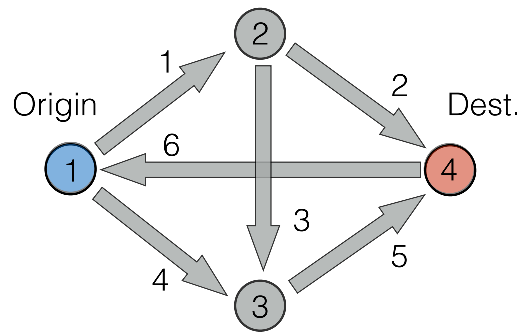
EE578B - Convex Optimization - Winter 2021

Homework 6

Due Date: Wednesday, Feb 24^{th}, 2021 at 11:59 pm

Consider the following graph structure for each of the following problems.



1. **Graph Constraints**

   - **(PTS:0-2)** Write down the incidence matrix for the graph, $E \in \mathbb{R}^{4\times 6}$. Write down the source-sink vector for the origin and the destination nodes, $b \in \mathbb{R}^4$.
   - **(PTS:0-2)** Write down a route indicator matrix $\mathbf{R}^{6\times 3}$ for the three routes from the origin to the destination that follow the direction of the edges given in the graph and do not include cycles. Compute $E\mathbf{R}$. How does this quantity relate to b?
   - **(PTS:0-2)** Show that E is not full row-rank. (Hint: compute $\mathbf{1}^T E$ where $\mathbf{1}$ is a vector of 1's.) Show also that $\mathbf{1}^T b = 0$. What does this say about the constraint $Ex = b$?
   - **(PTS:0-2)** Now consider the matrix

     $$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

     What is U^{-1}?
   - **(PTS:0-2)** The term $v^T(Ex-b)$ shows up in the Lagrangian of the network flow optimization problem we discussed in class, where $v \in \mathbb{R}^{|\mathcal{S}|}$ can be interpreted as a value function on the nodes. Rewrite this component of the Lagrangian as $v^T(Ex-b) = v'^T(E'x - b')$ where $v'^T = v^T U^{-1}$ What is E'? What is b'? Compute E' and b' for the example given. Intuitively, what do the values of v' represent relative to v? Find the useless row in the equation $E'x = b'$. After you remove this row is the equation full row rank?

2. **Shortest Path: Explicit Path Enumeration**

   Consider the shortest path linear program with the routes enumerated using the routing matrix computed in the previous part.

$$\begin{aligned}\min_{z\in\mathbb{R}^3} \quad & \ell^T z = c^T \mathbf{R} z \\ \text{s.t..} \quad & \mathbf{1}^T z = 1,\ z \geq 0\end{aligned} \tag{1}$$

- **(PTS:0-2)** What does z represent? What does each element of ℓ represent?
- **(PTS:0-2)** Write the dual of this optimization problem using $\lambda \in \mathbb{R}$ for the summation constraint and $u \in \mathbb{R}^3_+$ for the positivity constraint. What does λ represent? What does each element of u represent?
- **(PTS:0-2)** At optimum the complementary slackness condition is given by $z_i u_i = 0$ for $z_i \geq 0$, $u_i \geq 0$. Given the meaning of z_i and u_i, how does this condition intuitively ensure that the mass distribution vector z puts all mass on the optimal route?

3. **Shortest Path: Edge Formulation**

Consider the shortest path linear program using the incidence matrix.

$$\begin{aligned}\min_{x\in\mathbb{R}^6} \quad & c^T x \\ \text{s.t..} \quad & Ex = b,\ x \geq 0\end{aligned} \tag{2}$$

for $c \in \mathbb{R}^6$.

- **(PTS:0-2)** Write the Lagrangian of this optimization problem using $v \in \mathbb{R}^{|\mathcal{S}|}$ as the dual variable for the equality constraint and $\mu \in \mathbb{R}^{|\mathcal{E}|}$ as the dual variable for the positivity constraint.
- **(PTS:0-2)** Compute the KKT (optimality) conditions for this problem

$$\tfrac{\partial \mathcal{L}}{\partial x} = 0, \quad \tfrac{\partial \mathcal{L}}{\partial v} = 0, \quad x \geq 0, \quad \mu \geq 0, \quad \mu^T x = 0$$

- **(PTS:0-2)** Use the KKT conditions at optimum to show that for any route from the origin to the destination the total (summed) travel cost is greater than or equal to the minimum travel cost
- **(PTS:0-2)** Use the KKT conditions to show that, at optimum, mass only chooses optimal routes.
- **(PTS:0-2)** Derive the dual linear program. (You should end up with something closely related to the following.)

$$\begin{aligned}\max_{v} \quad & v^T b \\ \text{s.t.} \quad & v^T E \leq c^T\end{aligned}$$

- **(PTS:0-2)** Rewrite this linear program in terms of v', E', and b' from the first problem. Take a stab at interpreting the meaning of the constraints and objective in this reformulated linear program. How does this relate to the simplex optimization program we solved last week?

4. **Numerical optimization**

- **(PTS:0-4)** Use `cvx` or `cvxpy` to solve Problem (1) and it's dual for the cost vectors $c^T = [1\ 3\ 1\ 3\ 1\ 1]$ and $c^T = [1\ 2\ 1\ 3\ 1\ 1]$ and the graph structure given above. What are the optimal primal and dual variables for each case? Describe the meaning of each of these values intuitively. How are the two scenarios different?
- **(PTS:0-4)** Use `cvx` or `cvxpy` to solve Problem (2) and it's dual for the cost vectors $c^T = [1\ 3\ 1\ 3\ 1\ 1]$ and $c^T = [1\ 2\ 1\ 3\ 1\ 1]$ and the graph structure given above. What are the optimal primal and dual variables for each case? Describe the meaning of each of these values intuitively. Again, how are the two scenarios different?

  Note: You may have to use the constraint $E'x = b'$ if `cvx` chokes on the fact that the rows of E are not linearly independent. You can try both tho.