Topics

- Continuous time KF
- Cont./Discrete time KF
- Extended (nonlinear) KF
- Unscented KF

] → nonlinear sys.

Continuous Time:

Review

Linear ODES:

Linear time invariant $\boxed{LTI}$

Autonomous: $\dot{x} = Ax, \quad x(t_0) = x_0$

solution: $x(t) = e^{A(t-t_0)}x_0$

Controlled: $\dot{x} = Ax + Bu \quad x(t_0) = x_0$

SOLUTION: $x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{drift}} + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau$

LINEAR TIME VARYING $\boxed{LTV}$

→ Autonomous: $\dot{x} = \underline{A(t)}x \quad x(t_0) = x_0$

Solution $\underline{x(t)} = \underbrace{\phi(t, t_0)}_{\text{state transition matrix}} \overline{x_0}$ like $e^{A(t-t_0)}$

Properties:

- $\Phi(t,t) = I$ for $\forall t$
- $\Phi(t_0,t) = \Phi^{-1}(t,t_0)$
- $\Phi(t_2,t_1)\,\Phi(t_1,t_0) = \Phi(t_2,t_0)$
- $\frac{d}{dt}\Phi = A(t)\,\Phi$

for LTI sys:

$$\Phi(t,t_0) = e^{A(t-t_0)}$$

→ Controlled: $\dot{x} = A(t)x + B(t)u$, $x(t_0) = x_0$

solution: $x(t) = \underbrace{\Phi(t,t_0)x_0}_{\text{drift}} + \int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$

Continuous Time KF:

Dynamics:

$$\dot{x}(t) = F(t)x(t) + B(t)u(t) + G(t)w(t) \qquad w(t) \sim N(0, Q(t))$$

$$\tilde{y}(t) = H(t)x(t) + v(t) \qquad v(t) \sim N(0, R(t))$$

$$E[w(t)w(\tau)^T] = Q(t)\,\delta(t-\tau)$$

$$E[v(t)v(\tau)^T] = R(t)\,\delta(t-\tau)$$

$$E[v(t)w^T(\tau)] = 0$$

Estimator:

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + B(t)u(t) + K(t)\left[\tilde{y}(t) - H(t)\hat{x}(t)\right]$$

$$\hat{y}(t) = H(t)\,\hat{x}(t)$$

How do the error & error covariance update?

error: $\tilde{x} = e = \hat{x} - x$

Computing $\dot{e} = \dot{\hat{x}} - \dot{x}$

$$\boxed{\dot{e}(t) = E(t)e(t) + z(t)}$$

where:

$$E(t) = F(t) - K(t)H(t)$$

$$z(t) = -G(t)w(t) + K(t)v(t)$$

$$E[z(t)z(\tau)^T] = [G(t)Q(t)G(t)^T + K(t)R(t)K^T(t)]\delta(t-\tau)$$

error evolution:

$\phi(t,t_0)$ is the state transition matrix for $E = F - KH$

$$e(t) = \phi(t,t_0)e(t_0) + \int_0^t \phi(t,\tau)z(\tau)\,d\tau$$

$E(e(t_0)\cancel{z}(\tau))$ 0

$E(\cancel{z(\tau_1)}z(\tau_2))$ 0

$\tau_1 \neq \tau_2$

$$P(t) = E[e(t)e(t)^T]$$

$$P(t) = \phi(t,t_0)P(t_0)\phi^T(t,t_0)$$

$$+ \int_{t_0}^t \phi(t,\tau)[G(\tau)Q(\tau)G^T(\tau) + K(\tau)R(\tau)K^T(\tau)]\phi^T(t,\tau)\,d\tau$$

$P(t_0)$ 0

$$E(e(t)e(t)^T) = \Big(\phi \cancel{E}[e_0 e_0^T]\phi^T + \int_0^t \phi(t,\tau)\cancel{E}[z e_0^T]\phi\,d\tau$$

$$+ \phi E[e_0 \int_0^t z^T]\phi^T d\tau$$

$$+ \int_0^t \phi E[z\,d\tau \int_0^t z]\phi^T d\tau\Big)$$

$\dot{P}$ $(\Sigma -)(\Sigma -)$

$L(\Sigma(\cancel{t_1})\Sigma(t_2))$

$\tau_1 \neq \tau_2$

$$\frac{dP(t)}{dt} = \frac{d}{dt}\Big[\Phi(t,t_0)P(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)\big[G(\tau)Q(\tau)G^T(\tau) + K(\tau)R(\tau)K^T(\tau)\big]\Phi^T(t,\tau)\,d\tau\Big]$$

$$\dot{P}(t) = \frac{\partial \Phi}{\partial t}(t,t_0)P(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)P(t_0)\frac{\partial \Phi^T}{\partial t}$$

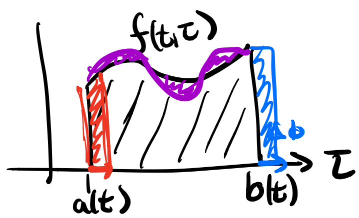
$$+\int_{t_0}^{t}\frac{d\Phi}{dt}(t,\tau)\big[G(\tau)Q(\tau)G^T(\tau) + K(\tau)R(\tau)K^T(\tau)\big]\Phi^T(t,\tau)\,d\tau$$

$$+\int_{t_0}^{t}\Phi(t,\tau)\big[G(\tau)Q(\tau)G^T(\tau) + K(\tau)R(\tau)K^T(\tau)\big]\frac{d}{dt}\Phi^T(t,\tau)\,d\tau$$

$$+\Phi(t,t)\big[G(t)Q(t)G^T(t) + K(t)R(t)K^T(t)\big]\Phi^T(t,t)$$

Leibnitz Rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau = \boxed{\frac{db}{dt} f(t,b(t))} - \boxed{\frac{da}{dt} f(t,a(t))} + \boxed{\int_{a(t)}^{b(t)} \frac{df}{dt}(t,\tau)\,d\tau}$$



$$\dot P(t) = E(t)\Phi(t,t_0)P(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)P(t_0)\Phi^T(t,t_0)E^T(t)$$

$$+ E(t)\int_{t_0}^{t} \Phi(t,\tau)\left[G(\tau)Q(\tau)G^T(\tau) + K(\tau)R(\tau)K^T(\tau)\right]\Phi^T(t,\tau)\,d\tau$$

$$+\int_{t_0}^{t} \Phi(t,\tau)\left[G(\tau)Q(\tau)G^T(\tau) + K(\tau)R(\tau)K^T(\tau)\right]\Phi^T(t,\tau)\,d\tau\, E^T(t)$$

$$+ \overset{I}{\cancel{\Phi(t,t)}}\left[G(t)Q(t)G^T(t) + K(t)R(t)K^T(t)\right]\overset{I}{\cancel{\Phi^T(t,t)}}$$

$$\dot P(t) = E(t)\Big[\overbrace{\Phi P(t_0)\Phi^T + \int_{t_0}^{t}\Phi\left[GQG^T + KRK^T\right]\Phi\, d\tau}^{P(t)}\Big]$$

$$+ \Big[\overbrace{\Phi P(t_0)\Phi^T + \int_{t_0}^{t}\Phi\left[GQG^T + KRK^T\right]\Phi\, d\tau}^{P(t)}\Big]E(t)^T$$

$$+ G(t)Q(t)G^T(t) + K(t)R(t)K^T(t)$$

$$\boxed{\begin{aligned}\dot P(t) = {} & E(t)P(t) + P(t)E^T(t) \\ & + G(t)Q(t)G^T(t) + K(t)R(t)K^T(t)\end{aligned}}$$

choose the optimal $K(t)$:

If $\dot P(t) = 0$: covariance stopped changing

want covariance to shrink...

$$\min_{K} J(K(t)) = Tr(\dot{P}(t))$$

Before

$\min Tr(P^{+})$

maximize the rate of decrease of covariance

$$\frac{\partial J}{\partial K} = 0 = 2K(t)R(t) - 2P(t)H^{T}(t)$$

$$\Longrightarrow \boxed{K(t) = P(t)H(t)^{T}R^{-1}(t)}$$

similar to the LQR Gain.

$$\dot{P}(t) = E(t)P(t) + P(t)E^{T}(t) + G(t)Q(t)G^{T}(t) + K(t)R(t)K^{T}(t)$$

$$\dot{P}(t) = E(t)P(t) + P(t)E^{T}(t) + G(t)Q(t)G(t)^{T} + P(t)H(t)^{T}R^{-1}(t)H(t)P(t)$$

$$\boxed{E = F - KH}$$

$$FP - PH^{T}R^{-1}HP + PF - PH^{T}R^{-1}HP$$
$$+ GQG^{T} + PH^{T}R^{-1}HP$$

$$\boxed{\dot{P}(t) = FP + PF + GQG^{T} - PH^{T}R^{-1}HP}$$

Riccati Eqn.

similar to the LQR eqn

Summary CTKF:

Model
$$\dot{x} = F(t)x + B(t)u + G(t)w(t) \qquad w(t) \sim N(0, Q(t))$$
$$\tilde{y} = H(t)x + v(t) \qquad v(t) \sim N(0, R(t))$$

Init
$$\hat{x}(t_0) = \hat{x}_0$$
$$P(0) = E[e(0)e(0)^T]$$

Gain
$$K(t) = P(t)H^T(t)R^{-1}(t)$$

Updates
$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G(t)^T - P(t)H^T(t)R^{-1}(t)H(t)P(t)$$

$$\dot{\hat{x}} = F(t)\hat{x}(t) + B(t)u(t) + K(t)[\tilde{y}(t) - H(t)\hat{x}(t)]$$

Lagrangian $f(x) + \lambda^T g(x) \Longleftarrow$

Mechanics $T - V$

T: kinetic, V: pot.

Feynman Lecture

action

$$\min_{x(t)} \int \underline{T(x(t)) - V(x(t))}\, dt$$

traj minimum action

calc of var $\Longrightarrow$ Euler-Lagrange

$$\frac{d}{dt}\frac{\partial T}{\partial x} + \frac{dV}{dx}$$

Comment: version of all of this for correlated meas. & process noise

Assumption before: $E\left[\underline{w(t)}\,\underline{v(t)}^T\right] = 0$

Now $E\left[\underline{w(t)}\,\underline{v(t)}^T\right] = \underline{S(t)}$ Eqns in books

Similar:

LQR $\displaystyle\int_0^t x(t)^T Q(t) x(t) + \underline{x(t)^T S(t) u(t)} + u(t)^T R(t) u(t)\, dt$

Continuous/Discrete KF:

Model $\dot{x} = F(t)x + B(t)u + G(t)w(t) \quad w(t) \sim N(0, Q(t))$

$\tilde{y}_k = H_k x_k + v_k \quad v_k \sim N(0, R_k)$

Init $\hat{x}(t_0) = \hat{x}_0$

$P(0) = E\left[e(0)e(0)^T\right]$

Gain: $K_k = P_k^- H_k^T \left[H_k P_k^- H_k^T + R_k\right]^{-1}$

Update discrete $\hat{x}_k^+ = \hat{x}_k^- + K_k\left[\tilde{y}_k - H_k \hat{x}_k^-\right]$

$P_k^+ = \left[I - K_k H_k\right] P_k^-$

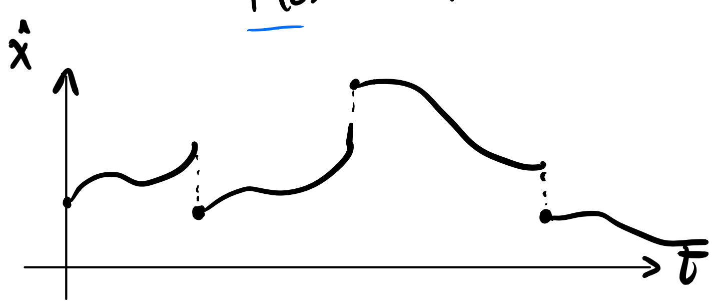
Come from continuous propagation

$\hat{x}_k^- \quad P_k^-$

Propagation continuous

$\dot{\hat{x}} = F(t)\hat{x}(t) + B(t)u(t)$

$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G(t)^T$



Extended Kalman Filter Nonlinear System:

Assumptions for regular Kalman filter
→ everything is linear

Linear way to propagate covariance matrix

$$e^+ = Ae \Leftarrow$$

$$\underbrace{E[e^+ e^{+T}]}_{P^+} = A \underbrace{E[ee^T]}_{P} A^T$$

$$P^+ = APA^T$$

Nonlinear Dynamics:

$$\dot{x} = f(x(t), u(t), t) + G(t)w(t)$$

$$\tilde{y} = h(x(t), t) + v(t)$$

use the nonlinear equations as much as possible...

linearize around $\hat{x}$ to update the covariance...

Extended Kalman Filter option 1

Model: $\dot{x} = f(x, u, t) + G(t)w$ $\quad w(t) \sim N(0, Q(t))$

$\tilde{y} = h(x, t) + v(t)$ $\quad v(t) \sim N(0, R(t))$

Init: $\hat{x}(t_0) = \hat{x}_0$

$P_0 = E[e(0)e(0)^T]$

Gain: $K(t) = P(t)H^T(t)R^{-1}(t)$

Covariance: $$\left[\begin{aligned} \dot{P}(t) = {} & F(t)P(t) + P(t)F^T(t) \\ & - P(t)H^T(t)R^{-1}(t)H(t)P(t) \\ & + G(t)Q(t)G(t)^T \end{aligned} \right]$$

$$\rightarrow F(t) = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}(t), u(t)} \quad H(t) = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}(t)} \leftarrow$$

Estimate $$\dot{\hat{x}}(t) = f(\hat{x}, u, t) + K(t)[\tilde{y}(t) - h(\hat{x}, t)]$$

Unscented Kalman Filter

works better than the extend KF for very nonlinear dynamics.

New perspective on K:

DTKF: $\hat{x}_k^+ = \hat{x}_k^- - K_k(\tilde{y}_k - H_k\hat{x}_k^-)$

$$P_k^+ = [I - K_k H_k] P_k^-$$

$$K_k = P_k^- H_k^T \underbrace{[H_k P_k^- H_k^T + R_k]}^{-1}$$

$$\longrightarrow \quad H_k P_k^- H_k^T + R_k = E\left[(\tilde{y} - H\hat{x})(\tilde{y} - H\hat{x})^T\right]$$

$$\hat{y} = Hx + v \Rightarrow E\left[(v - He)(v - He)^T\right]$$

for nonlinear case.

Covariance of

$\tilde{y} - h(\hat{x}, t)$

$$E\{vv^T - \cancel{Hev^T}^{\,0} + \cancel{ve^T}^{\,0}H^T + H(ee^T)H^T\}$$

$$E[vv^T] + H\,E(ee^T)H^T$$

$$E\left[(\hat{x}_k^+ - \hat{x}_k^-)(\hat{y}_k - H\hat{x})^T\right] = E\left[-K_k(\hat{y}_k - H_k\hat{x}_k^-)(\hat{y}_k - H\hat{x}_k)^T\right]$$

$$= -K_k \underbrace{E\left[(\hat{y}_k - H_k\hat{x}_k^-)(\hat{y}_k - H\hat{x}_k)^T\right]}_{H_k P_k^- H_k^T + R_k}$$

$$\downarrow$$

$$= P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1} (H_k P_k^- H_k^T + R_k)$$

$$E[(\hat{x}_k^+ - \hat{x}_k^-)(\tilde{y}_k - H\hat{x})^T] = -P_k^- H_k^T$$

$$K_k = P_k^- H_k^T [H_k P_k^- H_k^T + R_k]^{-1}$$

$$K_k = -E[(\hat{x}_k^+ - \hat{x}_k^-)(\hat{y}_k - H\hat{x}_k^-)^T]\, E[(\tilde{y} - H\hat{x})(\tilde{y} - H\hat{x})^T]^{-1}$$

$$\underbrace{(\hat{x}_k^+ - \hat{x}_k^-)}_{?} \quad \underbrace{(\hat{y}_k - H\hat{x}_k^-)}_{\hat{y} - h(\hat{x})} \quad \underbrace{(\tilde{y} - H\hat{x})(\tilde{y} - H\hat{x})^T}_{(\tilde{y} - h(\hat{x}))(\tilde{y} - h(\hat{x}))}$$

Unscented Transformation:

way to propagate probability distributions through nonlinear equations

Wikipedia notation.

$\hat{x}_k \in \mathbb{R}^n$. random variable

$\{s_j \in \mathbb{R}^n\}_{j=1}^N$: sigma points

$\{w_j^a\}_{j=1}^N$: first order weights

a: "average"

$\rightarrow \sum_j w_j^a = 1 \quad E[\hat{x}_k] = \sum_j w_j^a s_j$

c: "covariance"

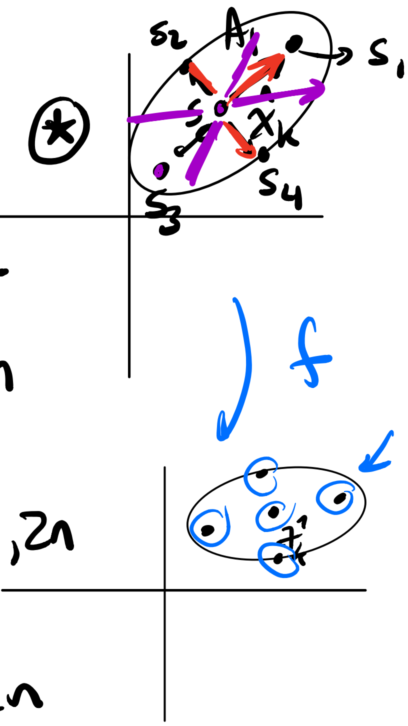
$\{w_j^c\}_{j=1}^N$: second order weights

$$\sum_j w_j^c = 1 \quad E[\hat{x}_k \hat{x}_k^T] = \sum_j w_j^c s_j s_j^T$$

$N = 2n+1$

Picking sigma points:

(*)



$$s_0 = \hat{x}_k \qquad -1 < W_0^a = W_0 = W_0^c < 1$$

$$S_j = \hat{x}_k + \sqrt{\frac{n}{1-W_0}}\, A_j \qquad i = 1, \ldots, n$$

$$S_j = \hat{x}_k - \sqrt{\frac{n}{1-W_0}}\, A_j \qquad i = n+1, \ldots, 2n$$

$$w_j^a = w_j^c = \frac{1-W_0}{2n} \qquad j = 1 \cdots 2n$$

A_j is the jth column of A where $P_k = AA^T$

Cholesky decomposition $\quad P_k = AA^T \quad A$ lower triangular

SVD or EIGEN DECOMP $\quad P_k = RDR^T = RD^{1/2}D^{1/2}R^T$

(*)

pos def.

right evecs

rows are left evecs

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \lambda_i > 0$$

$A = RD^{1/2}$

orthogonal cols scaled by

$\rightarrow \lambda_i^{1/2}$

$$x_j = f(s_j) \qquad j = 0, \ldots, 2n$$

$$\hat{x}_{k+1}^- = \sum_j w_j^a x_j$$

prediction step.

$$P_{k+1}^- = \sum_j w_j^c (x_j - \hat{x}_{k+1}^-)(x_j - \hat{x}_{k+1}^-)^T + Q_k$$

$\hat{x}_{k+1}^- \Rightarrow$ compute $2n+1$ more sigma points s_j

measurement step.

$$y_j = h(s_j)$$