Linearity:

$f(x)$ linear $\left[f(ax + bx') = af(x) + bf(x')\right]$

convex $f(ax+bx') \leq af(x)+bf(x')$

$a+b = 1 \quad a,b \geq 0$

concave $f(ax+bx') \geq af(x)+bf(x')$

$a+b = 1 \quad a,b \geq 0$

linear constraint:

$g(x) = 0$ $g$ is linear $\Rightarrow$ linear equality constraint

$g(x) \geq 0$ $g$ is linear $\Rightarrow$ linear inequality constraint

Matrices:

$f(x) \quad f: \mathbb{R}^n \to \mathbb{R}^m$ (basis, basis) $f$ is linear ]

there is a way to represent $f$ as a matrix...

$f(x) = Ax$

for some $A$. (dependent on bases that you pick)

---

Constraint:

$g(x) = Ax = 0$ linear eq. constraint

$g(x) = Ax \geq 0$ linear inequality constraint

$\underline{x}^T A \underline{x} \geq 0 \to$ not linear.

$\boxed{\text{Affine constraint:}}$ sometimes, call linear,

$\to Ax = b \longrightarrow g(x) = b$

$Cx \geq d \qquad g(x) \geq d$

Philosophical:

Modeling...

when

Physics $\to$ Nonlinear Eqns $\to$ Linear model

Nonlinear non convex $\to$ Convex model

Interior Pt Methods

$$\min_x \; f(x) \qquad \longrightarrow \qquad \min_x \; t f(x) - \mu \sum_i \ln(s_i)$$

$$\text{s.t.} \quad g(x) = 0 \qquad\qquad g(x) = 0$$

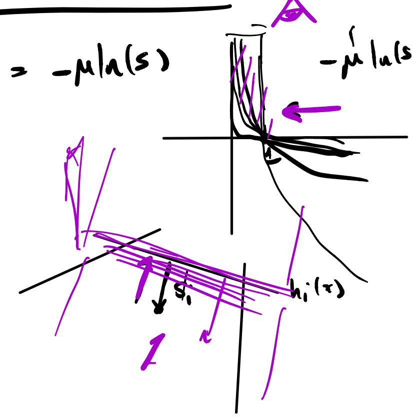
$$h(x) \geq 0 \;\longrightarrow\; h(x) = s, \; \boxed{\cancel{s \geq 0}}$$

$\mathbb{R}^n$ $\begin{bmatrix} x \\ h(x) \end{bmatrix}$ $\mu > 1$

Lagrangian:

$$L(x,s,v,w) = t f(x) - \mu \sum_i \ln(s_i) + v^T g(x) + w^T (h(x) - s)$$

$y = -\mu \ln(s)$



# Newton's Method on Lagrangian

$$\frac{\partial L}{\partial x} = t\frac{\partial f}{\partial x} + v^T\frac{\partial g}{\partial x} + w^T\frac{\partial h}{\partial x} \leftarrow$$

1st Derivatives

$$\left(\frac{\partial L}{\partial s_i} = -\mu\frac{1}{s_i} - w_i\right)$$

$$\frac{\partial L}{\partial s} = -\mu \mathbb{1}^T \operatorname{dg}(s)^{-1} - w^T \leftarrow$$

$$\frac{\partial L}{\partial v} = g(x)$$

$$\frac{\partial L}{\partial w} = h(x) - s$$

---

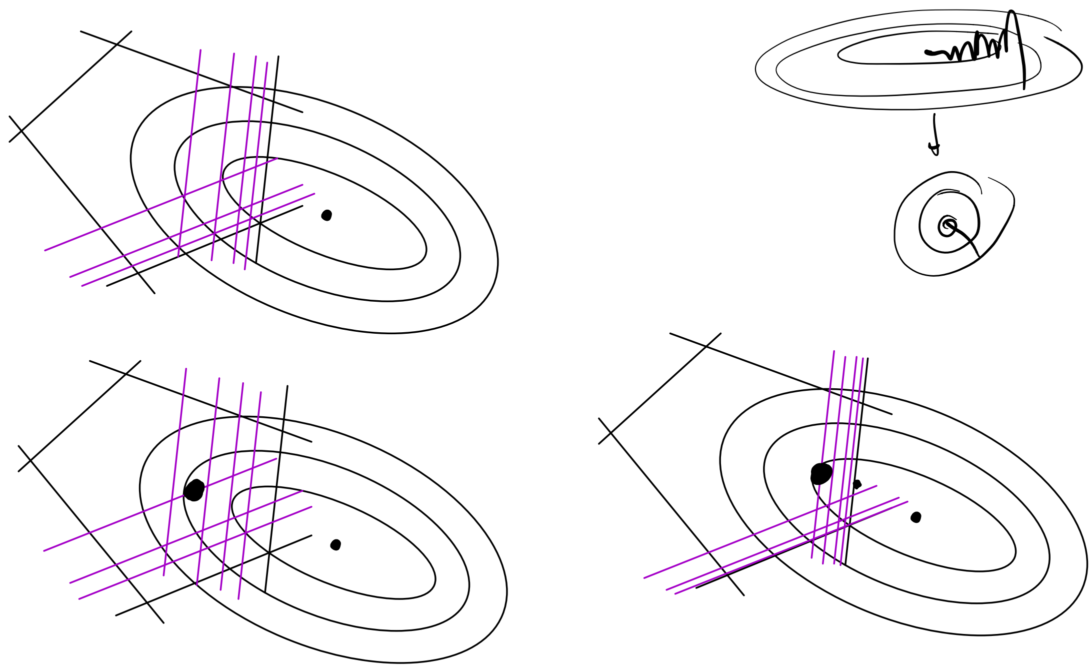
$$\frac{\partial^2 L}{\partial x,s,v,w^2} = \begin{bmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial s} & \frac{\partial^2 L}{\partial x \partial v} & \frac{\partial^2 L}{\partial w} \\ \frac{\partial^2 L}{\partial x \partial s} & \frac{\partial^2 L}{\partial s^2} & \cdot & \cdot \\ \frac{\partial^2 L}{\partial x \partial v} & \cdot & \frac{\partial^2 L}{\partial v^2} & \cdot \\ \frac{\partial^2 L}{\partial w} & \cdot & \cdot & \frac{\partial^2 L}{\partial w^2} \end{bmatrix} \leftarrow$$

2nd derivatives

$$H = \begin{bmatrix} t\frac{\partial^2 f}{\partial x^2} + \sum_i v_i \frac{\partial^2 g}{\partial x^2} + \sum_i w_i \frac{\partial^2 h}{\partial x^2} & 0 & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ 0 & +\mu \operatorname{dg}(s)^{-2} & 0 & -I \\ \partial g/\partial x^T & 0 & 0 & 0 \\ \partial h/\partial x^T & -I & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x^+ \\ s^+ \\ v^+ \\ w^+ \end{bmatrix} = \begin{bmatrix} x \\ s \\ v \\ w \end{bmatrix} - \gamma H^{-1} \begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial s} \\ \frac{\partial L}{\partial v} \\ \frac{\partial L}{\partial w} \end{bmatrix} \leftarrow$$

iterate on this eqn till converges.



$\mu \uparrow$ $\qquad\qquad$ $\mu \to 0$

relative magnitude of $\left[ f(x) \ \& \ -\mu \sum_i \ln(s_i) \right]$ determines how close you are to the boundary

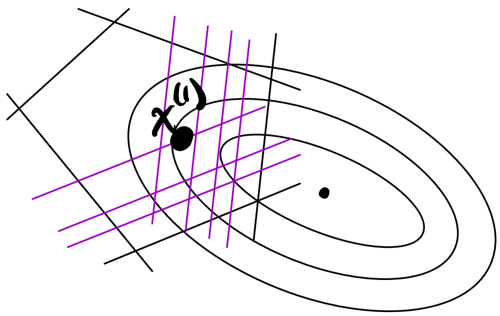
→ reduce $\mu$ to get closer to the boundary

0.001 0.0001

Better idea:
increase the magnitude of $f$

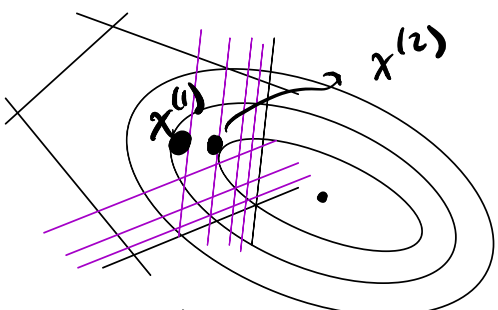
$f(x) \to t f(x)$ $\quad t \geq 0 \quad$ $t$ grows...

- set $\mu > 1$ :   $t = 1$
- run Newton's → till converges



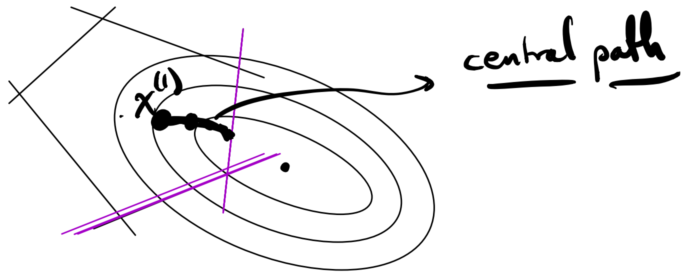

$\mu \uparrow$

- update $t^{+} = \mu t$
- run Newton's method again.
  initialize at $x^{(1)}, s^{(1)}, v^{(1)}, w^{(1)}$

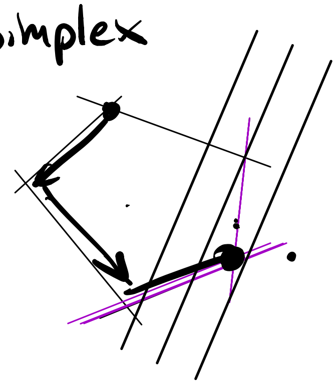


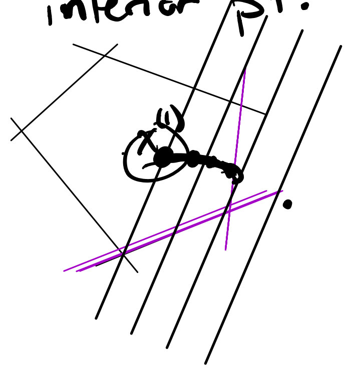

- update $t^{+} = \mu t$
- run Newton's method again.
  initialize at $x^{(2)}, s^{(2)}, v^{(2)}, w^{(2)}$

$\vdots$ repeat.




outer loop: updates $t$
inner loop: run Newton's to find fixed pt of the Lagrangian.




Simplex Method: LP

standard form:

$$\max_x \; r^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

$$- \min_z \; -c^T z \quad \text{s.t.} \quad Ez = g, \quad Cz \geq d$$

$$x = \begin{bmatrix} z_+ \\ z_- \\ s \end{bmatrix} \qquad x \geq 0$$

$$r^T = [c \;\; -c \;\; 0]$$

Transform

$$Cz - s = d \qquad s \geq 0$$

$$\to z = z_+ - z_- \qquad z_+ \geq 0, \quad z_- \geq 0$$

$$[E \;\; -E] \begin{bmatrix} z_+ \\ z_- \end{bmatrix} = g$$

$$A = \begin{bmatrix} E & -E & 0 \\ C & -C & -I \end{bmatrix} \qquad -c^T z = \begin{bmatrix} -c & +c \end{bmatrix} \begin{bmatrix} z_+ \\ z_- \end{bmatrix}$$

$$b = \begin{bmatrix} g \\ d \end{bmatrix} \qquad \min -(*) = \max(*)$$

$$\max_x \; c^T x = \begin{bmatrix} c & -c & 0 \end{bmatrix} x$$

$$\text{s.t.} \; \underbrace{\begin{bmatrix} E & -E & 0 \\ C & -C & -I \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} g \\ d \end{bmatrix}}_{b} \qquad x = \begin{bmatrix} z_+ \\ z_- \\ s \end{bmatrix} \geq 0$$

Constraints → ineq

$0 \leq z_1 \leq 2$

$0 \leq z_2 \leq 2$

$0 \leq z_3 \leq 3$

$z_1 - z_2 \leq 1$

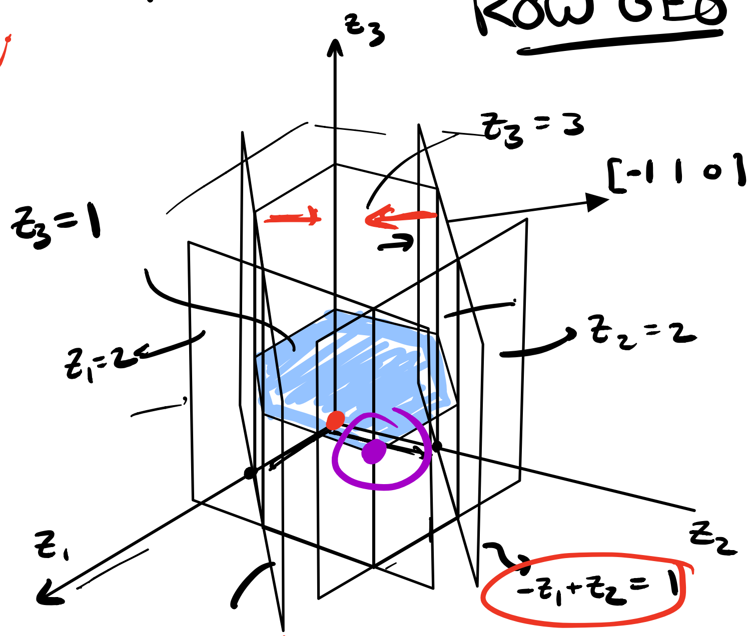
$-z_1 + z_2 \leq 1$

$z_3 = 1$

objective

$\rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

ROW GEO




$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} z = 1$

$z_1 - z_2 = 1$

$0 - 0 \leq 1$

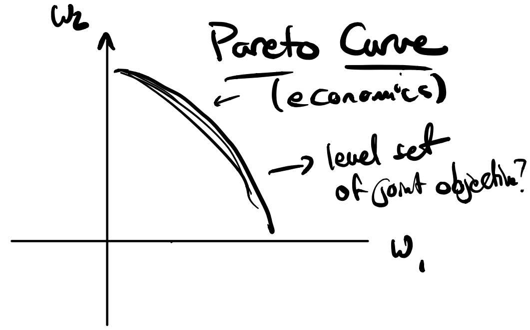
$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 1$

$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix} = z_2$

$-0 + 0 \leq 1$

## Multi objective opt:

$$[w_1\ w_2]\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix} = w_1[1\ 1\ 0]z + w_2[0\ 0\ -1]z$$




## LP solving...

$$\left[\begin{array}{l} 0 \le z_1 \le 2 \\ 0 \le z_2 \le 2 \end{array}\right] \text{ ineq.} \longrightarrow \begin{array}{l} z_1 \ge 0 \\ z_2 \ge 0 \\ z_3 \ge 0 \\ -z_1 \ge -2 \\ -z_2 \ge -2 \\ -z_3 \ge -3 \\ -z_1 + z_2 \ge -1 \\ +z_1 \mp z_2 \ge -1 \end{array}$$

$$[\, 0 \le z_3 \le 3 \,] \rightarrow \text{useless}$$

$$\left[\begin{array}{l} z_1 - z_2 \le 1 \\ -z_1 + z_2 \le 1 \end{array}\right] \text{ ineq}$$

$$[\, z_3 = 1 \,] \text{ eq.} \leftarrow$$

$$z_2 \le 2.$$
$$-z_2 \ge -2.$$
$$-z_2 + s$$
$$-z_2 = -2 + s$$
$$-z_2 - s = -2$$
$$-z_2 - s = -2$$
$$\boxed{z_2 + s = 2}$$

objective

$$\rightarrow [1\ 1\ 1]\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix}$$

max $[1\ 1\ 1]z$

s.t. $\boxed{[0\ 0\ 1]z = 1}$

$\underline{Cz \ge d.}$ $\Rightarrow$

$$\underbrace{\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\\ -1 & 1 & 0\\ 1 & -1 & 0\end{bmatrix}}_{C} z \ge \underbrace{\begin{bmatrix}-2\\ -2\\ -3\\ -1\\ -1\end{bmatrix}}_{d}$$

$1$

$$\max_{x} \; \overset{d}{\downarrow}\, r^T x \qquad \swarrow$$

$$\text{s.t.} \quad Ax = b \quad \underline{x \geq 0}$$

$$Cz \geq d \qquad z \geq 0$$

$$Cz - s = d \qquad z \geq 0, \; s \geq 0$$

$$x = \begin{bmatrix} z \\ s \end{bmatrix}$$

$$\boxed{[C \;\; -I]\, x = d}$$

$$\underbrace{\begin{bmatrix} 0\;0\;1 & -0- \\ C & -I \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ d \end{bmatrix}}_{b}$$

$$r^T x = [\,\underbrace{1\;1\;1}_{r}\;\; -0-\,]\begin{bmatrix} z \\ s \end{bmatrix}$$

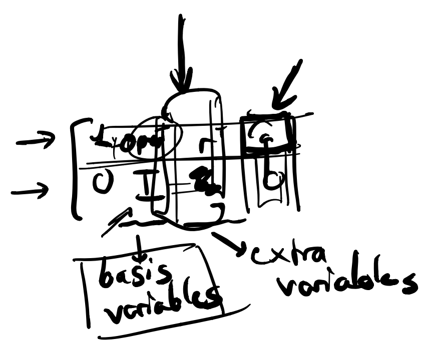
---

$$\max_{x} \; [1\;1\;1\;0\;0\;0\;0\;0]\, x \qquad \swarrow$$

$$\text{s.t.} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & & & & 0 \\ 0 & -1 & 0 & & \ddots & & & \\ 0 & 0 & -1 & & & & & \\ -1 & 1 & 0 & & & & & \\ 1 & -1 & 0 & 0 & & & & -1 \end{bmatrix} x = \begin{bmatrix} 1 \\ -2 \\ -2 \\ -3 \\ -1 \\ -1 \end{bmatrix} \qquad x \geq 0$$

TABLEAU: (Matrix)

$$\begin{bmatrix} 1 & -r^T & 0 \\ \hline 0 & \underbrace{A}_{x} & \underset{\uparrow}{b} \end{bmatrix} \begin{matrix} \leftarrow \text{objective} \\ \leftarrow \text{constraints} \end{matrix}$$




$$\begin{bmatrix} 1 & -r^T & 0 \\ \hline 0 & \underbrace{A}_{x} & \underset{\uparrow}{b} \end{bmatrix} = \begin{bmatrix} 1 & [-1\;-1\;-1\;0\;0\;0\;0\;0] & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & & & & 0 \\ 0 & -1 & 0 & & \ddots & & & \\ 0 & 0 & -1 & & & & & \\ -1 & 1 & 0 & & & & & \\ 1 & -1 & 0 & 0 & & & & -1 \end{bmatrix} & \begin{bmatrix} 1 \\ -2 \\ -2 \\ -3 \\ -1 \\ -1 \end{bmatrix} \\ \vdots & & \\ 0 & & \end{bmatrix}$$

Starting point

Big Picture:

ways we can improve the objective

$$
\left[\begin{array}{c|cccccccc|c}
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\vdots & -1 & 0 & 0 & -1 & & & & 0 & -2 \\
\vdots & 0 & -1 & 0 & & \ddots & & & & -2 \\
\vdots & 0 & 0 & -1 & & & \ddots & & & -3 \\
\vdots & -1 & 1 & 0 & & & & \ddots & & -1 \\
0 & 1 & -1 & 0 & 0 & & & & -1 & -1
\end{array}\right]
$$

row reduction

$$
\left[\begin{array}{c|cccccc|cc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & r^T x \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & * & * & + \\
\vdots & & +1 & & & & 0 & \vdots & \vdots & \vdots \\
\vdots & & & \ddots & & & & & & \\
0 & & & & & & +1 & * & * & +
\end{array}\right]
$$

$r^T x$ — at optimum

$Ax = b$ always needs to be true

basis variables → extra

$x$

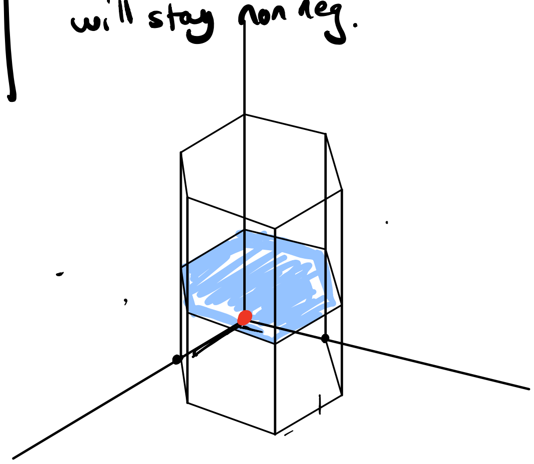
Start:

$$
\left[\begin{array}{c|cccccccc|c}
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\vdots & -1 & 0 & 0 & -1 & & & & 0 & -2 \\
\vdots & 0 & -1 & 0 & & \ddots & & & & -2 \\
\vdots & 0 & 0 & -1 & & & \ddots & & & -3 \\
\vdots & -1 & 1 & 0 & & & & \ddots & & -1 \\
0 & 1 & -1 & 0 & 0 & & & & -1 & -1
\end{array}\right]
$$

make non negative will stay non neg.

$$
\left[\begin{array}{c|ccc|ccccc|c}
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\vdots & +1 & 0 & 0 & +1 & & & & 0 & +2 \\
\vdots & 0 & +1 & 0 & & \ddots & & & & +2 \\
\vdots & 0 & 0 & +1 & & & \ddots & & & +3 \\
\vdots & +1 & -1 & 0 & & & & \ddots & & +1 \\
0 & -1 & +1 & 0 & 0 & & & & +1 & +1
\end{array}\right]
$$

$z$ $\qquad$ $s$



row5 = row5 − row1 ...

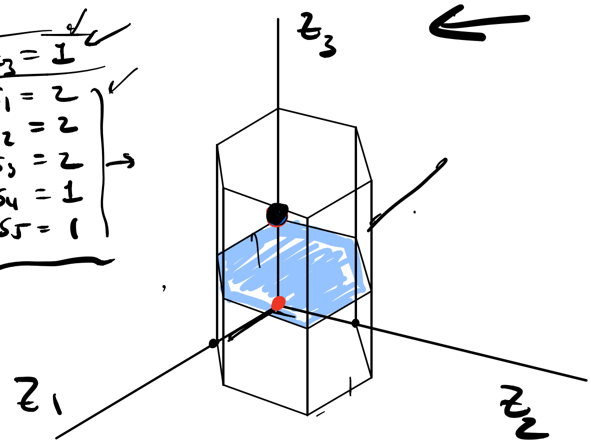
$z_1 \; z_2 \; z_3 \; s_1 \; \cdots \; s_5$

$$\left[\begin{array}{c|cccccccc|c} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \vdots & +1 & 0 & 0 & +1 & & & & 0 & +2 \\ & 0 & +1 & 0 & & +1 & & & & +2 \\ & 0 & 0 & 0 & & & +1 & & & 2 \\ & +1 & -1 & 0 & & & & +1 & & +1 \\ 0 & -1 & +1 & 0 & 0 & & & & +1 & +1 \end{array}\right]$$

$\underbrace{z}$   $\underbrace{1}_{z_3}$   $s$ basis cols

slack variables

$z_3 = 1$
$s_1 = 2$
$s_2 = 2$
$s_3 = 2$
$s_4 = 1$
$s_5 = 1$




cashing out...

row 1 = row 1 + row 2

$$\left[\begin{array}{c|cccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \vdots & +1 & 0 & 0 & +1 & & & & 0 & +2 \\ & 0 & +1 & 0 & & +1 & & & & +2 \\ & 0 & 0 & 0 & & & +1 & & & 2 \\ & +1 & -1 & 0 & & & & +1 & & +1 \\ 0 & -1 & +1 & 0 & 0 & & & & +1 & +1 \end{array}\right]$$

$z_1 \; z_2 \; z_3 \; s_1 \; \cdots \; s_5$

which non basis col can we add to our basis to improve reward ← look for - signs

pick $z_2$ to swap in as a basis variable (arbitrary choice over $z_1$)

entering variable

$$\left[\begin{array}{c|cccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \vdots & +1 & 0 & 0 & +1 & & & & 0 & +2 \\ & 0 & \boxed{+1} & 0 & & \boxed{1} & & & & \boxed{+2} \\ & 0 & 0 & 0 & & & 1 & & & 2 \\ & +1 & -1 & 0 & & & & & & +1 \\ 0 & -1 & \boxed{+1} & 0 & 0 & & & & \boxed{+1} & +1 \end{array}\right]$$

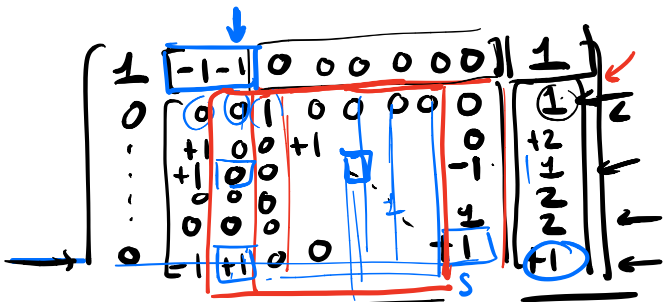
$s$

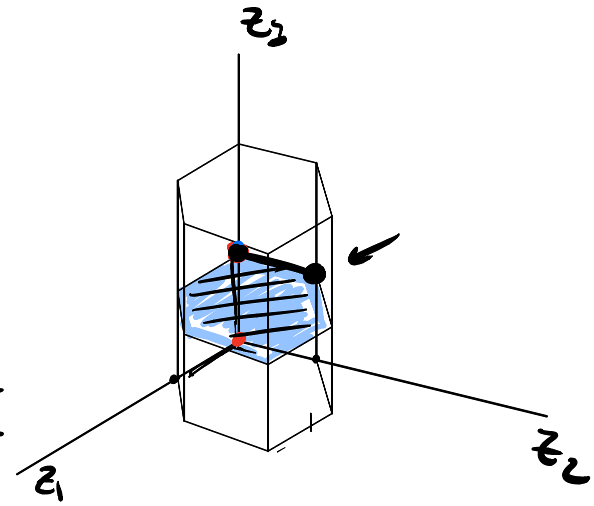
need to pick leaving variable - basis variable to remove as a basis variable

- corresponds to a row.

row 4 = row 4 − row 7

row 6 = row 6 + row 7






$z_2\ z_3\ s_1\ \cdots\ s_4\ 1$

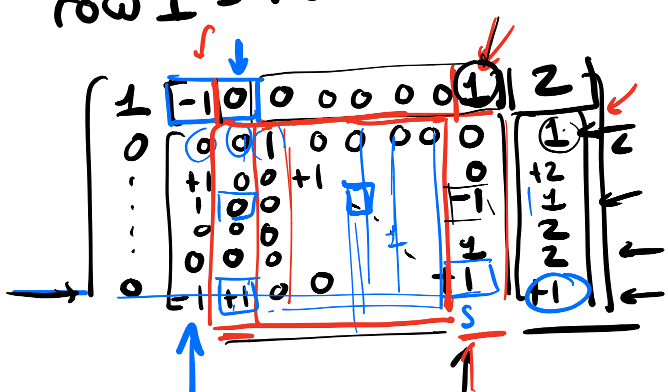
$z_2 = 1$ $z_3 = 1$

cashout

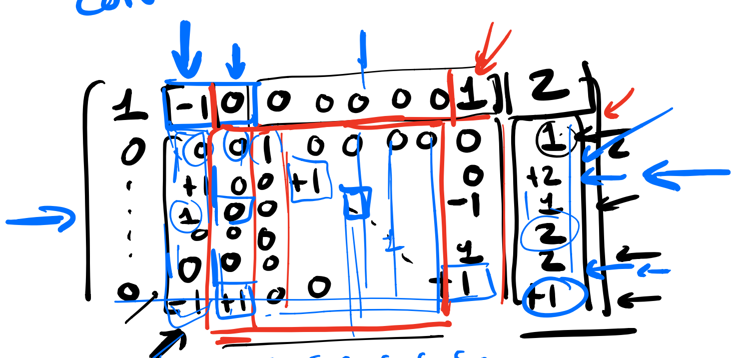
row 1 = row 1 + row 7

$z_2 = 1$ $z_3 = 1$

$1 + 1 = 2$



new column to add...



$z_3\ s_1\ s_2\ s_3\ s_4\ s_5$

# CARNAGE – TRAINWRECK

row 6 = row6 − row 3

row 7 = row 7 + row 3

cash out...

row1 = row1 + row 3

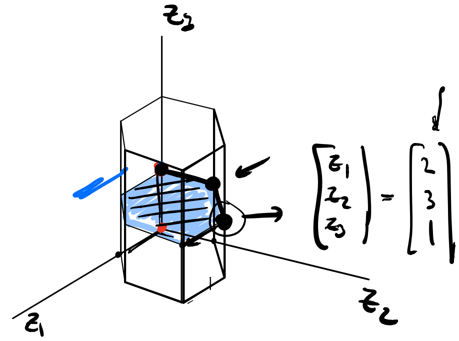
$$\left[\begin{array}{c|cccccccc|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cdot & +1 & 0 & 0 & +1 & & & & 0 & +2 \\ \cdot & 1 & 0 & 0 & & & & & -1 & 1 \\ \cdot & 0 & 0 & 0 & & & & & & 2 \\ \cdot & 0 & 0 & 0 & -1 & & & & 1 & 0 \\ 0 & 0 & +1 & 0 & 1 & & & & +1 & 3 \end{array}\right]$$

$z_1 = 2$ $z_3 = 1$

$z_2 = 3$

$\boxed{A}\boxed{x} = b$

$x = A^{-1} b$




$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Row 3 = Row3 − Row4

Row 7 = Row 7 + Row 4

CASH OUT...

Row 1 = Row 1 + Row 4

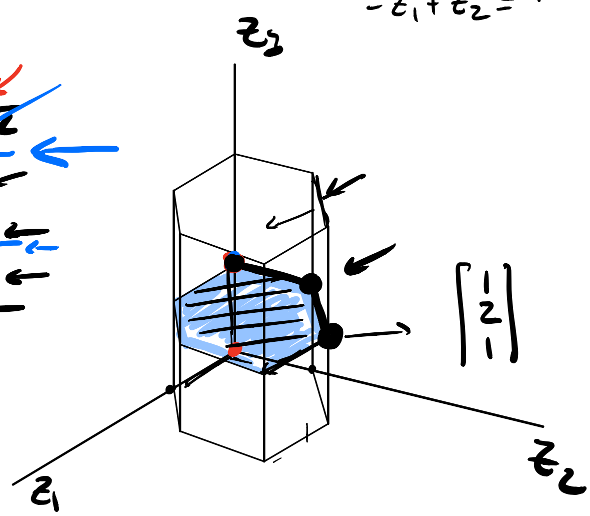
$$\left[\begin{array}{c|cccccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cdot & 0 & 0 & 0 & +1 & -1 & & & 1 & +1 \\ \cdot & 1 & 0 & 0 & & & & & -1 & 1 \\ \cdot & 0 & 0 & 0 & & & & & & 2 \\ \cdot & 0 & 0 & 0 & & & & & 1 & 2 \\ 0 & 0 & +1 & 0 & 0 & 1 & & & 0 & 2 \end{array}\right]$$

$z_3$ $s_1$ $s_2$ $s_3$ $s_4$ $s_5$

$z_1 = 1$ $z_2 = 2$ $z_3 = 1$

$[-1 \quad 1]z = 1$

$-z_1 + z_2 \leq 1$




$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$