 9 & 9 & 3 \\ 9 & 9 & 4 & 9 \end{bmatrix}$$

← **Cost matrix** C

Optimal Assignment

One task per worker

Optimal matching

Overall Cost: *Sum of costs for each assigned worker* $\mathrm{Tr}(C^T P)$ **do you see why?**

for some permutation matrix: $P \in \mathcal{P}$ ← not a convex set **do you see why?**

A permutation matrix is an identity matrix with the columns (or rows) shuffled… *ex.* $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Convex Relaxation: $\mathcal{P} \rightarrow \mathcal{X}$ Doubly stochastic matrices: *rows and columns sum to 1* $\mathcal{X}$ is convex ✔

Birkhoff-von Neumann Theorem: *"the convex hull of permutation matrices is doubly stochastic matrices* **do you see why?**

Doubly stochastic matrices: $X \in \mathcal{X}$ satisfies $X\mathbf{1} = \mathbf{1}$ $X^T\mathbf{1} = \mathbf{1}$ $X \geq 0$

Convex Relaxation

$$\min_X \ \mathrm{Tr}(C^T X)$$
$$\text{s.t.} \quad X\mathbf{1} = \mathbf{1} \quad X \geq 0$$
$$X^T\mathbf{1} = \mathbf{1}$$

Dual Program

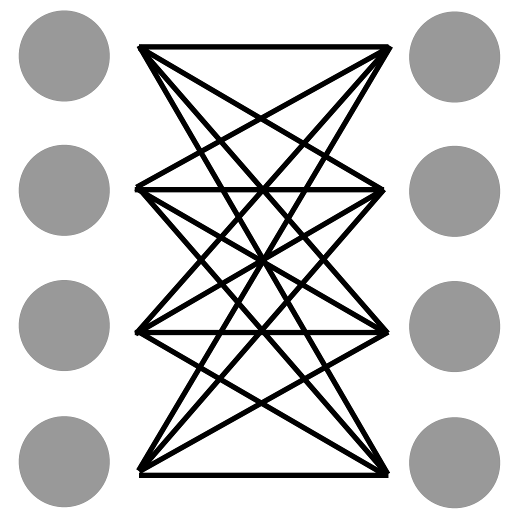
$$\max_{v,w,U} \ v^T\mathbf{1} + w^T\mathbf{1}$$
$$\text{s.t.} \quad C^T = \mathbf{1}v^T + w\mathbf{1}^T + U^T$$
$$U \geq 0$$

Note: this convex relaxation works cause linear programs are always optimized at the vertices

do you see why?

Bipartite Matching Perspective

N tasks **N workers**



…Costs are edge weights

If a worker can't perform a task… either remove the edge… or make the cost infty

Dual Variable Interpretation

v_i : *row potential - lower bound on cost of performing task* i

w_j : *column potential - lower bound on cost incurred by worker* j

U_{ij} : *Inefficiency of worker* j *performing task* i