EE578B - Convex Optimization - Winter 2021

Homework 3

Due Date: Sunday, Jan 31^{st}, 2020 at 11:59 pm

1. **Quadratic Functions**

   Consider the quadratic function

   $$f(x) = \frac{1}{2}x^TQx + c^Tx$$

   - **(PTS:0-2)** Rewrite $f(x)$ in the form

     $$f(x) = \frac{1}{2}(x - x_c)^TQ(x - x_c) + \text{CONST}$$

   - **(PTS:0-2)** Compute the derivative of both forms of $f(x)$ and show that they are the same.

2. **Minimum Norm Problem**

   Consider the following optimization problem for finding the minimum norm solution to a linear system of equations

   $$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) = \frac{1}{2}|x|_2^2 = \frac{1}{2}x^Tx \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

   for $A \in \mathbb{R}^{m \times n}$ full row rank with $m < n$ and $b \in \mathbb{R}^m$. The optimality conditions for this optimization problem are given by

   $$\frac{\partial f}{\partial x}^T = x = -A^Tv \quad (1)$$
   $$Ax = b \quad (2)$$

   with dual variable $v \in \mathbb{R}^m$. Let x^*, v^* refer to x and v at optimum.

   - **(PTS:0-2)** Solve for v^* in terms of b. (Hint: start by left multiplying (1) by A and substituting in $Ax = b$).
   - **(PTS:0-2)** Solve for x^* in terms of b.
   - **(PTS:0-2)** Let the columns of $N \in \mathbb{R}^{n \times (n-m)}$ form a basis for the nullspace of A. Compute $z_1^* \in \mathbb{R}^m$ and $z_2^* \in \mathbb{R}^{n-m}$ such that

     $$x^* = \underbrace{\begin{bmatrix} A^T & N \end{bmatrix}}_{P} \begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$$

     ie. write x^* in terms of the coordinates with respect to the columns of P. Interpret z_1^* and z_2^* in terms of projections of x^* onto $\mathcal{R}(A^T)$ and $\mathcal{R}(N)$. How does z_1^* relate to v^*? Explain the value of z_2^* intuitively.

- **(PTS:0-2)** Consider the above problem for $A = [1\ 1]$ and $b = 1$. Draw a picture of the optimization space labeling

$$\left\{x \in \mathbb{R}^2 \;\middle|\; Ax = b\right\}, \quad \mathcal{R}(A^T), \quad x^*, \quad -A^T v^*, \quad \text{level sets of } f(x), \quad \left.\frac{\partial f}{\partial x}\right|_{x^*}$$

3. **Spherical Level Sets**

Now consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) = \tfrac{1}{2}|x|_2^2 + c^T x = \tfrac{1}{2}x^T x + c^T x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

for $A \in \mathbb{R}^{m \times n}$ full row rank with $m < n$ and $b \in \mathbb{R}^m$. The optimality conditions are given by

$$\frac{\partial f}{\partial x}^T = x + c = -A^T v \tag{3}$$

$$Ax = b \tag{4}$$

with dual variable $v \in \mathbb{R}^m$. Let x^*, v^* refer to x and v at optimum.

- **(PTS:0-2)** Solve for v^* in terms of b. (Hint: start by left multiplying (3) by A and substituting in $Ax = b$). Using the solution for v^* solve for x^*.
- **(PTS:0-2)** Write the objective function in the form from Problem 1.

$$\tfrac{1}{2}x^T x + c^T x = \tfrac{1}{2}z^T z + \text{CONST}$$

for $z = x - \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Rewrite the constraint in terms of z, ie. compute $\bar{b}$ such that

$$Ax = b \qquad \Rightarrow \qquad Az = \bar{b}$$

- **(PTS:0-2)** Show that

$$z^* = x^* - \bar{x} = A^T(AA^T)^{-1}\bar{b}$$

- **(PTS:0-2)** Consider the above problem for $A = [1\ 1]$ and $b = 1$ and $c^T = [-1\ 1]$ Draw a picture of the optimization space labeling

$$\left\{x \in \mathbb{R}^2 \;\middle|\; Ax = b\right\}, \quad \mathcal{R}(A^T), \quad \text{level sets of } f(x), \quad \bar{x}$$

- **(PTS:0-2)** Also label

$$x^*, \quad z^* = x^* - \bar{x}, \quad -A^T v^*, \quad \left.\frac{\partial f}{\partial x}\right|_{x^*},$$

and interpret the location of x^* relative to $\bar{x}$

4. **Ellipsoidal Level Sets**

Now consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) = \tfrac{1}{2}x^T Q x + c^T x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

for $A \in \mathbb{R}^{m \times n}$ full row rank with $m < n$ and $b \in \mathbb{R}^m$. The optimality conditions are given by

$$\frac{\partial f}{\partial x}^T = Qx + c = -A^T v \tag{5}$$

$$Ax = b \tag{6}$$

with dual variable $v \in \mathbb{R}^m$. Let x^*, v^* refer to x and v at optimum.

- **(PTS:0-2)** Solve for v^* in terms of b. (Hint: start by left multiplying (5) by AQ^{-1} and substituting in $Ax = b$). Using the solution for v^* solve for x^*.
- **(PTS:0-2)** Rewrite the optimization problem using the coordinate transformation $x = Q^{-\frac{1}{2}}x'$ (equivalently $x' = Q^{\frac{1}{2}}x$).
- **(PTS:0-2)** Re-solve the optimization problem using the form from Problem 3 in the x' coordinates and show that you get the same solution as your solution above in the x coordinates.
- **(PTS:0-2)** Consider the above problem for

  $$Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = [1\ 1], \quad b = 1, \quad c^T = [-1\ 1]$$

  Compute the center of the ellipsoidal level sets $\bar{x}$.
- **(PTS:0-2)** Draw a picture of the optimization space labeling

  $$\left\{x \in \mathbb{R}^2 \;\middle|\; Ax = b\right\}, \quad \mathcal{R}(A^T), \quad \bar{x}, \quad \text{level sets of } f(x), \quad x^*, \quad -A^T v^*, \quad \left.\frac{\partial f}{\partial x}\right|_{x^*}$$