EE578B - Convex Optimization - Winter 2021

Homework 8

<u>**Due Date**</u>: Sunday, Mar 14^{th}, 2021 at 11:59 pm

1. **Newton's Method**

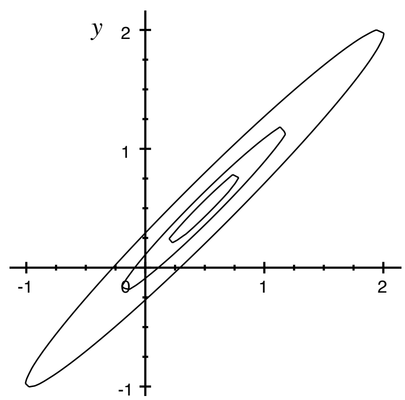
   Consider the following unconstrained quadratic program.

$$\min_x \quad f(x) = \tfrac{1}{2}x^T Q x + c^T x$$

   for $x \in \mathbb{R}^2$

$$Q = \begin{bmatrix} 101 & -99 \\ -99 & 101 \end{bmatrix}, \qquad c = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

   Note that the level sets look something like



   - **(PTS:0-2)** Perform regular first order gradient descent using the update equation

$$x^+ = x - \gamma \frac{\partial f}{\partial x}^T$$

     starting from the initial condition $x = \begin{bmatrix} 10 & 0 \end{bmatrix}^T$ with a fixed step size γ. (Pick the step size.) Note: if you want you can use a more sophisticated step-size method.
   - **(PTS:0-2)** Plot the trajectory of x and describe the behavior intuitively.
   - **(PTS:0-2)** Perform Newton's Method starting from the same initial condition with the same step size.

$$x^+ = x - \gamma \left(\frac{\partial^2 f}{\partial x^2}\right)^{-1} \frac{\partial f}{\partial x}^T$$

   - **(PTS:0-2)** Plot the trajectory of x and compare the qualitative performance to the first-order gradient descent method.

Now consider the following constrained convex program

$$\begin{aligned} \min_x \quad & 10x_1^4 + 2x_2^4 + 2x_3^4 + 2x_4^4 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

for $x \in \mathbb{R}^4$ and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- **(PTS:0-2)** Use Newton's method to perform gradient descent on this constrained optimization problem to solve for the optimal $x \in \mathbb{R}^n$ and the optimal dual variable $v \in \mathbb{R}^2$
- **(PTS:0-2)** Compare $\frac{\partial f}{\partial x}$ and $v^T A$ at optimum. How do they relate?

2. **Interior Point Method**

Consider the constrained optimization problem

$$\begin{aligned} \min_x \quad & f(x) = 10x_1^4 + x_2^4 \\ \text{s.t.} \quad & h(x) \leq 0 \end{aligned}$$

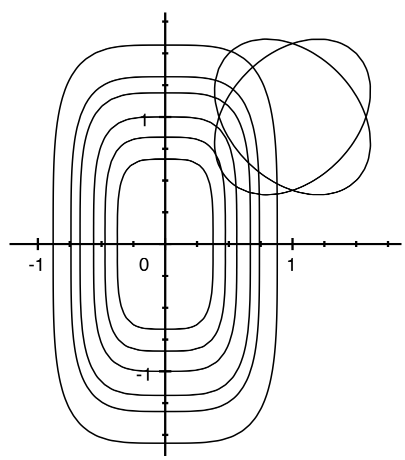
for $x \in \mathbb{R}^2$

$$h(x) = \begin{bmatrix} (x - \mathbf{1})^T Q_1 (x - \mathbf{1}) - 1 \\ (x - \mathbf{1})^T Q_2 (x - \mathbf{1}) - 1 \end{bmatrix}$$

where $\mathbf{1}^T = [1 \ 1]$

$$Q_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

The constraint level sets and objective level sets are shown in the following figure.



- **(PTS:0-2)** Replace the objective function $f(x)$ with $tf(x)$ for some $t > 0$. Replace the inequality constraints with equality constraints of the form $h_i(x) = s_i$ and add barrier function terms of the form $\mu \ln(s_i)$ to the objective for some $\mu > 1$.

- **(PTS:0-2)** Write the Lagrangian for this new optimization problem (with barrier functions) with dual variables $v \in \mathbb{R}^2$ for the equality constraints.
- **(PTS:0-4)** Write code to perform Newton's method for gradient descent to solve for the optimal $x \in \mathbb{R}^2$, $s \in \mathbb{R}^2$, and $v \in \mathbb{R}^2$ for a given value of t.
- **(PTS:0-4)** For each value of t, run Newton's method till $|x - x^+| < \delta$ for some tolerance $\delta > 0$. After x converges, update t as $t^+ = \mu t$ and repeat solving for x. (Note, it's important that $\mu > 1$ so that t will grow, ie. the objective gets more weight as you approach the boundary.) Iterate on this process till the optimal x is found. How does this method perform for different values of μ?

3. **Simplex Method - Row Geometry**

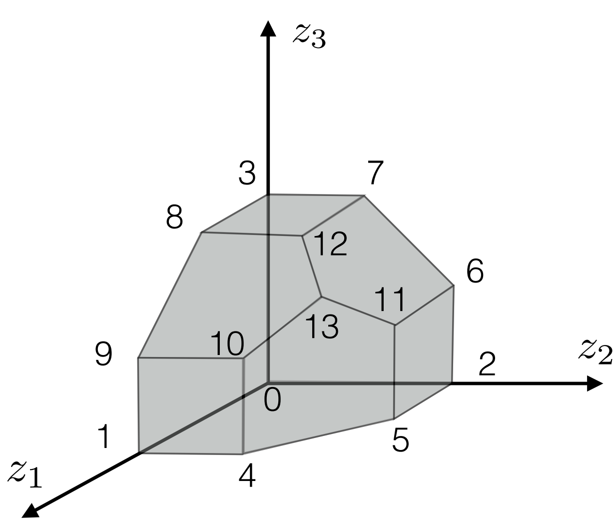
Consider the following linear program for $z \in \mathbb{R}^3$

$$\begin{aligned} \max_{x} \quad & c^T z \\ \text{s.t.} \quad & Cz \leq d, \; x \geq 0 \end{aligned}$$

where

$$c = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad d = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

Note that this set is the inside of the following geometric shape (with the vertices numbered).



- **(PTS:0-2)** Use a slack variable $s \in \mathbb{R}^6$ to rewrite the LP in standard form for the simplex method

$$\begin{aligned} \max_x \quad & r^T x \\ \text{s.t.} \quad & Ax = b, \; x \geq 0 \end{aligned}$$

What is x? A? b? What feasible x corresponds to $z = 0$?

- **(PTS:0-2)** Write a tableau for the linear program in the form

$$\begin{bmatrix} 1 & -r^T & 0 \\ \mathbf{0} & A & b \end{bmatrix}$$

- **(PTS:0-4)** Starting at the initial solution $z = 0$ (vertex 0), perform pivot steps to find the optimal solution to the linear program. What is the optimal x? What is the corresponding optimal z? Which rows of the constraint $Cz \leq d$ are satisfied with equality?
- **(PTS:0-2)** What route did you follow through the polytope? You can list the route referencing the vertex numbers.

4. **Simplex Method - Column Geometry**

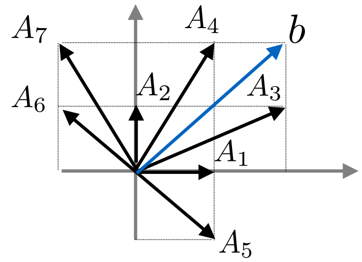
Consider the following linear program for $x \in \mathbb{R}^7$

$$\begin{aligned} \max_x \quad & r^T x \\ \text{s.t.} \quad & Ax = b, \; x \geq 0 \end{aligned}$$

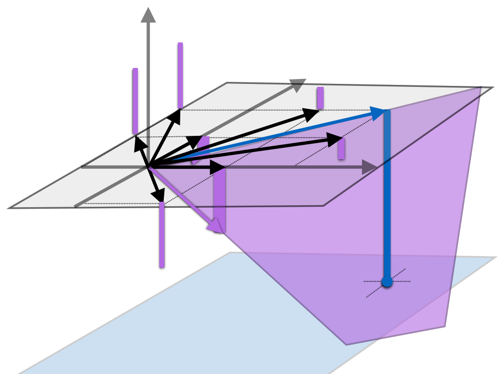
where

$$A = \begin{bmatrix} A_1 & \cdots & A_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 2 & -1 & 1 & 2 \end{bmatrix}$$

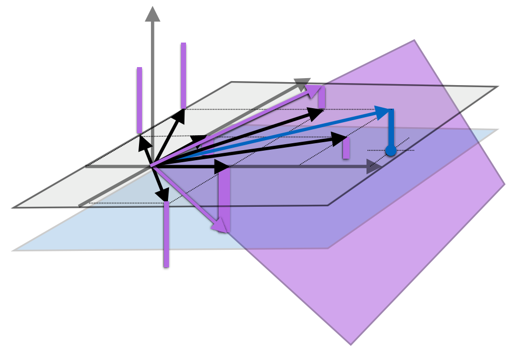
- **(PTS:0-2)** Draw the columns of A as vectors in R^2.
- **(PTS:0-2)** Suppose $b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find all possible pairs of basis vectors (A_i and $A_{i'}$) such that $\begin{bmatrix} A_i & A_{i'} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i'} \end{bmatrix} = b$. for $x \geq 0$. (Hint: there are 4 pairs. Drawing b with the columns of A may help.)
- **(PTS:0-2)** Suppose $b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find all possible pairs of basis vectors (A_i and $A_{i'}$) such that $\begin{bmatrix} A_i & A_{i'} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i'} \end{bmatrix} = b$. for $x \geq 0$. (Hint: there are 7 pairs. Drawing b with the columns of A may help.)
- **(PTS:0-4)** Now consider the reward vector $r^T = [-3 \;\; -1 \;\; -1 \; 1 \;\; -3 \; 3 \; 3]$ for $b^T = [2 \; 2]^T$. Write the tableau for the linear program to maximize $r^T x$. Perform the pivot steps shown in the following illustrations.



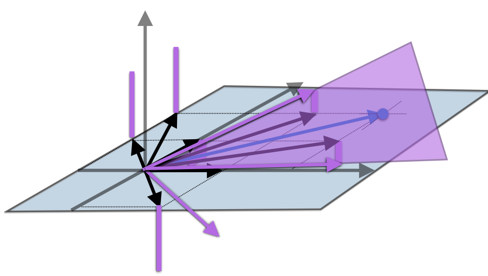
Start:



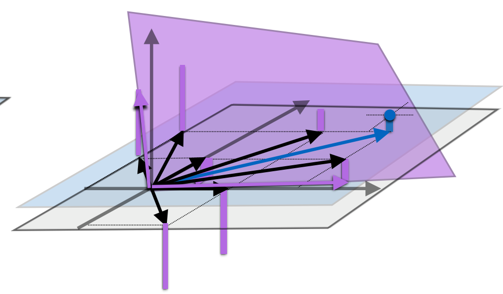
Step 1:



Step 2:

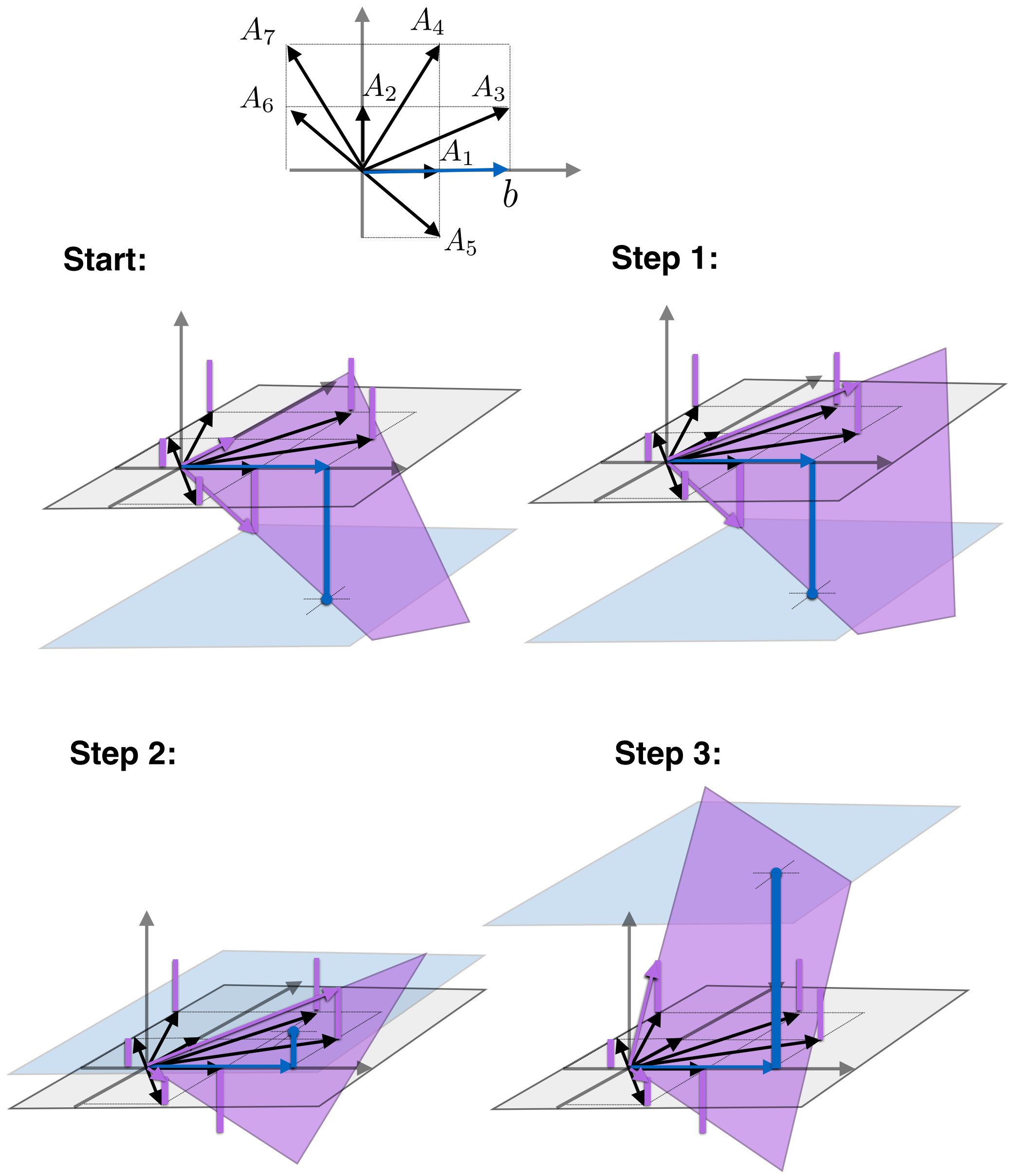


Step 3:



What is the optimal x and $r^T x$?

- **(PTS:0-4)** Now consider the reward vector $r^T = [-3\ 0\ 1\ 2\ 1\ \ -1\ 2]$ for $b^T = [2\ 0]^T$ Write the tableau for the linear program to maximize $r^T x$. Perform the pivot steps shown in the following illustrations.



What is the optimal x and $r^T x$?

- **(PTS:0-2)** Which individual x_i's could correspond to the positive and negative part of a single unconstrained variable?