EE578B - Convex Optimization - Winter 2021

Homework 5

Due Date: Wednesday, Feb 17^{th}, 2021 at 11:59 pm

1. **Linear Program Duality**

   Consider the linear program

$$p^* = \min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b, \; Cx \geq d$$

   for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{p \times n}$, $d \in \mathbb{R}^p$.

   (a) **(PTS: 0-2)** Write the linear program in it's game form

$$p^* = \min_{x \in \mathbb{R}^n} \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \mathcal{L}(x, v, w)$$

   for dual variables $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^p_+$

   (b) **(PTS: 0-2)** The game form of the dual problem is given by swapping the minimum and maximum

$$d^* = \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \min_{x \in \mathbb{R}^n} \mathcal{L}(x, v, w)$$

   How does p^* relate to d^*?

   (c) **(PTS: 0-2)** Now suppose x is chosen to solve $\min_x \; \mathcal{L}(x, v, w)$. What constraints does this imply on v and w? (Hint: use the condition $\frac{\partial \mathcal{L}}{\partial x} = 0$ to compute the constraints.).

   (d) **(PTS: 0-2)** Replace the $\min_x \; \mathcal{L}(x, v, w)$ with the constraints computed in the previous part and the appropriate objective function of v and w, $\ell(v, w)$. ie. write the dual problem in the form

$$\max_{v \in \mathbb{R}^m, \; w \in \mathbb{R}^p_+} \quad \ell(v, w)$$
$$\text{s.t.} \quad g(v, w) = 0,$$
$$h(v, w) \geq 0$$

   (e) **(PTS: 0-2)** For the following matrices, solve both the primal and dual versions of the linear program using `cvx` (in Matlab) or `cvxpy` in Python for $x \in \mathbb{R}^5$.

$$c^T = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} I_{5\times 5} \\ C' \end{bmatrix}, \quad C' = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ -0.5 \end{bmatrix}$$

(f) **(PTS: 0-2)** Show numerically that the optimal dual variables for the constraints of the primal problem are the same as the optimizers of the dual problem. Show also that the optimizers of the primal problem are the optimal dual variables for the constraints of the dual problem.

2. **Quadratic Program Duality**

Consider the quadratic program

$$\begin{aligned} p^* = \max_x \quad & \tfrac{1}{2} x^T Q x + r^T x \\ \text{s.t.} \quad & Ax = b,\ Cx \geq d \end{aligned}$$

for $Q = Q^T \prec 0$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{p\times n}$, $d \in \mathbb{R}^p$.

(a) **(PTS: 0-2)** Write the quadratic program in it's game form

$$p^* = \max_{x\in\mathbb{R}^n} \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \mathcal{L}(x, v, w)$$

(b) **(PTS: 0-2)** The game form of the dual problem is given by swapping the minimum and maximum

$$d^* = \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \max_{x\in\mathbb{R}^n} \mathcal{L}(x, v, w)$$

How does p^* relate to d^*?

(c) **(PTS: 0-2)** Now suppose x is chosen to solve $\max_x \mathcal{L}(x, v, w)$. What constraints does this imply on v and w? (Hint: use the condition $\frac{\partial \mathcal{L}}{\partial x} = 0$ to compute the constraints.)

(d) **(PTS: 0-2)** Replace the $\max_x \mathcal{L}(x, v, w)$ with the constraints computed in the previous part and the appropriate objective function of the dual variables to write the dual problem.

(e) **(PTS: 0-2)**
For the following matrices, solve both the primal and dual versions of the linear program using `cvx` (in Matlab) or `cvxpy` in Python for $x \in \mathbb{R}^5$.

$$Q = -\text{diag}\left(\begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}\right), \quad r^T = \begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix},$$

$$A = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 1 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}, \quad b = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$C = \begin{bmatrix}I_{5\times 5}\\ C'\end{bmatrix}, \quad C' = \begin{bmatrix}-1 & -1 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & -1\end{bmatrix}, \quad d = \begin{bmatrix}0\\0\\0\\0\\0\\-0.5\\-0.5\end{bmatrix}$$

where diag(y) is the diagonal matrix with the vector y on the diagonal.

(f) **(PTS: 0-2)** Show numerically that the optimal dual variables for the constraints of the primal problem are the same as the optimizers of the dual problem. Show also that the optimizers of the primal problem are the optimal dual variables for the constraints of the dual problem.

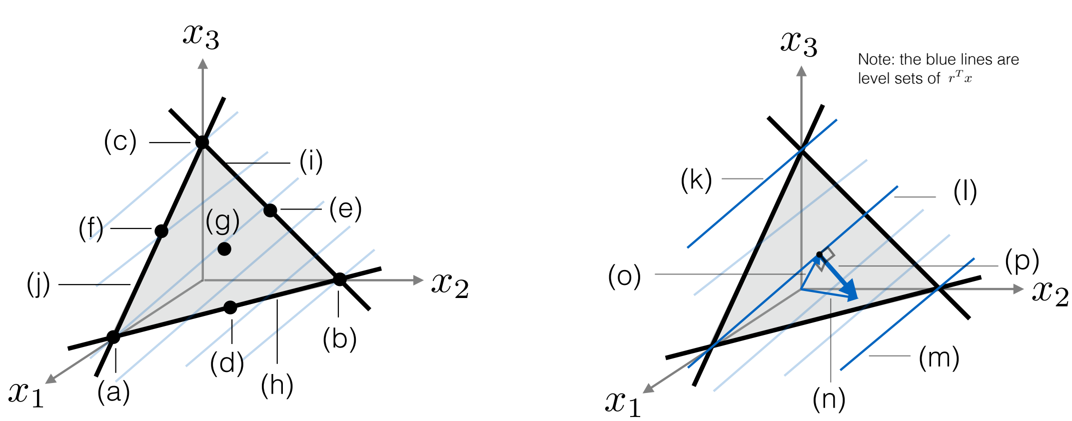
3. **Simplex Optimization**

- Consider the primal form of optimization on a simplex in $\mathbb{R}^3$.

$$\max_{x \in \mathbb{R}^3} \quad r^T x$$
$$\text{s.t.} \quad \mathbf{1}^T x = 1, \; x \geq 0$$

for $\mathbf{1}^T = [1\ 1\ 1]$. Consider the following illustrations of the primal problems. Label each indicated part of the diagrams.

(PTS: 0-2) Label: (a)-(j) **(PTS: 0-2)** Label: (k)-(p)

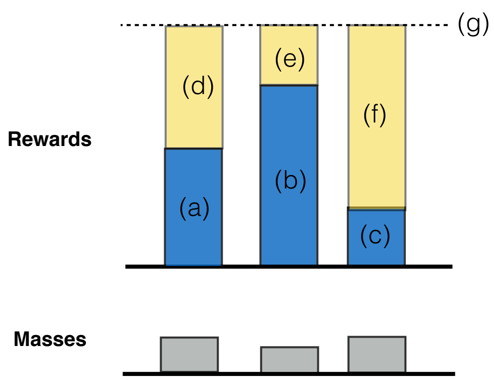


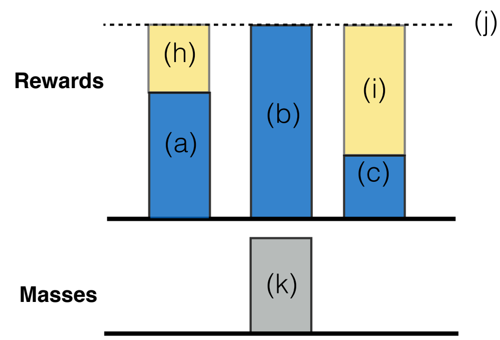
- Consider the dual form of the optimization problem

$$\min_{\lambda \in \mathbb{R},\ \mu \in \mathbb{R}^3_+} \quad \lambda$$
$$\text{s.t.} \quad \lambda \mathbf{1}^T = r^T + \mu^T,\ \ \mu \geq 0$$

  Consider the following illustrations of the dual problem. Label each indicated part of the diagrams.

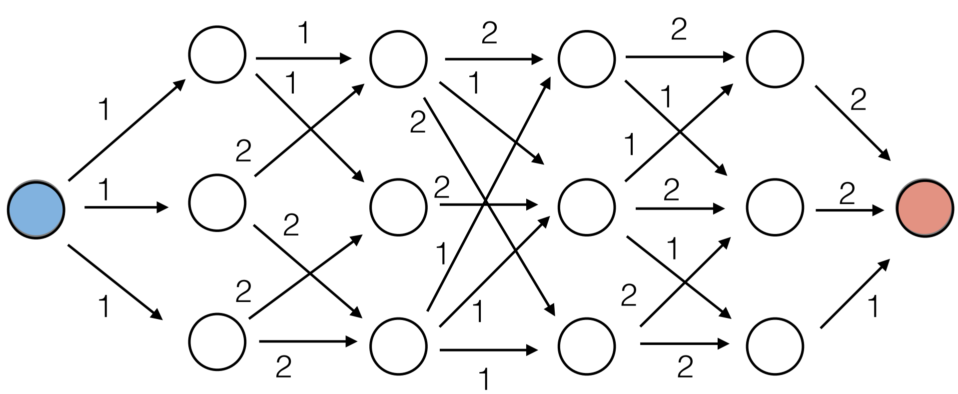
  (PTS: 0-2) Label: (a)-(f) **(PTS: 0-2)** Label: (h),(i),(j),(k)





4. **Dynamic Programming**

   Use dynamic programming to compute the shortest path from the blue node to the red node (given the travel costs on each edge given in the diagram).



- **(PTS:0-2)** Compute the optimal "cost-to-go" from each node to the end.
- **(PTS:0-2)** What is the shortest path?