EE578B - Convex Optimization - Winter 2021

Homework 1

Due Date: Sunday, Jan 17^{th}, 2021 at 11:59 pm

1. **Projections (PTS: 0-2)**

   (a) Compute the projection of $x = [1, 2, 3]^T$ onto $y = [1, 1, -2]^T$.

   (b) Compute the projection of $x = [1, 2, 3]^T$ onto the range of

$$Y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. **Block Matrix Computations**

   Multiply the following block matrices together. In each case give the required dimensions of the sub-blocks of B. If the dimensions are not determined by the shapes of A, then pick a dimension that works.

   (a) **(PTS: 0-2)**

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1K} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NK} \end{bmatrix}, \qquad AB =? \tag{1}$$

   where $A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{1N} \in \mathbb{R}^{m_1 \times n_N}$, $A_{M1} \in \mathbb{R}^{m_M \times n_1}$, and $A_{MN} \in \mathbb{R}^{m_M \times n_N}$.

   (b) **(PTS: 0-2)**

$$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad AB =? \tag{2}$$

   where $A_1 \in \mathbb{R}^{1 \times n}$ and $A_m \in \mathbb{R}^{1 \times n}$.

   (c) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad AB =? \tag{3}$$

   where $A_1 \in \mathbb{R}^{m \times 1}$ and $A_n \in \mathbb{R}^{m \times 1}$.

(d) **(PTS: 0-2)**

$$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad D \in \mathbb{R}^{n\times n}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad ADB = ? \tag{4}$$

where $A_1 \in \mathbb{R}^{1\times n}$, $A_m \in \mathbb{R}^{1\times n}$.

(e) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad ADB = ? \tag{5}$$

where $A_1 \in \mathbb{R}^{m\times 1}$, $A_n \in \mathbb{R}^{m\times 1}$, $d_{ij} \in \mathbb{R}$.

(f) **(PTS: 0-2)**

$$A \in \mathbb{R}^{m\times n}, \quad \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}, \qquad AB = ? \tag{6}$$

(g) **(PTS: 0-2)**

$$A = \begin{bmatrix} -A_1- \\ \vdots \\ -A_m- \end{bmatrix}, \quad B, \qquad AB = ? \tag{7}$$

where $A_1, A_m \in \mathbb{R}^{1\times n}$.

3. **Linear Transformations of Sets**

(a) **Affine Sets: (PTS: 0-4)**

Consider the affine sets for $x \in \mathbb{R}^2$.

$$\mathcal{X}_1 = \left\{x \;\middle|\; x_1 + x_2 = 1,\ x \in \mathbb{R}^2\right\}, \qquad \mathcal{X}_2 = \left\{x \;\middle|\; x_1 - x_2 = 1,\ x \in \mathbb{R}^2\right\},$$

Draw the set of points Ax for $x \in \mathcal{X}_1$ and $x \in \mathcal{X}_2$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) **Unit Balls: (PTS: 0-4)**

Consider the unit-balls defined by the 1-norm, the 2-norm, and the ∞-norm.

$$\mathcal{X}_1 = \left\{x \;\middle|\; |x|_1 \le 1,\ x \in \mathbb{R}^2\right\}, \qquad \mathcal{X}_2 = \left\{x \;\middle|\; |x|_2 \le 1,\ x \in \mathbb{R}^2\right\}, \qquad \mathcal{X}_\infty = \left\{x \;\middle|\; |x|_\infty \le 1,\ x \in \mathbb{R}^2\right\}$$

Draw the set of points Ax for $x \in \mathcal{X}_1$, $x \in \mathcal{X}_2$, and $x \in \mathcal{X}_\infty$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(c) **Convex Hulls: (PTS: 0-4)**
Consider the simplicies in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ respectively

$$\Delta_2 = \left\{x \;\middle|\; \mathbf{1}^T x = 1,\ x \geq 0,\ x \in \mathbb{R}^2\right\},$$
$$\Delta_3 = \left\{x \;\middle|\; \mathbf{1}^T x = 1,\ x \geq 0,\ x \in \mathbb{R}^3\right\},$$
$$\Delta_4 = \left\{x \;\middle|\; \mathbf{1}^T x = 1,\ x \geq 0,\ x \in \mathbb{R}^4\right\}$$

where $\mathbf{1}$ is the vector of all ones of the appropriate dimension and $\geq$ is an element-wise inequality.
Draw the set of points Ax for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ x \in \Delta_2$$
$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix},\ x \in \Delta_3$$
$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix},\ x \in \Delta_4$$

4. **Affine and Half Spaces**

Plot each of the following sets and indicate whether or not each space is a *subspace*, an *affine space*, or a *half space*.

- **(PTS: 0-2)** For $a^T = \begin{bmatrix} 1 & -1 \end{bmatrix}$.
  $\mathrm{X} = \left\{ \mathrm{x} \in R^2 \;\middle|\; a^T x = 0\right\}, \quad X = \left\{x \in R^2 \;\middle|\; a^T x = 1\right\}, \quad X = \left\{x \in R^2 \;\middle|\; a^T x \leq 1\right\},$
- **(PTS: 0-2)** For $a^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.
  $\mathrm{X} = \left\{ \mathrm{x} \in R^2 \;\middle|\; a^T x = 0\right\}, \quad X = \left\{x \in R^2 \;\middle|\; a^T x = 1\right\}, \quad X = \left\{x \in R^2 \;\middle|\; a^T x \leq 1\right\},$
- **(PTS: 0-2)**
  For $\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  $\mathrm{X} = \left\{ \mathrm{x} \in R^2 \;\middle|\; Ax = 0\right\}, \quad X = \left\{x \in R^2 \;\middle|\; Ax = b\right\}, \quad X = \left\{x \in R^2 \;\middle|\; Ax \leq b\right\},$

5. **Coordinates**

Let y be the coordinates of a vector with respect to the standard basis in $\mathbb{R}^2$. In each case below consider a different basis for $\mathbb{R}^2$ given by the columns of the matrix T. Compute the coordinates of the vector y with respect to the new basis 1) by graphically drawing the columns of T and y as vectors and 2) by inverting the matrix T, ie. by solving $y = Tx$.

(a) **(PTS: 0-2)** Solve graphically and then by inverting T.

$$y = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) **(PTS: 0-2)** Solve graphically and then by inverting T.

$$y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

(c) **(PTS: 0-2)** Solve graphically and then by inverting T.

$$y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

6. **Finding a Nullspace Basis**

(a) **Basis Derivation**

Consider a fat matrix $A \in \mathbb{R}^{m \times n}$ $(m < n)$ that is partitioned as $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ with $A_1 \in \mathbb{R}^{m \times m}$ invertible. Show that the columns of $B \in \mathbb{R}^{n \times n-m}$

$$B = \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix}$$

form a basis for the nullspace of A, $\mathcal{N}(A)$ by performing the following two steps.

i. **(PTS: 0-2)** Show that any vector $v \in \mathcal{N}(A)$ can be written as $v = Bw$ for some $w \in \mathbb{R}^{n-m}$, ie. v is linear combination of the columns of B (the columns of B span the nullspace).

ii. **(PTS: 0-2)** Show that the columns of B are linearly independent.

(b) **Computation**

For the following matrices explicitly compute a basis for their nullspaces BY HAND, ie. do not use computational software.

i. **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

ii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{bmatrix}$$