# EE578B - Convex Optimization - Winter 2021

# Homework 4

**Due Date**: Sunday, Feb $7^{th}$, 2021 at 11:59 pm

1. **Constraint Reformulations**

   Consider the optimization problem

$$\begin{aligned} \min_x \quad & f(x) = \tfrac{1}{2}x^T Q x + c^T x \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned}$$

   where $\mathcal{X}$ is expressed in two forms.

$$\begin{aligned} \text{Form 1:} \quad & \mathcal{X} = \left\{ x \in \mathbb{R}^n \;\middle|\; Ax = b \right\} \\ \text{Form 2:} \quad & \mathcal{X} = \left\{ x \in \mathbb{R}^n \;\middle|\; x = Nz + x^0,\ z \in \mathbb{R}^{n-m} \right\} \end{aligned}$$

   for $A \in \mathbb{R}^{m \times n}$ full row rank, $b \in \mathbb{R}^m$, $N \in \mathbb{R}^{n-m}$ where the columns of $N$ form a basis for $\mathcal{N}(A)$ and $x^0$ satisfies $Ax^0 = b$.

   - **(PTS:0-2)** Solve the optimization problem using the optimality conditions

$$\tfrac{\partial f}{\partial x} = v^T A, \quad Ax = b$$

     for Lagrange multiplier $v \in \mathbb{R}^n$. Solve for $x$ and $v$.
   - **(PTS:0-2)** Solve the optimization problem using the optimality conditions

$$\left[\tfrac{\partial f}{\partial x} \ \tfrac{\partial f}{\partial z}\right] = \tau^T [I \ \ -N], \quad x = Nz + x^0$$

     for Lagrange multiplier $\tau \in \mathbb{R}^n$. Solve for $x$,$z$, and $\tau$.
   - **(PTS:0-2)** Use Form 2 to turn the optimization problem into an unconstrained optimization problem and solve for $z$.
   - **(PTS:0-2)** Show that the answers for all three optimizaton problems are the same. How do $x$ and $z$ relate to each other? How do $\tau$ and $v$ relate to each other?

2. **Inequality constraint reformulations**

   Consider the affine equality and inequality constraints of the form

$$Ax = b, \quad Cx \geq d$$

   where $x \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \qquad b = 1$$

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \qquad d = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- **(PTS: 0-2)** Plot the set of $x$'s that satisfy these constraints.
- **(PTS: 0-2)** Represent the set $Ax = b$ in the form $x = Nz + x^0$ where the columns of $N$ are a basis for the nullspace of $A$. Given $N$ and $x^0$, $Cx \geq d$ implies inequality constraints on $z$. Write down these constraints, ie. find $E$ and $h$ such that $Ez \geq h$. If you chose a different $N$ and/or $x^0$ would these constraints be the same?
- **(PTS: 0-2)** Plot the set of $z$'s such that $Ez \geq h$.

3. **Inequality Constraints and Positive Matrices (EXTRA CREDIT) (PTS: 0-2)**

   The inequality constraint $Cx \geq d$ for $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$ can be rewritten as

$$Cx \geq d \qquad \Rightarrow \qquad Cx = d + s, \quad s \geq 0$$

   using a slack variable $s \in \mathbb{R}^p$. How does the constrained set change if $s \geq 0$ is replaced with the constraint $Us \geq 0$ for some matrix $U$ with positive elements. Suggestion: experiment with an example where $C \in \mathbb{R}^{2\times 2}$.

4. **Complementary Slackness**

   For $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^n$

   - **(PTS: 0-2)** Show that

$$\mu_i x_i = 0, \; x_i \geq 0, \; \mu_i \geq 0, \qquad \Longleftrightarrow \qquad \mu_i = 0 \;\; \text{OR} \;\; x_i = 0$$

   - **(PTS: 0-2)** Show that if $x_i \geq 0$ and $\mu_i \geq 0$ for all $i$ then

$$\mu_i x_i = 0, \; \forall i \qquad \Longleftrightarrow \qquad \mu^T x = 0$$

5. **Lagrangians**

   Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x) = \tfrac{1}{2} x^T x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

   where

$$A = \begin{bmatrix} - & \bar{A}_1^T & - \\ - & \bar{A}_2^T & - \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix} \tag{1}$$

   - **(PTS:0-2)** Write the Lagrangian for the optimization problem $\mathcal{L}(x, v)$ with a dual variable $v \in \mathbb{R}^2$. (You can leave your answer in terms of $A$ and $b$.)
   - **(PTS:0-2)** Plot each of the following surfaces $h(x)$ in the $[x_1 \; x_2 \; h(x)]^T$ space. For the expressions that include $v_1$ and $v_2$, hold $v_1$ and $v_2$ fixed for each plot.

$$h(x) = f(x), \quad h(x) = v_1(\bar{A}_1^T x - b_1), \quad h(x) = v_2(\bar{A}_2^T x - b_2), \quad h(x) = \mathcal{L}(x, v)$$

     Now vary the values of $v_1$ and $v_2$ over the range $[-2, 2]$. How does each surface change?

- **(PTS:0-2)** Now replace $Ax = b$ with $Ax \geq b$. Does the Lagrangian change? What new constraints do we have to impose on the dual variables $v_1$ and $v_2$? How does this affect how the surfaces

$$h(x) = v_1(\bar{A}_1^T x - b_1), \quad h(x) = v_2(\bar{A}_2^T x - b_2), \quad h(x) = \mathcal{L}(x, v)$$

  change with $v_1$ and $v_2$?