NETWORK FLOW PROBLEMS

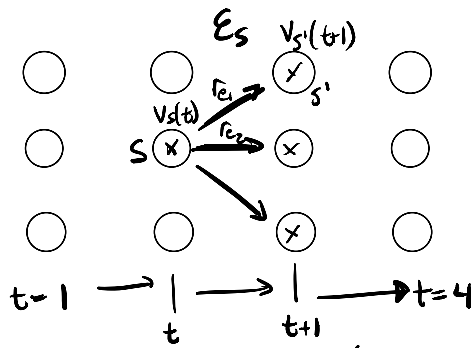
- SHORTEST PATH LP (QP, CONVEX)
- MARKOV DECISION PROCESSES LP

ROUTING GAMES

Dynamic Programming

- "Cost-to-go"
  (Reward-to-go)
  Value function

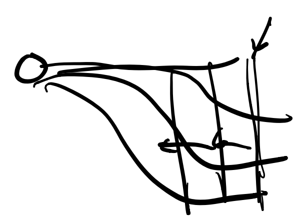
Deterministic



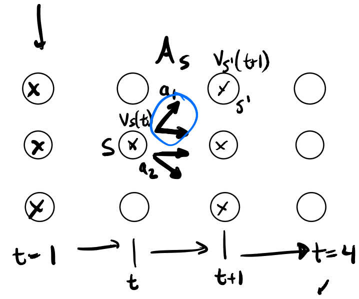
Bellman Eqn:

$$V_s(t) = \max_{e \in \mathcal{E}_s} \{ r_e + \underbrace{V_{s'}(t+1)}_{e: s \to s'} \}$$

$e_{opt}(t) \in \mathcal{E}_s = \arg\max$



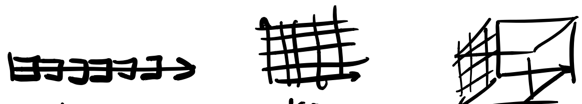
Stochastic



Model our position as a prob. distribution over the states

$$V_s(t) = \max_{a \in A_s} \{ r_a^t + \sum_{s'} P(s' | s,a) V_{s'}(t+1) \}$$

$P(s'|s,a)$ prob of transitioning to s' if you take action a in state s



16 —— 160 1000 4D 5D

10^4 10^5

Dynam Prog : → reachability quadrotor

1

6 DOF

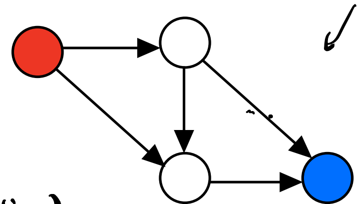
→ 12 states

SHORTEST PATH PROBLEMS:

Graph (directed)

$G = (S, \mathcal{E})$

states edges (have direction)



$|S|$: # of states $|\mathcal{E}|$: # of edges

● : $s_o \in S$ origin node

● : $s_d \in S$ destination node

Ea. edge has a travel cost : $c_e \in \mathbb{R}_+$

GRAPH MATRICES:

Incidence Matrices

- $E_o \in \mathbb{R}^{|S| \times |\mathcal{E}|}$ $[E_o]_{se} = \begin{cases} 1; & \text{if } e \text{ originates at } s \\ 0; & \text{otherwise} \end{cases}$

- $E_i \in \mathbb{R}^{|S| \times |\mathcal{E}|}$ $[E_i]_{se} = \begin{cases} 1; & \text{if } e \text{ terminates in } s \\ 0; & \text{otherwise} \end{cases}$

→ $\boxed{E = E_i - E_o}$

↳ "node edge incidence matrix"

SIDE NOTE : Laplacian $L = EE^T \in \mathbb{R}^{|S| \times |S|}$ → Grammian of E

eigenvalues of L, consensus dynamics

"how does information spread through graph

$\dot{x} = -L\,x$

Laplacian : "shape of a graph"

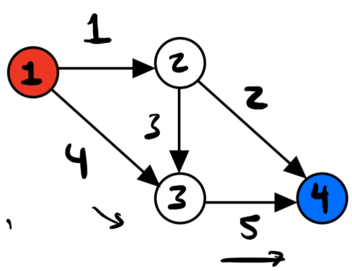
$(EE^T)^{1/2}$: shape of a graph

$$E = \underbrace{(EE^T)^{1/2}}_{\text{shape of rows}} \; \underbrace{(EE^T)^{-1/2}E}_{\text{rows are orthonormal}}$$

SVD of $E = U \Sigma V^T$

$$\underbrace{(EE^T)^{1/2} = U\Sigma U^T}_{\text{positive def.}} \qquad (EE^T)^{-1/2}E = \underbrace{UV^T}_{\text{rotation}}$$

$$E = 4\begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



always not full row rank...

$\underset{\text{vec 1's}}{\mathbb{1}^T} E = \underset{\text{vec 0's}}{0^T}$ $\qquad \mathbb{1}^T E_o = \mathbb{1}^T$, $\quad \mathbb{1}^T E_i = \mathbb{1}^T$

$x \in \mathbb{R}_+^{|E|}$ mass flow on edges

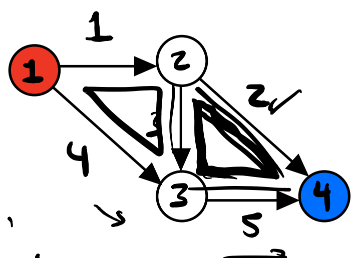
$Ex = 0$: conservation of mass at eq. state $\Leftarrow$

Ex. $\quad [0 \;\; 0 \;\; 1 \;\; 1 \;\; -1]\,x$

$x_3 + x_4 - x_5 = 0 \implies x_5 = x_3 + x_4$

$Ex = 0 \implies$

x is a linear combination of cyclic flows



C : set of cycles of graph

cycle: directed flow around a loop of the graph

Constructed a basis for nullspace of E

$E\,\mathbf{C} = 0 \quad \text{span}(\mathbf{C}) = \mathcal{N}(E)$

$\Rightarrow$ the cols of $\mathbf{C}$ will be indicator vectors for cycles of the graph..

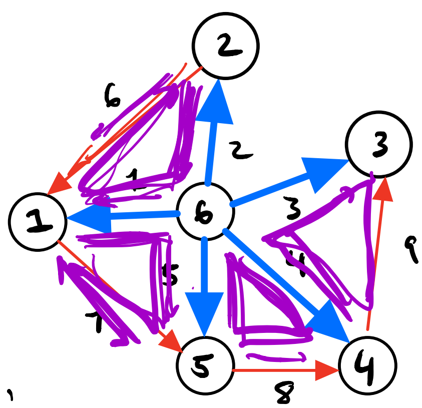
$\underline{\mathbf{C}} \in \mathbb{R}^{|\mathcal{E}| \times |C|}$ $\quad [\mathbf{C}]_{ec} = \begin{cases} 1 & \text{if edge } e \text{ is in cycle } c \text{ and points w/ the cycle} \\ -1 & \text{if } e \text{ is in cycle } c \text{ and points against the cycle} \\ 0 & \text{otherwise} \end{cases}$

construction of a cycle indicator:

Before: $A = [\underbrace{A_1}_{\text{lin ind cols}} \ \underbrace{A_2}_{\text{lin dep cols}}]$

$A = A_1 [I \ \underbrace{B}_{\text{coeffs of } A_2 \text{ w.r.t. } A_1}]$

$= [A_1 \ \underbrace{A_1 B}_{A_2}]$



$E =_{|S|} [\overbrace{\underbrace{E_1}_{\text{lin ind.}}}^{|S|-1} \ E_2]$

spanning tree

$\rightarrow$ spanning : set of edges that touch ea. node

tree : no cycles

$= \underline{E_1} [\underline{I} \ C]$

$\hookrightarrow$ coeffs of the cols of E_2 w.r.t. E_1

c

$E_2 = E_1 C$

$\vert \ , \ ,$

$$E = \Big[\underbrace{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{matrix}}_{E_1} \; \underbrace{\begin{matrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}_{E_2}\Big]$$

Every edge in E_2 corresponds to the one missing from a cycle in the graph

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix} \Big[I \; \underbrace{\begin{matrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{matrix}}_{C} \Big]$$

each col is identified w/ a cycle

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$E = E_1 [I \; C]$$

basis for nullspace of E $\Rightarrow$ $\underbrace{\begin{bmatrix} C \\ -I \end{bmatrix}}$

cols are basis for a nullspace

ea. col of $\begin{bmatrix} C \\ -I \end{bmatrix}$ is an indicator vector for a cycle

$$\mathbb{C} = \begin{bmatrix} C \\ -I \end{bmatrix}$$

$$\{x \mid Ex = 0\} = \left\{x \;\middle|\; x = \begin{bmatrix} C \\ -I \end{bmatrix} z \right\}$$

source - sink vector
(Origin - destination)

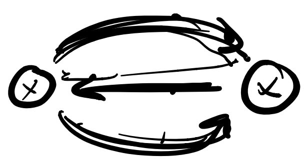
$b \in \mathbb{R}^{|S|}$

$$b_s = \begin{cases} -1 & \text{if } s \text{ source / origin} \\ 1 & \text{if } s \text{ sink / dest.} \\ 0 & \text{otherwise} \end{cases}$$

$[\ Ex = bm \Leftarrow$ set of edge flows from the source to the sink (+ cyclic flows)

ea. row is mass conservation at a node

m : total amount of mass through network



SHORTEST PATH LP:

cost of traveling on edge $e \in \mathcal{E}$ is $c_e > 0$

$c \in \mathbb{R}_+^{|\mathcal{E}|}$

$$\begin{cases} \min\limits_{x \in \mathbb{R}^{|\mathcal{E}|}} \sum\limits_e c_e x_e = c^T x \\ \text{s.t. } Ex = b, \quad x \geq 0 \end{cases}$$

$m = 1$

$Ex = b$: conservation of mass from source to sink

$x \geq 0$: ensures flow follows direction of edges

$x \in \mathbb{R}^{|\mathcal{E}|}$: indicates which edges we travel on.

$Ex = b \Leftarrow$ guarantees all mass flows from source to sink
(can't just pick random edges)

$x:$ $\begin{bmatrix}1\\0\\ \vdots\\0\\1\end{bmatrix} \to$ one route $\quad \begin{bmatrix}0.5\\0.5\\ \vdots\end{bmatrix} + \begin{bmatrix}0.5\\ \\0.5\end{bmatrix} \to$ 2 routes $\quad \begin{bmatrix}0.3\\0.3\end{bmatrix} + \begin{bmatrix}0.7\\0.7\end{bmatrix}$

$\mathbb{1}^T x \neq 1$

$$\min_x \; c^T x$$
s.t. $Ex = b$, $x \geq 0$

dual $\quad v \in \mathbb{R}^{|S|} \quad \mu = \mathbb{R}_+^{|E|}$

$\frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial v} = 0 \quad \frac{\partial L}{\partial \mu} \geq 0$

OPTIMALITY (KKT)

Stationarity : $c^T + v^T E - \mu^T = 0$

Feasibility : $Ex = b$

Positivity : $x \geq 0$, $\mu \geq 0$

slackness : $\mu^T x = 0 \quad (\mu_e x_e = 0)$

Lagrangian : $\mathcal{L}(x, v, \mu) = c^T x + v^T(Ex - b) - \mu^T x$

Intuition:

c: cost vector

x: edge flow

v: value func.

μ_e: ineff. of edge e

$c^T + v^T E - \mu^T = 0$ one eqn for ea. edge

$v^T E[\;]$

$e: s \to s'$ $\quad v_s\, \underset{s}{\circ} \xrightarrow{e} \underset{s'}{\circ}\, v_{s'}$

$\begin{bmatrix}0\\1\\-1\end{bmatrix}$

$$\to \left[c_e + v_{s'} - v_s - \mu_e = 0 \right] \; \forall e \in \mathcal{E}$$

Sum this eqn over edges in a route from origin to dest. $\Leftarrow$ $v_s = c_e + v_{s'} - \mu_e$ → ineff.

c_e: cost on edge e $\quad v_{s'}$: cost-to-go from s'

route : $r \subset \mathcal{E}$

$$\sum_{e \in r} (c_e + v_{s'} - v_s - \mu_e) = 0$$

$$\sum_{e \in r} c_e + \sum_{e \in r} v_{s'} - v_s - \sum_{e \in r} \mu_e = 0$$

$$\sum_{e\in r} c_e + \boxed{v_d - v_o} - \sum_{e\in r} \mu_e = 0$$

$\sum_{e\in r} c_e$: total travel cost

$v_d - v_o$: diff in cost to go from origin to dest.

$\sum_{e\in r}\mu_e$: sum of coeffs.

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_4$$

$$-v_1 + \cancel{v_2} - \cancel{v_2} + \cancel{v_3} - \cancel{v_3} + v_4$$

Complementary slackness

$$\mu_e x_e = 0$$

$$x_e \geq 0$$

$$\sum_{e\in r} c_e = \boxed{v_o - v_d} + \boxed{\sum_{e\in r}\mu_e}$$

$\sum_{e\in r} c_e$: total travel cost

$v_o - v_d$: cost to go from origin

for an optimal route r^* s.t. $x_e^* > 0$ for all $e \in r^*$

$$\mu_e x_e^* = 0 \quad x_e > 0 \implies \mu_e = 0$$

$$\sum_{e\in r^*} c_e = \boxed{v_o - v_d} + \cancel{\sum_{e\in r^*}\mu_e}$$

suboptimal route r s.t. $x_e = 0$ for some $e \in r$

$$\sum_{e\in r} c_e = \boxed{v_o - v_d} + \underbrace{\sum_{e\in r}\mu_e}_{\geq 0} \quad \text{for } x_e = 0$$

μ_e can be greater than 0

$$c^T + v^T E - \mu^T = 0 \Leftarrow$$

take feasible flow x. $\Rightarrow x \geq 0 \quad \boxed{Ex = b}$

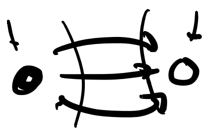
$$c^T x + v^T E x - \mu^T x$$

$$c^T x + v^T b - \mu^T x \implies \underbrace{c^T x}_{\sum_{e\in r} c_e x_e} = -\underbrace{v^T b}_{v_o - v_d} + \underbrace{\mu^T x}_{0 \geq 0}$$

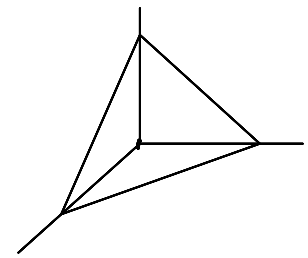
Pictures

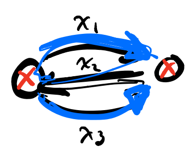
$f(x) = c^T x \qquad \frac{\partial f}{\partial x} = \underline{c^T}$
const.

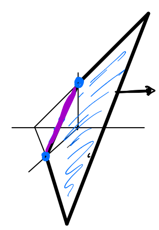
$$\min_x \; c^T x$$
$$\text{s.t.} \; \boxed{Ex = b} \; \boxed{x \geq 0}$$

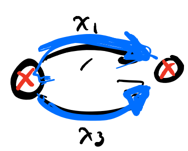


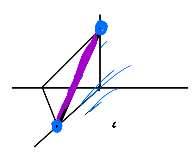
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$(1\;1\;1)\,x = 1$$

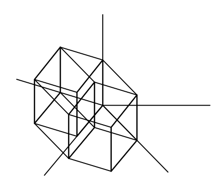


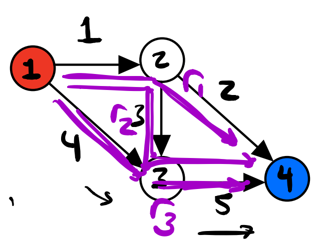


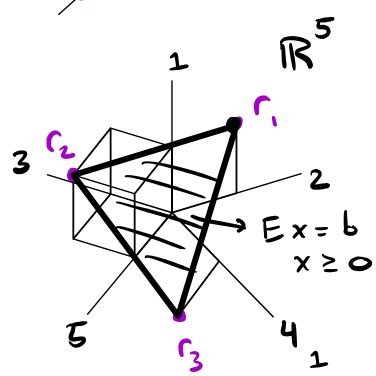


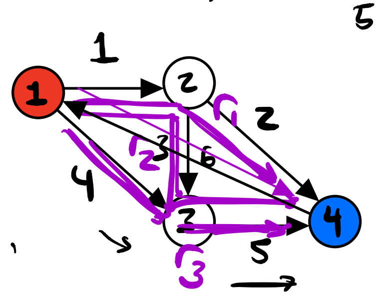


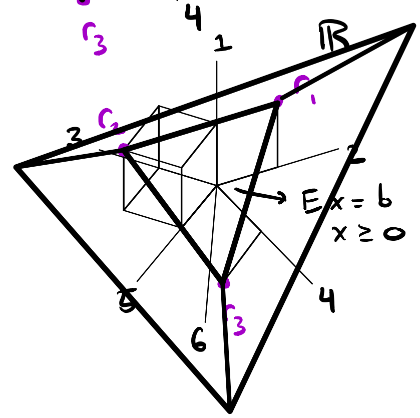




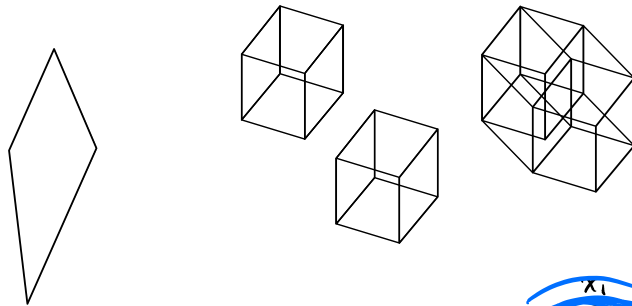


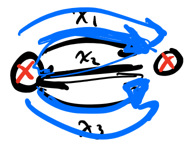






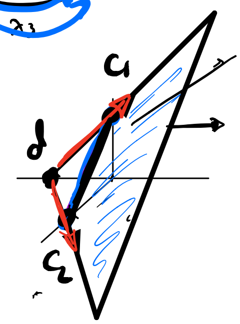
4 nodes :
complete graph w/
4 nodes





$$\min_{x} \; c^T x \quad \text{s.t.} \quad Ex = b \quad x \geq 0$$

$$\mathbb{C} = \begin{bmatrix} C \\ -I \end{bmatrix}$$



$$\min_{x,z} \; c^T x \quad \text{s.t.} \quad x = \mathbb{C} z + d \quad x \geq 0$$

d : $Ed = b$ $\qquad \mathbb{C} = [c_1 \; c_2]$

$\Downarrow$

list out the routes $\qquad \Leftarrow$ Not equivalent

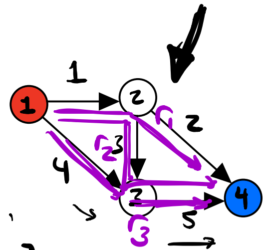
$R_1, R_2 \qquad \mathbb{R} = [R_1 \, R_2]$

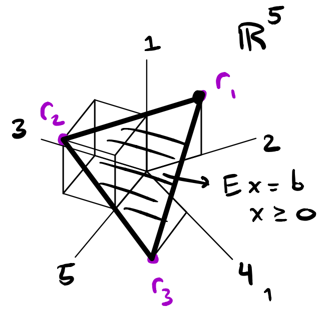
$$\min_{x,z} \; c^T x \quad \text{s.t.} \quad x = \mathbb{R} z \quad \mathbb{1}^T z = 1 \quad z \geq 0$$

$$\min_{z} \; c^T \mathbb{R} z \quad \text{s.t.} \quad \mathbb{1}^T z = 1 \quad z \geq 0$$

$\rightarrow$ explicitly listed routes

$$\boxed{\begin{aligned}&\min_{z}\ \underline{c^T\mathbb{R}}z \Leftarrow\\ &\text{s.t. } [\mathbb{1}^T z = 1,\ z \geq 0\end{aligned}}$$





$\underline{c}^T\mathbb{R} = \ell^T = [c^TR_1\ c^TR_2\ c^TR_3]$

$\ell \in \mathbb{R}^{|\mathcal{R}|}$

ℓ_R: cost of taking route R

$\underline{\mathbb{R}} = [R_1 R_2 R_3]$

$\mathcal{R}$: set of routes

Dual of Edge Flow optimization

$$\min_{x}\ c^Tx \leftarrow$$
$$\text{s.t. } Ex = b,\ x \geq 0$$

$$L(x,v,\mu) = c^Tx + v^T(Ex-b) - \mu^Tx$$

$$\begin{aligned}\min_x c^Tx \\ \text{s.t. } Ex=b,\ x\geq 0\end{aligned} = \min_x \Big(\underbrace{\max_{v,\mu \geq 0}\ c^Tx + v^T(Ex-b) - \mu^Tx}_{\text{func of } x} \Big)$$

$$\geq \max_{v,\mu\geq 0} \Big(\underbrace{\min_x\ c^Tx + v^T(Ex-b) - \mu^Tx}_{\text{func of } v,\mu} \Big) = \begin{aligned}&\max_{v,\mu\geq 0}\ \_\_\\ &\text{s.t. } \underline{\ *\ }\end{aligned}$$

$$\longrightarrow$$

$$\Big(\min_x\ c^Tx + v^T(Ex-b) - \mu^Tx \Big) \Rightarrow \underline{c^T + v^TE - \mu^T} = 0 \ *$$

dual objective

dual constraint

{ func of v,μ

$$0\,(\underline{c^T + v^TE} - \mu^T)x - \boxed{v^Tb} - \underline{\mu^T}x$$

Dual Problem:

$$\boxed{\begin{array}{ll} \max\limits_{v,\mu\geq 0} & -v^T b = v_o - v_d \\ \text{s.t.} & c^T + \underline{v^T E} - \mu^T = 0 \quad \underline{\mu \geq 0} \end{array}}$$

V: cost to go at ea state

μ: ineff. of edge

$$\begin{array}{ll} \max\limits_{v,\mu\geq 0} & -v^T b = v_o - v_d \\ \text{s.t.} & c^T + \underline{v^T E} - \mu^T = 0 \quad \mu \geq 0 \\ & v^T E_i - v^T E_o \end{array}$$

max difference in value func between origin & dest.

$$\boxed{v^T E_o = c^T + v^T E_i - \mu^T}$$

More next week

for ea. edge

$$s \xrightarrow{e} s'$$

$$v_s = \underline{c_e + v_{s'}} - \underline{\mu_e}, \quad \underline{\mu_e \geq 0}$$

like * at every state

$$\rightarrow \boxed{v_s \leq c_e + v_{s'}}$$

cost to go at state s $\leq$ immediate future edge cost + cost to go at the next state s'

like the Bellman eqn but forces V_s to be a lower bound on the cost to go.

if we explicitly list out the routes...

$$\min_{z} \; c^T R z \quad \text{s.t. } \mathbb{1}^T z = 1,\; z \geq 0 \quad = \quad \min_{z} \left(\max_{\lambda, u \geq 0} c^T R z - \lambda(\mathbb{1}^T z - 1) - u^T z \right)$$

$\lambda \in \mathbb{R}$ $\quad u \in \mathbb{R}^{|R|}$

$l^T = c^T R$

$$\geq \max_{\lambda, u \geq 0} \left(\min_{z} \; c^T R z - \lambda(\mathbb{1}^T z - 1) - u^T z \right)$$

$$c^T R - \lambda \mathbb{1}^T - u^T = 0$$

Dual problem

$$\max_{\lambda, u \geq 0} \; \lambda$$

$$\text{s.t. } \lambda \mathbb{1}^T + u^T = c^T R \qquad u \geq 0$$

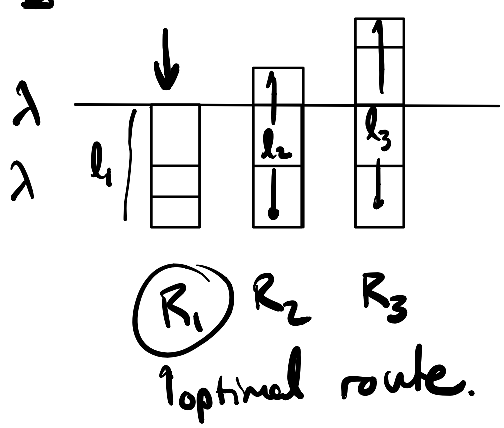
$l = [l_1 \; l_2 \; l_3]$

$$\left[\begin{array}{l} \max_{\lambda} \; \lambda \\ \text{s.t. } \lambda \mathbb{1}^T \leq c^T R = l^T \end{array} \right.$$

cost for ea route

$$\lambda \leq l_r$$

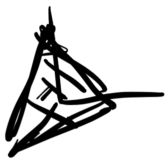
Similar to the simplex setup we did last week



From last lecture

simplex.

$$\max_{z} \ \ell^T z$$
$$\text{s.t.} \ \ \mathbb{1}^T z = 1, \ z \geq 0$$



dual

$$\min_{\lambda} \ \lambda$$
$$\text{s.t.} \ \ \lambda \mathbb{1}^T \geq \ell^T$$