EE578B - Convex Optimization - Fall 2021

Homework 2

<u>**Due Date**</u>: Sunday, Jan 24^{th}, 2021 at 11:59 pm

1. **Matrix Rank**

   The column rank of a matrix is the number of linearly independent columns. The row rank of a matrix is the number of linearly independent row.

   (a) **(PTS: 0-2)** Show that the row rank is less than or equal to the column rank.

   (b) **(PTS: 0-2)** Show that the col rank is less than or equal to the row rank.

2. **Grammian Rank**

   (PTS: 0-2) Show that $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(A^T A) = \text{rank}(AA^T)$

3. **Basis for Domain from Nullspace of A and Range of A^T**

   Consider $A \in \mathbb{R}^{m \times n}$ with $m < n$ and full row rank and a matrix $N \in \mathbb{R}^{n \times n-m}$ with full column rank whose columns span the nullspace of A. Suppose we write a vector $x \in \mathbb{R}^n$ as a linear combination of the rows of A and the columns of N, ie.

   $$x = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

   for $x_1' \in \mathbb{R}^m$ and $x_2' \in \mathbb{R}^{n-m}$

   (a) **(PTS: 0-2)** Symbollically compute $\begin{bmatrix} A^T & N \end{bmatrix}^{-1}$.

   Hint: Start by checking if $\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} A^T & N \end{bmatrix}^T$...

   (b) **(PTS: 0-2)** Solve for x_1' and x_2' given A,N, and x.

4. **Range and Nullspace**

   Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent the range and nullspace of A (and similarly let $\mathcal{R}(A^T)$ and $\mathcal{N}(A^T)$ be the range and nullspace of A^T).

   (a) **(PTS: 0-2)** Suppose $y \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$. Show that $x \perp y$, ie. $x^T y = 0$.

   (b) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{5 \times 10}$. Suppose A has only 3 linearly independent columns (the other 7 are linearly dependent on the first 3). What is the dimension of $\mathcal{R}(A)$? What is the dimension of $\mathcal{N}(A^T)$ What is the dimension of $\mathcal{N}(A)$? What is the dimension of $\mathcal{R}(A^T)$? (You can state your answers without proof.)

5. **Fundamental Theorem of Linear Algebra Pictures**

For each of the following matrices draw a picture of the domain (either $\mathbb{R}^2$ or $\mathbb{R}^3$) labeling $\mathcal{R}(A^T)$ and $\mathcal{N}(A)$ and a picture of the co-domain (either $\mathbb{R}^2$ or $\mathbb{R}^3$) labeling the $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$.

(PTS: 0-2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **(PTS: 0-2)** $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(PTS: 0-2) $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$ **(PTS: 0-2)** $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

6. **Representations of Affine Sets**

Consider two representations of the same affine set.

$$\textbf{Representation 1:} \quad \mathcal{X} = \left\{ x \in \mathbb{R}^n \mid Ax = b, \right\}$$
$$\textbf{Representation 2:} \quad \mathcal{X} = \left\{ x \in \mathbb{R}^n \mid x = Nz + d,\ z \in \mathbb{R}^{n-m} \right\}$$

where

- $A \in \mathbb{R}^{m \times n}$ is fat $(m < n)$ and full row rank $(\text{rk}(A) = m)$ and $b \in \mathbb{R}^m$
- $N \in \mathbb{R}^{n \times (n-m)}$ is tall, $\mathcal{R}(N) = \mathcal{N}(A)$, and $d \in \mathbb{R}^n$

(a) For each $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, compute $N \in \mathbb{R}^{n \times (n-m)}$ and $d \in \mathbb{R}^n$.
Note: There are many possible N's and d's that work.

(PTS: 0-2) $A = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}, \quad b = 1,$

(PTS: 0-2) $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$

(PTS: 0-2) $A = \begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) For each $N \in \mathbb{R}^{n \times (n-m)}$ and $d \in \mathbb{R}^n$, compute $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Note 1: There are many possible A's and b's that work.
Note 2: Note that $N^T A^T = 0$, ie. the rows of A should form a basis for the nullspace of N^T.

(PTS: 0-2) $$N = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(PTS: 0-2) $$N = \begin{bmatrix} 1 & 0 \\ -2 & 0 \\ 1 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(PTS: 0-2) $$N = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

7. **Equivalent representations of spaces**

- **(PTS: 0-2)** For $A \in \mathbb{R}^{m \times n}$ and $U \in \mathbb{R}^{m \times m}$, invertible, show that $\mathcal{N}(A) = \mathcal{N}(UA)$. OPTIONAL: Comment on how the rows of A relate to the rows of UA.
- **(PTS: 0-2)** For $A \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{n \times n}$, invertible, show that $\mathcal{R}(A) = \mathcal{R}(AV)$. OPTIONAL: Comment on how the columns of A relate to the columns of AV.

8. **Vector Derivatives**

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. Compute $\frac{\partial f}{\partial x}$ for the following functions:

- **(PTS: 0-2)**

$$f(x) = x_1^4 + 3x_1x_2^2 + e^{x_2} + \frac{1}{x_1x_2}$$

- **(PTS: 0-2)**

$$f(x) = \begin{bmatrix} \beta x_1 + \alpha x_2 \\ \beta(x_1 + x_2) \\ \alpha^2 x_1 + \beta x_2 \\ \beta x_1 + \frac{1}{\alpha} x_2 \end{bmatrix}$$

for $\alpha, \beta \in \mathbb{R}$

- **(PTS: 0-2)**

$$f(x) = \begin{bmatrix} e^{x^TQx} \\ (x^TQx)^{-1} \end{bmatrix}$$

for some $Q = Q^T \in \mathbb{R}^{2 \times 2}$.