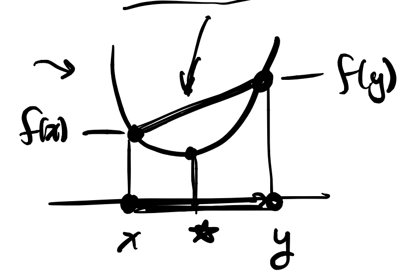
Convex Functions

$f: \mathbb{R}^n \to \mathbb{R}$ is convex

$x, y \in \mathbb{R}^n$ $\quad f(\underbrace{\alpha x + (1-\alpha)y}_{*}) \le \alpha f(x) + (1-\alpha) f(y)$

$0 \le \alpha \le 1$

$\forall\, x, y \in \operatorname{dom} f$



strictly convex

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$$

$f: \mathbb{R}^n \to \mathbb{R}$ is concave

$x, y \in \mathbb{R}^n$ $\quad f(\alpha x + (1-\alpha)y) \ge \alpha f(x) + (1-\alpha) f(y)$

$0 \le \alpha \le 1$ $\quad$ strictly concave

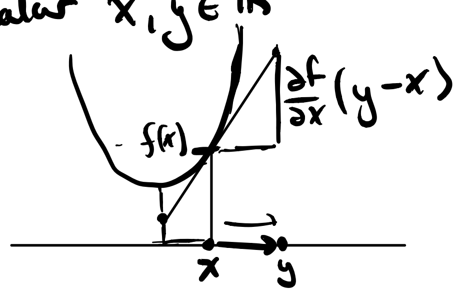
$\forall\, x, y \in \operatorname{dom} f$ $\quad f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha) f(y)$

1ST ORDER CONDS:

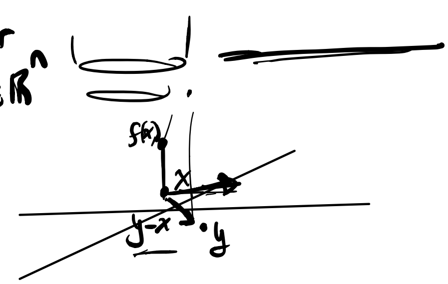
if f is differentiable

f is convex iff $\quad f(y) \ge f(x) + \frac{\partial f}{\partial x}(y - x)$

Scalar $x, y \in \mathbb{R}$



Vector $x, y \in \mathbb{R}^n$



2ND ORDER CONDS:

twice differentiable f.

Hessian: $\dfrac{\partial^2 f}{\partial x^2} = H$ } → measure of curvature of a function

f is convex iff $H \succeq 0$

strictly convex iff $H \succ 0$

$$f = \frac{1}{2} x^T Q x + c^T x$$

$$\frac{\partial f}{\partial x} = x^T Q + c^T$$

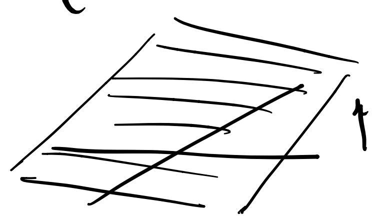
$$\frac{\partial^2 f}{\partial x^2} = Q \succ 0$$

$\Rightarrow$ f is strictly convex

Examples of Convex Functions:

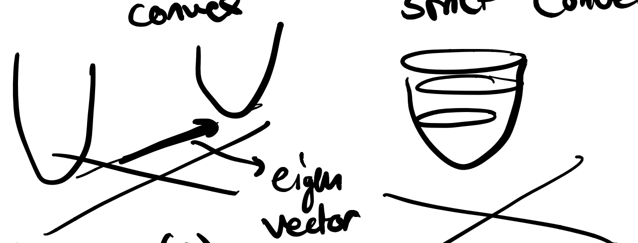
- linear/affine functions (also concave)

  $f(x) = c^T x + d$



- Quadratic functions

  $f(x) = \frac{1}{2} x^T Q x + c^T x$ $\quad Q \succeq 0$ convex $\quad Q \succ 0$ strict convex



$\lambda_i \in \text{eigs}(Q)$

$\lambda_i \geq 0$

$\lambda_i \in \text{eigs}(Q)$

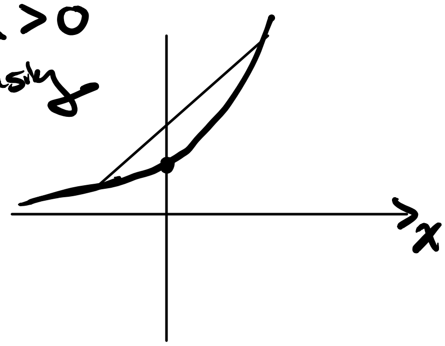
$\lambda_i > 0$

- exponentials

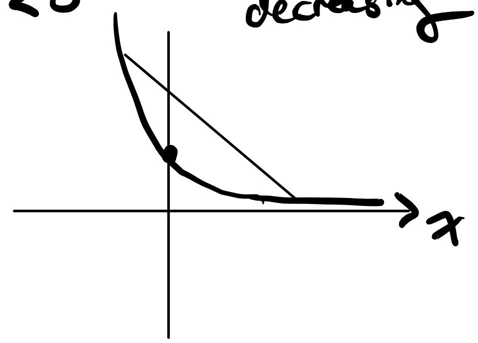
e^{ax} $\quad a \in \mathbb{R}$, $x \in \mathbb{R}$ $\quad$ convex in x

$a > 0$

increasing

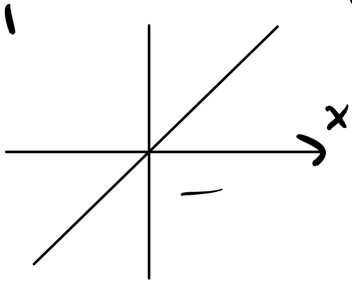


$a < 0$ $\quad$ decreasing

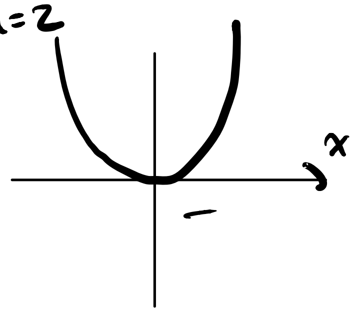


- x^a on $\mathbb{R}_{++}$ $\quad$ convex if $\boxed{a \geq 1}$

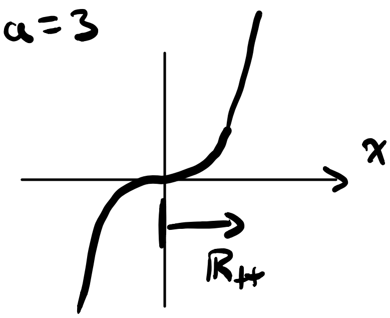
$a = 1$



$a = 2$

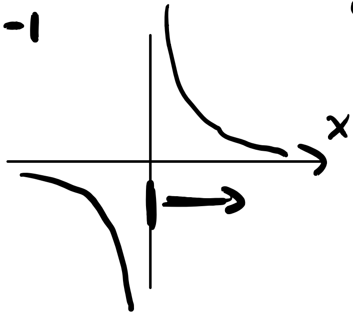


$a = 3$

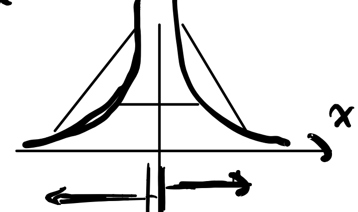


convex if $\boxed{a \leq 0}$

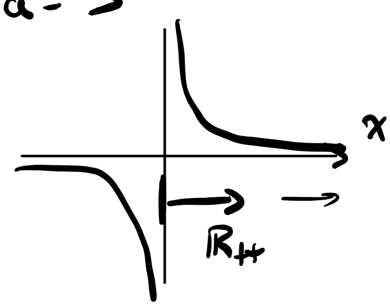
$a = -1$



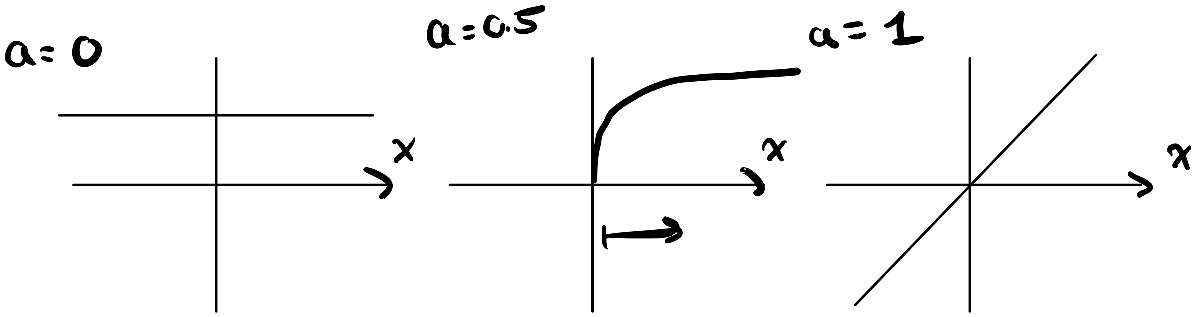
$a = -2$



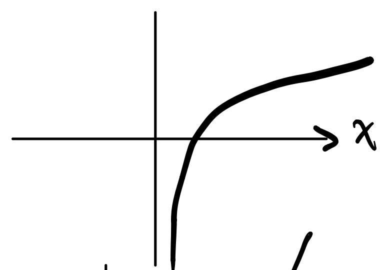
$a = -3$

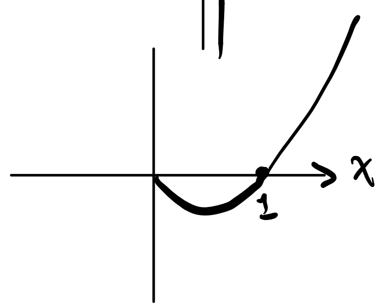


x^a on $\mathbb{R}_{++}$ concave $0 \leq a \leq 1$



- $|x|^p$ for $p \geq 1$ convex on $\mathbb{R}$
- $\log(x)$ concave on $\mathbb{R}_{++}$
- $x\log(x)$ (negative entropy) convex on $\mathbb{R}_{++}$





Vector valued

- every norm in $\mathbb{R}^n$ is convex
- $f(x) = \max\{x_1, \dots, x_n\}$ convex
- Quadratic over linear $f(x,y) = \frac{x^2}{y}$ $y > 0$ convex

- log sum exp $\qquad x_n = 10 >> x_1, x_2 \dots$

convex $\longrightarrow$ $f(x) = \log\left(e^{\alpha x_1} + \dots + e^{\alpha x_n}\right) \qquad \alpha > 0$

softmax function

$$\frac{1}{\alpha}\frac{\partial f}{\partial x} = \frac{1}{\sum_i e^{\alpha x_i}}\left[e^{\alpha x_1}, \dots, e^{\alpha x_n}\right]$$

soft argmax function

- geometric mean

$$f(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

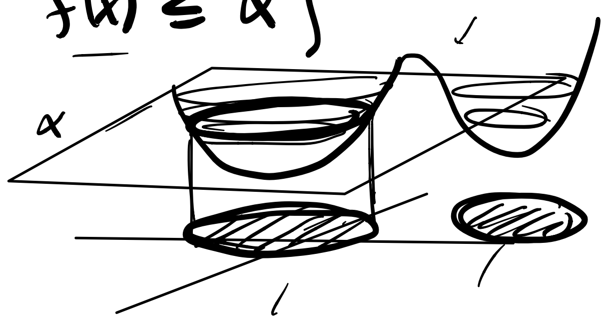
concave for $x \in \mathbb{R}^n_{++}$

compare arithmetic mean

$$f(x) = \left(\sum_{i=1}^{n} x_i\right)\frac{1}{n}$$

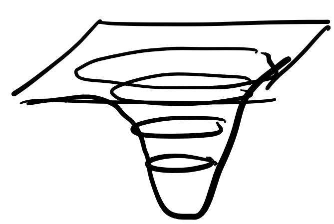
SUBLEVEL SETS OF CONVEX FUNCTIONS:

$$C_\alpha = \{x \in \mathrm{dom} f \mid f(x) \le \alpha\}$$

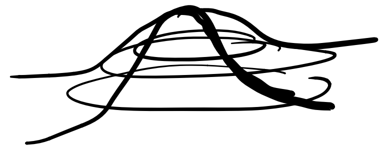


if f is convex $\Longrightarrow$ sublevel sets of f are convex

sublevel sets are convex $\not\Longrightarrow$ f is convex



Multivariable Gaussian



Operations that preserve convexity.

f : convex

- $\alpha f \quad \alpha \geq 0 \Rightarrow$ convex
- $f_1, \ldots, f_m$ convex ✓

  $f = w_1 f_1 + \cdots + w_m f_m \quad w_i \geq 0 \Rightarrow$ convex.
- $f(x,y)$ convex in x all y.

  $w(y) \geq 0$ ✓

  $$g(x) = \int_y w(y) f(x,y)\, dy \Rightarrow \text{convex}$$

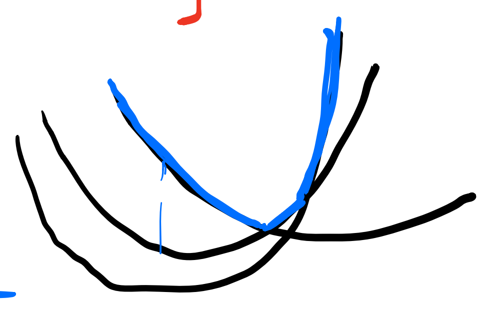
Composition Rules

- $f(x)$ convex. ✓

  $g(x) = f(Ax+b) \Rightarrow$ convex
- pointwise max

  $f_1, \ldots, f_m$ convex

  $f(x) = \max_i \{f_i(x)\} \Rightarrow$ convex

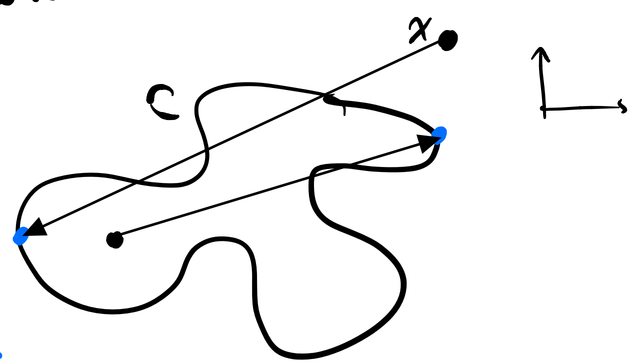


$$g(x) = \underbrace{\sup_y f(x,y)}_{\text{convex function}} \qquad f(x,y) \text{ is convex in } x \quad \forall y$$

Ex. set C

$$f(x) = \sup_{y \in C} |x-y|$$

distance to the farthest point in C



for any y

$x - y \longrightarrow$ linear

$|\cdot| \longrightarrow$ convex

$\sup_y \longrightarrow$ sup over convex functions

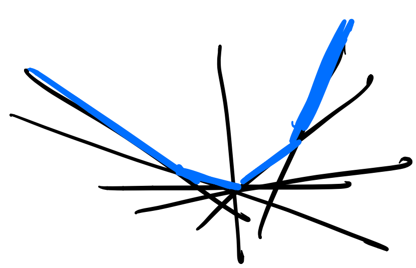
- max eigenvalue of symmetric matrix

$$f(X) = \lambda_{max}(X) \Longleftarrow X = X^T$$

$$= \sup\{\underbrace{y^T X y}_{\text{linear in } X} \mid \|y\|_2 = 1\}$$

convex

sup of linear functions in X



- max singular value of X any matrix

$$f(X) = \sigma_{max}(X)$$

$$= \sup\{ \underline{u^T X v} \mid \|u\|_2 = 1, \|v\|_2 = 1 \}$$

$$X = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \quad u^T U = [1\ 0 \cdots 0] \quad V^T v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

rot. rot. σ_1

$$\Sigma = \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}}_{\text{all positive}}$$

$f(X)$: convex function

General Composition Rules:

$$\underline{f}(x) = \underline{h}(\underline{g}(x)) \text{ —}$$

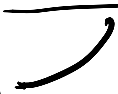
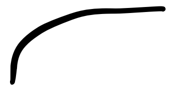
Note: $g(x) \in \text{dom } h$ etc.

h is convex non decreasing, g convex $\Longrightarrow$ f convex

h is convex non increasing, g concave $\Longrightarrow$ f convex

h is concave non decreasing, g concave $\Longrightarrow$ f concave

h is concave non increasing, g convex $\Longrightarrow$ f concave

h	convex	concave
non decreasing		
non increasing