Rank of Matrix:

row rank = col rank = rank

\# lin ind rows   \# of lin ind cols

$$\boxed{\text{rank} = \dim \mathcal{R}(A)}$$

$A \in \mathbb{R}^{m \times n}$

row rank = col rank $\boxed{r = k}$

r c

$\rightarrow r \leq k$

$\rightarrow r \geq k$

A: k lin ind cols

$A = [A_1 \cdots A_n]$

coords of cols of A

$$A = \underbrace{[c_1 \cdots c_k]}_{\text{basis for colspace}} [w_1 \cdots w_n]$$

$C = [c_1 \cdots c_k]$

$A_1 = C w_1$

$\vdots$

$A_n = C w_n$

$$A = {}_m\overset{k}{C}\,{}_k\overset{n}{W}$$

$$\begin{bmatrix} -\bar{c}_1^T- \\ \vdots \\ -\bar{c}_m^T- \end{bmatrix}\begin{bmatrix} -\bar{w}_1^T- \\ \vdots \\ -\bar{w}_k^T- \end{bmatrix} = \begin{bmatrix} \bar{c}_1^T W \\ \bar{c}_m^T W \end{bmatrix}$$

\# of lin ind rows of A = row rank = $r \leq k$

$A = VR$ $A = \begin{bmatrix} -\bar{A}_1^T- \\ -\bar{A}_m^T- \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -v_1^T- \\ \vdots \\ -v_m^T- \end{bmatrix}}_{\text{coords}}\begin{bmatrix} -R_1^T- \\ \vdots \\ -R_r^T- \end{bmatrix}$$

basis for row space of A

$\bar{A}_1^T = v_1^T R$

$\bar{A}_m^T = v_m^T R$

$$= {}_m\overset{r}{\begin{bmatrix} V \end{bmatrix}}\,{}_r\begin{bmatrix} R \end{bmatrix}$$

\# of lin ind cols of A = $k \leq r$

Rank Nullity Thm: $A \in \mathbb{R}^{m \times n}$

$$\underbrace{\dim R(A)}_{\text{rank } A} + \underbrace{\dim N(A)}_{\text{nullity } A} = n$$

$$k + n-k = n$$

$$A \in \mathbb{R}^{m\times n} \longrightarrow \underbrace{A^TA \in \mathbb{R}^{n \times n}}, \; AA^T \in \mathbb{R}^{m \times m}$$

$$A^TA = \begin{bmatrix} -A_1^T- \\ -A_n^T- \end{bmatrix}\begin{bmatrix} A_1 \cdots A_n \end{bmatrix}$$

$$= \begin{bmatrix} A_1^TA_1 & \cdots & A_1^TA_n \\ A_n^TA_1 & \cdots & A_n^TA_n \end{bmatrix} \leftarrow$$ elements of this matrix are dot products of cols

AA^T = elements are dot products of rows

Grammians of A.

- symmetric, positive semi definite
- describe "shape of A"

$$\underline{rk(A)} = \underline{rk(A^T)} = \underline{rk(A^TA)} = rk(AA^T)$$

Proof: rank-nullity thm
equate the nullspaces

$A \in \mathbb{R}^{m\times n}$ $\quad A^TA \in \mathbb{R}^{n \times n}$ $\quad N(A) = N(A^TA)$

WTS: $N(A) = N(A^TA)$

$$Ax = 0 \Rightarrow A^T\underbrace{Ax}_{0} = 0 \quad ✓$$

$$A^TAx = 0 \Rightarrow x^TA^TAx = 0 \qquad Ax = 0$$

$$|Ax|^2 = 0 \quad ✓$$

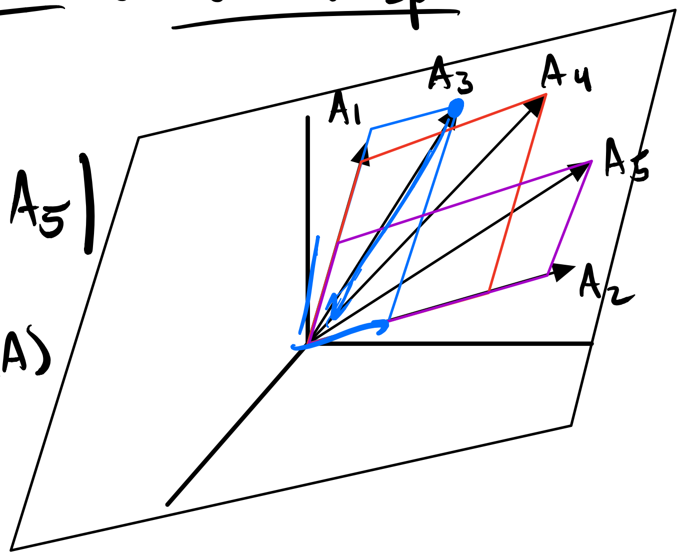
$$|Ax| = 0 \rightarrow Ax = 0$$

since $|v| = 0 \Rightarrow v = 0$

Constructing a basis for a Nullspace

$A \in \mathbb{R}^{3\times 5}$

$A = [A_1\ A_2\ A_3\ A_4\ A_5]$



$[A_1 A_2]$ basis for $\mathcal{R}(A)$

$A_3 = [A_1 A_2]\, B_3$

$A_4 = [A_1 A_2]\, B_4$

$A_5 = [A_1 A_2]\, B_5$

$$A = [A_1 A_2 \mid A_3 A_4 A_5] \begin{bmatrix} B_3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = [A_1 A_2] B_3 - A_3 = 0$$

similarly for A_4 A_5

$$N = \begin{bmatrix} B_3 & B_4 & B_5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow AN = 0$$

$A \in \mathbb{R}^{m\times n}$ rank$(A) = k$ $\quad B = [B_{k+1} \cdots B_n]$

$$A = [\,{}_m\underbrace{\overset{k}{C}}\ \mid CB] = C[I\ \ B] \qquad N = \begin{bmatrix} B \\ -I \end{bmatrix} ✓$$

k lin ind cols basis for $\mathcal{R}(A)$

coeffs for other $n-k$ cols

$AN = 0$

cols of N form a basis for $\mathcal{N}(A)$

- lin ind.

  cols of N are lin ind if $Nz = 0 \Rightarrow z = 0$

  the only lin comb of cols of N that gives 0 is 0

$$Nz = \begin{bmatrix} Bz \\ \hline -z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow z = 0 \Leftarrow$$

$$N = \begin{bmatrix} B \\ \hline -I \end{bmatrix} = \begin{bmatrix} & B & \\ -1 & & 0 \\ 0 & & -1 \end{bmatrix}$$

$\uparrow$

- cols of N span $\mathcal{N}(A)$.

$x \in \mathcal{N}(A) \quad \exists z, \ x = Nz$

$$Ax = C[I|B]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= C(x_1 + Bx_2) = 0$$

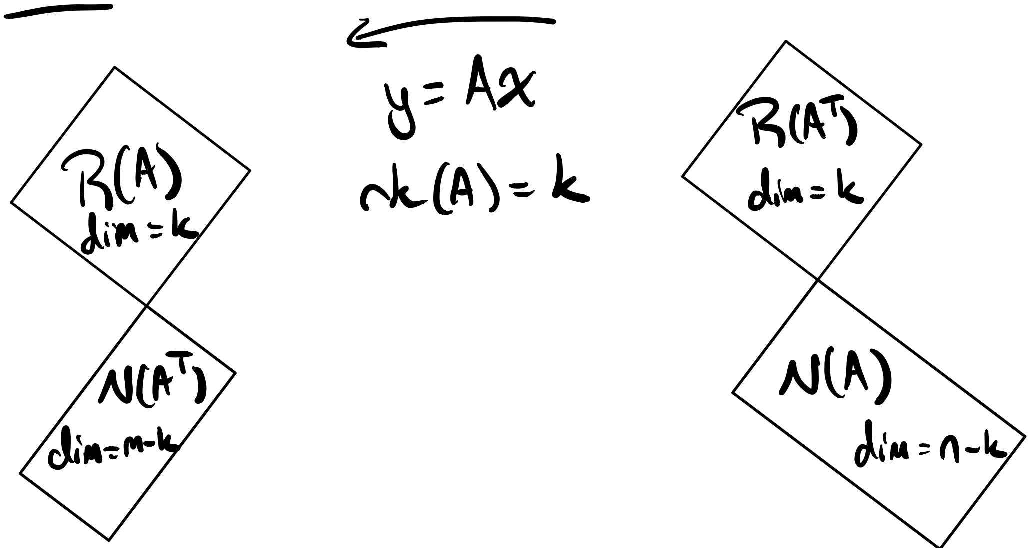
since cols of C are lin ind.

$$\Rightarrow x_1 + Bx_2 = 0 \Rightarrow x_1 = -Bx_2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -Bx_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix}(-x_2) = N(-x_2)$$

Fundamental Thm of Lin Algebra:

CODOMAIN $\mathbb{R}^m$ $\quad$ $A \in \mathbb{R}^{m\times n}$ $\quad$ DOMAIN $\mathbb{R}^n$



$R(A)$ orthogonal complement $N(A^T)$

$R(A^T)$ orthogonal complement of $N(A)$

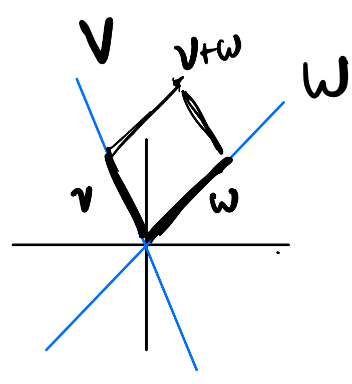
$$R(A) \oplus^{\perp} N(A^T) = \mathbb{R}^m \qquad R(A^T) \oplus^{\perp} N(A) = \mathbb{R}^n$$

Direct Sum of 2 vector spaces...

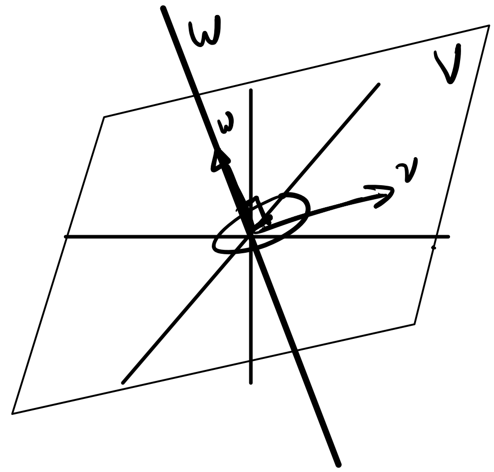
V, W

$V \oplus W = \{v+w \mid v \in V, w \in W\}$

combined span of V & W



2 vector spaces are orthogonal if
$V \perp W \Rightarrow v \in V \quad w \in W \quad v^T w = 0$



$$R(A^T) \oplus^{\perp} N(A) = \mathbb{R}^n$$

$x \in \mathbb{R}^n \qquad A \in \mathbb{R}^{m \times n}$

cols of $N \in \mathbb{R}^{n \times n-k}$ span $N(A)$

$$x = A^T \underline{x_1'} + N \underline{x_2'}$$

$A \in \mathbb{R}^{2\times 3} \qquad A = \begin{bmatrix} -A_1^T- \\ -A_2^T- \end{bmatrix}$

$$x = \underbrace{\begin{bmatrix} A^T & N \end{bmatrix}}_{\text{cols as a basis for domain}} \underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\text{coeffs of } x.}$$

$N = \begin{bmatrix} N_1 \end{bmatrix}$

$AN = 0$

$$\begin{bmatrix} A_1^T N_1 \\ A_2^T N_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

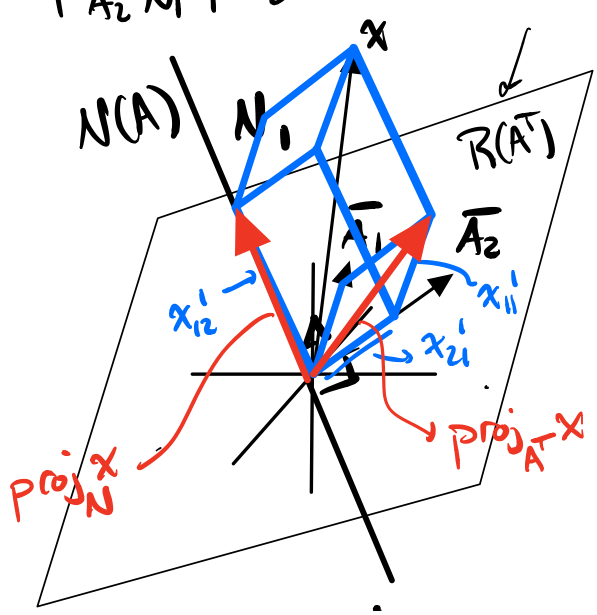
$x_1' = \begin{bmatrix} x_{11}' \\ x_{21}' \end{bmatrix}$

$x_2' = \begin{bmatrix} x_{12}' \end{bmatrix}$

$$x = \underbrace{\begin{bmatrix} A^T & N \end{bmatrix}}_{\text{want invert}} x'$$

$$\begin{array}{cc} m & n-k \\ \end{array}$$
$$\begin{bmatrix} A^T & N \end{bmatrix}$$

k is rank of A



DOMAIN

$m + n - k = n$

$m = k \Rightarrow$ A needs to have full row rank $\to$ m lin ind rows

Note: $\begin{bmatrix} \underline{A^T} & \underline{N} \end{bmatrix}$ is only square and invertible if A is full row rank

$\begin{bmatrix} A^T & \underline{N} \end{bmatrix}^{-1} \stackrel{?}{=} \begin{bmatrix} A \\ N^T \end{bmatrix}$ Note M^T relates to M^{-1}

$M^T = M^{-1} \leftarrow$ M is a rotation

$\begin{bmatrix} A^T & N \end{bmatrix}^{-1} \begin{bmatrix} A^T & N \end{bmatrix} = I$

$\begin{bmatrix} A \\ N^T \end{bmatrix} \begin{bmatrix} A^T & N \end{bmatrix} = \begin{bmatrix} AA^T & \cancel{AN}^{\,0} \\ \cancel{N^TA^T}_{\,0} & N^TN \end{bmatrix} = \begin{bmatrix} AA^T & 0 \\ 0 & N^TN \end{bmatrix}$

$\begin{bmatrix} A \\ N^T \end{bmatrix} \begin{bmatrix} A^T & N \end{bmatrix} = \begin{bmatrix} AA^T & \boxed{0} \\ \boxed{0} & N^TN \end{bmatrix}$ $\begin{bmatrix} AA^T & 0 \\ 0 & N^TN \end{bmatrix}^{-1} = \begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^TN)^{-1} \end{bmatrix}$

$\underline{\begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^TN)^{-1} \end{bmatrix} \begin{bmatrix} A \\ N^T \end{bmatrix}} \begin{bmatrix} A^T & N \end{bmatrix} = I$

$\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} (AA^T)^{-1}A \\ (N^TN)^{-1}N^T \end{bmatrix}$

$$\underline{x} = [A^T N] x'$$

$$x' = [A^T N]^{-1} x = \begin{bmatrix} (AA^T)^{-1} A x \\ (N^T N)^{-1} N^T x \end{bmatrix}$$

plugging x' back in

$$x = [A^T N] x' = \underbrace{A^T (AA^T)^{-1} A x}_{\text{proj}_{A^T} x} + \underbrace{N (N^T N)^{-1} N^T x}_{\text{proj}_N x}$$

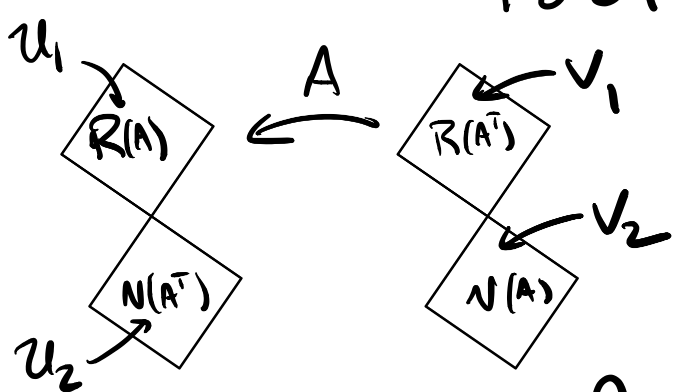
$$x = \text{proj}_{A^T} x + \text{proj}_N x$$

SVD: $_m[\hat{A}] = \mathcal{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$ $\mathcal{U}, V$ unitary

$\mathcal{U}^T \mathcal{U} = I$

$V^T V = I$

$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \underline{\sigma_k} \end{bmatrix}$



$\sigma_1 \cdots \sigma_k$
singular values
(positive)

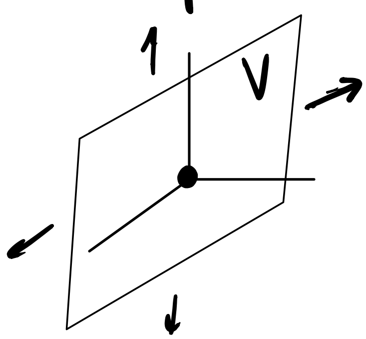
$$A = \begin{array}{c} \\ m \end{array} \begin{array}{c} \quad k \quad\; m-k \\ \left[\mathcal{U}_1 \middle| \mathcal{U}_2 \right] \end{array} \left[\begin{array}{cc} {}_k\hat{\Sigma}^k & 0 \\ 0 & 0 \end{array} \right] \left[\begin{array}{c} V_1^T \\ \hline V_2^T \end{array} \right] \begin{array}{l} k \\ n-k \end{array}$$

$$= \mathcal{U}_1 \Sigma V_1^T \leftarrow \text{reduced SVD}$$

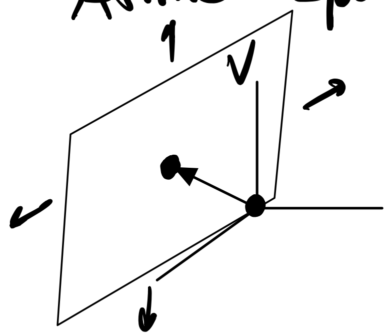
$\left. \begin{array}{l} A^T A \\ AA^T \end{array} \right\} \rightarrow$ eigenvalues $\sigma_1^2 \cdots \sigma_k^2$, $0 \cdots 0$

Characterizing Linear Sets:

Subspaces



Affine Spaces



Characterization 1: Nullspace

$$V = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

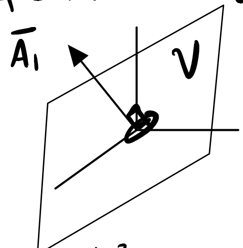
defining a plane as the nullspace of A

$\bar{A}_1^T x = 0$

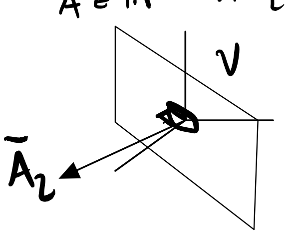
$\bar{A}_2^T x = 0$

$$Ax = \begin{bmatrix} - \bar{A}_1^T - \\ \vdots \\ - \bar{A}_m^T - \end{bmatrix} x = \begin{bmatrix} \bar{A}_1^T x \\ \vdots \\ \bar{A}_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

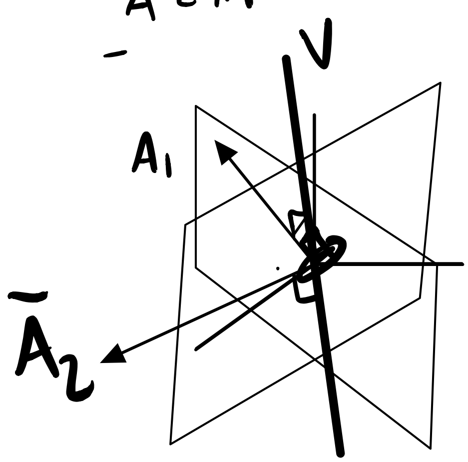
$A \in \mathbb{R}^{1 \times 3}$ $\quad A = [\bar{A}_1^T]$

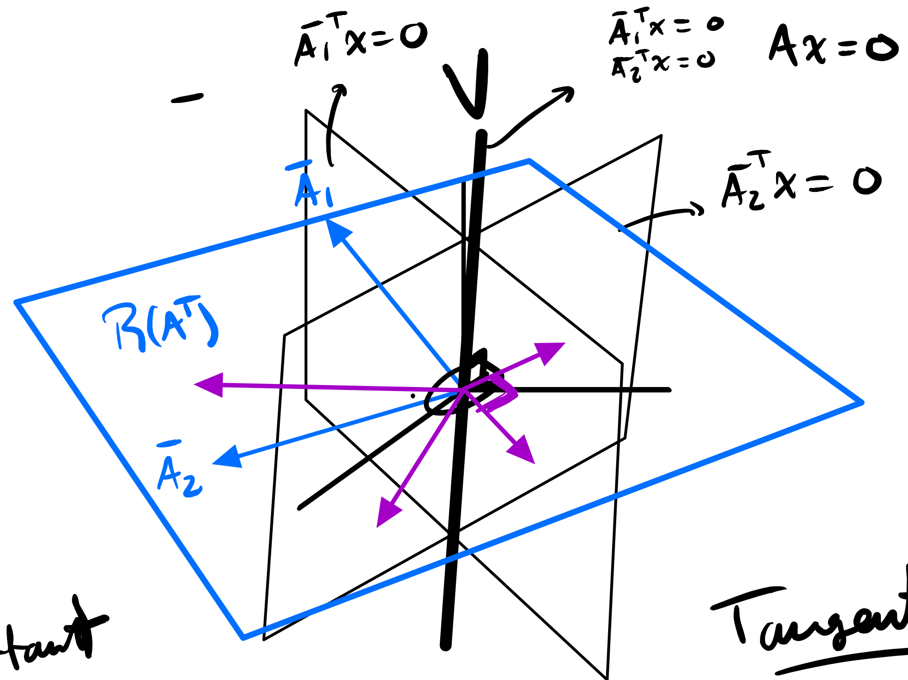


$A \in \mathbb{R}^{1 \times 3}$ $\quad A = [\bar{A}_2^T]$



$A \in \mathbb{R}^{2 \times 3}$ $\quad A = \begin{bmatrix} \bar{A}_1^T \\ \bar{A}_2^T \end{bmatrix}$





Important

$$V = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$U[Ax = 0] \Rightarrow UAx = U \cdot 0 = 0$$

$$\downarrow \qquad UAx = 0$$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} - \bar{A}_1^T - \\ - \bar{A}_2^T - \end{bmatrix} x = 0$$

$$\rightarrow \begin{bmatrix} \boxed{u_{11} \bar{A}_1^T + u_{12} \bar{A}_2^T} \\ \boxed{u_{21} \bar{A}_1^T + u_{22} \bar{A}_2^T} \end{bmatrix} x = 0$$

$$U = \begin{bmatrix} 1 & 2 \\ 1/4 & 1/2 \end{bmatrix} \longrightarrow U \text{ is rank 1 matrix}$$

problem

if U is square & invertible
$\Rightarrow$ doesn't change the constraints

Tangent

$$Ax = 0$$

$$\downarrow$$

$$UAx = 0$$

rows are orthonormal

$-1/2$ of eigenvalues

$$\rightarrow (AA^T)^{-1/2} A \qquad (AA^T)^{-1/2} (AA^T)^{-1/2} = (AA^T)^{-1}$$

$$M^{1/2} = \quad M^{1/2} M^{1/2} = M$$

$\hookrightarrow$ square roots of eigenvalues
spectral mapping thm.

$$(AA^T)^{-1/2} \underline{A A^T} (AA^T)^{-1/2} = I$$

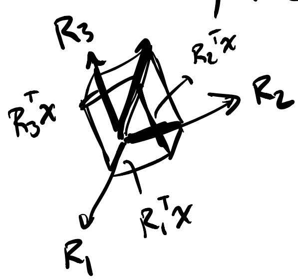
$$\underbrace{(AA^T)^{-1/2} (AA^T)^{1/2}}_{I} \underbrace{(AA^T)^{1/2} (AA^T)^{-1/2}}_{I}$$

Tangent

subspace w orthonormal basis
cols of R

$$\text{proj}_R x = R\underline{R}^T x \quad \leftarrow$$

$$[R_1 R_2 R_3]\begin{bmatrix} R_1^T \\ R_2^T \\ R_3^T \end{bmatrix} x$$



$$R_1 R_1^T x + R_2 R_2^T x + R_3 R_3^T x$$

subspace w any basis
cols of A

$$\text{proj}_A x = A(A^TA)^{-1}A^T x$$

$$= \underline{A(A^TA)^{-1/2}}\ \underline{(A^TA)^{-1/2}A^T} x$$

Affine Space

$$V = \{ x \in \mathbb{R}^n \mid Ax = b \}$$

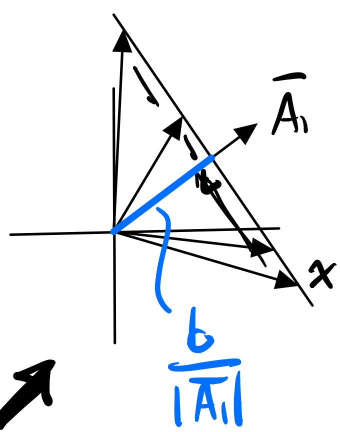
$A = [\bar{A}_1^T]$

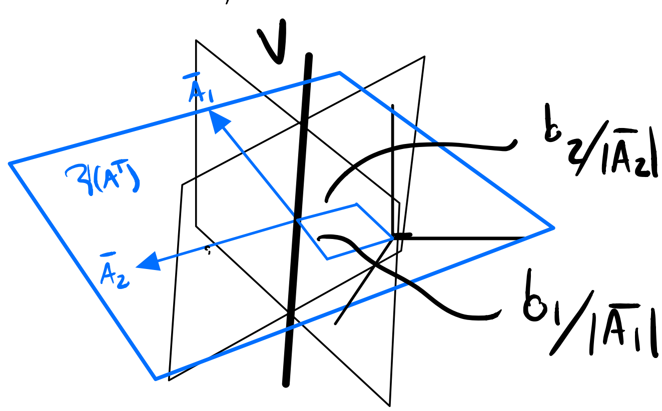
$\bar{A}_1^T x = b \qquad b \in \mathbb{R}^1$

$$\rightarrow \frac{\bar{A}_1^T}{|\bar{A}_1|} x = \frac{b}{|\bar{A}_1|}$$

$$A = \begin{bmatrix} \bar{A}_1^T \\ \bar{A}_2^T \end{bmatrix} \qquad Ax = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} - \frac{1}{|\bar{A}_1|}\bar{A}_1^T - \\ - \frac{1}{|\bar{A}_2|}\bar{A}_2^T - \end{bmatrix} x = \begin{bmatrix} b_1/|\bar{A}_1| \\ b_2/|\bar{A}_2| \end{bmatrix}$$





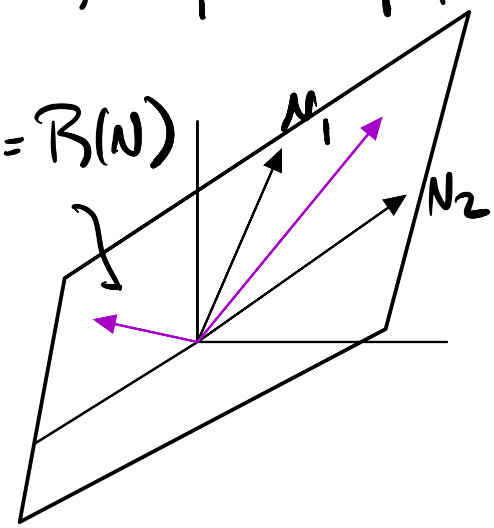
Characterization 2: range $\mathcal{R}(N)$ before $\mathcal{N}(A)$

N basis for $\mathcal{N}(A)$

$$V = \{x \in \mathbb{R}^n \mid x = Nz,\ z \in \mathbb{R}^{n-k}\} = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

$V = \mathcal{R}(N)$



$$x = Nz$$

$$x = NUz$$

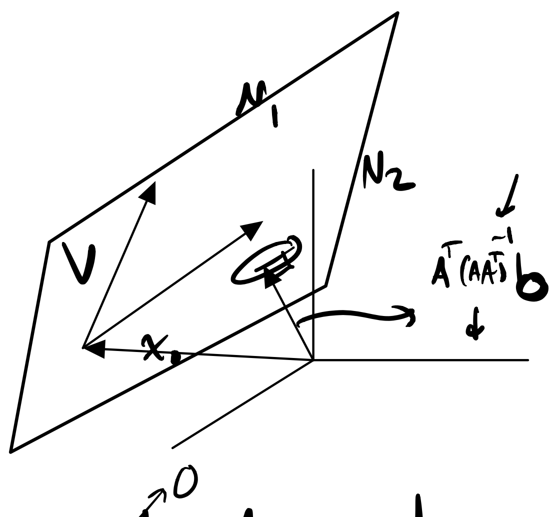
$$= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \boxed{N_1 u_{11} + N_2 u_{21}} & \boxed{N_1 u_{12} + N_2 u_{22}} \end{bmatrix}$$

Affine space

$$V = \{x \in \mathbb{R}^n \mid Ax = b\}$$
$$= \{x \in \mathbb{R}^n \mid x = Nz + x_0,\ z \in \mathbb{R}^{n-k}\}$$

$Ax = b$ $\quad \boxed{x = Nz + x_0}$

- N basis for $\mathcal{N}(A)$
- $Ax_0 = b$



$$Ax = \cancel{ANz}^{\,0} + Ax_0 = b$$

$$x = \underline{x_0} + Nz' + Nz$$

want to find x_0

s.t. $x_0 \perp N(A)$

$$Ax_0 = b \Leftarrow$$

$$x_0 = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \leftarrow$$

$$A(x_0 + Nz) = b$$

$$\cancel{AN}z^{\nearrow 0}$$

$$b = Ax_0 = \begin{bmatrix} AA^T & AN \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$b = AA^T x_1'$$

$$x_1' = (AA^T)^{-1} b$$

$$x_0 = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} x_1' \\ 0 \end{bmatrix} = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} (AA^T)^{-1} b \\ 0 \end{bmatrix}$$

$$x_0 = A^T (AA^T)^{-1} b$$

Vector Derivatives

$$f(x) \qquad f: \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\underline{\frac{\partial f}{\partial x}} : \Delta x \longmapsto \Delta f \qquad \Delta f = \frac{\partial f}{\cancel{\partial x}} \cancel{\Delta x}$$

$$\begin{bmatrix} \Delta f \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{\partial f}{\partial x}}_{\text{row vector}} \end{bmatrix} \begin{bmatrix} \Delta x \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$= \frac{\partial f}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f}{\partial x_n} \Delta x_n$$

$$f(x) = r^T x \Longrightarrow \Delta f = \underbrace{r^T}_{\frac{\partial f}{\partial x}} \Delta x \qquad \frac{\partial f}{\partial x} = r^T$$

$$f : \mathbb{R}^n \to \mathbb{R}^m \qquad \underbrace{\Delta f}_{\mathbb{R}^m} = \underbrace{\frac{\partial f}{\partial x}}_{n} \underbrace{\Delta x}_{\mathbb{R}^n}$$

$$m \left[\frac{\partial f}{\partial x} \right]$$

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} - \frac{\partial f_1}{\partial x} - \\ \vdots \\ - \frac{\partial f_m}{\partial x} - \end{bmatrix} = \begin{array}{c c} & \longleftarrow \text{"}x\text{"} \longrightarrow \\ \text{"}f\text{"} \downarrow & \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \end{array}$$

$$f(x) = Ax \qquad \Delta f = \underbrace{A}_{\frac{\partial f}{\partial x}} \Delta x$$

$$\frac{\partial f}{\partial x} = A$$

Product rule: perturb ea. instance of a variable separately & then add

$$f(x) = x^T Q x \Leftarrow$$

$$\Delta f = \underline{\frac{\partial f}{\partial x}} \Delta x$$

$$\Delta f = \underbrace{\Delta x^T Q x}_{x^T Q^T \Delta x} + x^T Q \underline{\Delta x}$$

$$= x^T (Q + Q^T) \Delta x$$

$$\frac{\partial f}{\partial x} = x^T (Q + Q^T)$$

$$\frac{\partial f}{\partial x} = 2x^T Q$$

always assume Q is symmetric

Chain Rule:

$g(y)$, $f(x)$ $\qquad \Delta g = \frac{\partial g}{\partial y}\Delta f \qquad \Delta f = \frac{\partial f}{\partial x}\Delta x$

$g(f(x)) \qquad \Delta g = \underbrace{\frac{\partial g}{\partial y}\frac{\partial f}{\partial x}}_{\frac{\partial g}{\partial x}}\Delta x$

$\downarrow$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}$$

$f(x) = e^{x^T Q x} \qquad g(\cdot) = e^{(\cdot)} \qquad f(x) = x^T Q x$

$\uparrow$

Gaussian form

$$\frac{\partial g}{\partial y} = e^{(\cdot)} \qquad \frac{\partial f}{\partial x} = x^T(Q+Q^T)$$

$$\frac{\partial g}{\partial x} = e^{(x^T Q x)} x^T(Q+Q^T)$$

$$= [e^{*}]\,[\ x^T\ ]\,[Q+Q^T]$$

$$f(x) = \begin{bmatrix} e^{x^T Q x} \\ \sin(r^T x) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \end{bmatrix} = \begin{bmatrix} [e^{*}]\, x^T(Q+Q^T) \\ \cos(r^T x)\, r^T \end{bmatrix}$$