Kalman gain & covariance converge to something constant.

Note: doesn't mean that the state $x(t)$ is constant or control $u(t)$ is constant.

Note: $F, B, G, Q, R, H \longrightarrow$ all constant w time

$\dot{P} = 0 \Longleftarrow$ covariance has stopped changing

<u>Steady State Continuous Time KF</u>

Model: $\dot{x}(t) = Fx(t) + Bu(t) + Gw(t) \quad w(t) \sim N(0,Q)$

(LTI) $\tilde{y}(t) = Hx(t) + v(t) \quad v(t) \sim N(0,R)$

Init: $\hat{x}(t_0) = \hat{x}_0$

Gain: $K = PH^TR^{-1}$

Covariance: (Algebraic Riccati Eqn) $\underline{FP + PF^T} - \underline{PH^TR^{-1}HP} + \underline{GQG^T} = 0$

conditions: LTI system, observable

Estimate: $\dot{\hat{x}}(t) = F\hat{x}(t) + Bu(t) + K[\tilde{y}(t) - H\hat{x}(t)]$

Numerical solutions:

Matlab: icare (A, B, Q, R, S, E, G)

where $A^TXE + E^TXA + E^TXGXE - (E^TXB + S)R^{-1}(B^TXE + S^T) + Q = 0$

Other numerical ways:

- integrate diff eq till it converges
- 2n dim Hamiltonian system → eigenvectors $\begin{bmatrix} A & -PH^TR^{-1}HP \\ Q & -A^T \end{bmatrix}$
- semi definite programming method based on Schur complement → SDP: convex optimization for pos def matrices

robust control

$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ $\quad A - BD^{-1}C$

Steady State Discrete Time KF:

Model: $x_{k+1} = \Phi x_k + \Gamma u_k + \Upsilon w_k \quad w_k \sim N(0,Q)$

(LTI) $\quad \tilde{y}_k = H x_k + v_k \quad v_k \sim N(0,R)$

Init: $\hat{x}(t_0) = \hat{x}_0$

GAIN: $K = PH^T[HPH^T + R]^{-1}$

Covariance: $P = \Phi P \Phi^T - \Phi P H^T (HPH^T+R)^{-1} H P \Phi^T + \Upsilon Q \Upsilon^T$

Estimate: $\hat{x}_{k+1} = \Phi \hat{x}_k + \Gamma u_k + \Phi K[\tilde{y}_k - H\hat{x}_k]$

Numerical solutions:

Matlab: idare (A, B, Q, R, S, E)

where $\quad A^TXA + E^TXE - (A^TXB+S)(B^TXB+R)^{-1}(A^TXB+S)^T + Q = 0$

Unscented Kalman Filter: a powerful way to deal w nonlinearities

"sampling a mean & updating covariance & updating"

DISCRETE TIME LINEAR KF:

$\hat{x}_k^+ , P_k^+$ $\xrightarrow{\text{pushed through dynamics}}$ $\hat{x}_{k+1}^- = \phi_k \hat{x}_k^+ + B_k u_k$

1 $P_{k+1}^- = \phi_k P_k^+ \phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

writing this out is new

$\hat{x}_{k+1}^- \; P_k^-$ $\xrightarrow{\text{pushed through measurement.}}$ $(e_y)_{k+1} = \tilde{y}_{k+1} - \hat{y}_{k+1}$

$\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1}^-$

$P_{k+1}^{e_y e_y} = H_{k+1} P_k^- H_{k+1}^T + R_{k+1}$

$(e_y)_{k+1}, P_{k+1}^{e_y e_y}$ $\xrightarrow{\text{push optimal gain}}$ $\hat{x}_{k+1}^+ = \hat{x}_{k+1}^- + K_{k+1} (e_y)_{k+1}$

$$P_k^{e_y e_y} = E\left[(\tilde{y}_k - \hat{y}_k)(\tilde{y}_k - \hat{y}_k)^T\right]$$

$$E\left[(H_k x_k + v_k - H_k \hat{x}_k)(H_k x_k + v_k - H_k \hat{x}_k)^T\right]$$

$$E\left[(v_k - H_k(\hat{x}_k - x_k))(v_k - H_k(\hat{x}_k - x_k))^T\right]$$

$$\Rightarrow E[v_k v_k^T] + H_k E\left[(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T\right] H_k^T$$

$$R_k + H_k P_k^- H_k^T$$

$$P_k^{e_x e_y} = E\left[(\hat{x}_k^- - x_k)(\tilde{y}_k - \hat{y}_k)^T\right]$$

$$= E\left[(\hat{x}_k^- - x_k)(e_y)_k^T\right]$$

$$P_{k+1}^+ = E\left[(\hat{x}_{k+1}^+ - x_k)(\hat{x}_{k+1}^+ - x_k)^T\right]$$

$$= E\left[\left((\hat{x}_{k+1}^- - x_k) + K_{k+1}(e_y)_{k+1}\right)\left((\hat{x}_{k+1}^- - x_k) + K_{k+1}(e_y)_{k+1}\right)^T\right]$$

$$= E\left[(\hat{x}^-_{k+1} - x_k)(\hat{x}^-_{k+1} - x_k)^T\right] + K_{k+1} E\left[(e_y)_{k+1}(e_y)^T_{k+1}\right] K^T_{k+1}$$

$$+ E\left[K_{k+1}(e_y)_{k+1}(\hat{x}^-_{k+1} - x_k)^T\right] + E\left[(\hat{x}^-_{k+1} - x_k)(e_y)^T_{k+1} K^T_{k+1}\right]$$

$$= P^-_{k+1} + K_{k+1} P^{e_y e_y}_{k+1} K^T_{k+1}$$

$$+ K_{k+1} P^{e_x e_y T}_{k+1} + P^{e_x e_y}_{k+1} K^T_{k+1}$$

$$\min_{K_{k+1}} \operatorname{Tr}(P^+_{k+1})$$

$$\frac{\partial}{\partial K_{k+1}} = 0 = K_{k+1}(2P^{e_y e_y}_{k+1}) + 2P^{e_x e_y}_{k+1}$$

? sign ?

$$\Rightarrow \boxed{K_{k+1} = -P^{e_x e_y}_{k+1}\left(P^{e_y e_y}_{k+1}\right)^{-1}}$$

Identities:

$$\frac{\partial}{\partial A}\operatorname{Tr}(BAC) = B^T C^T$$

$$\frac{\partial}{\partial A}\operatorname{Tr}(ABA^T) = A(B + B^T)$$

$$P^+_{k+1} = P^-_{k+1} - P^{e_x e_y}_{k+1}\left(P^{e_y e_y}_{k+1}\right)^{-1} P^{e_x e_y T}_{k+1}$$

state meas 2 2

2

$$= P^-_{k+1} - K_{k+1} P^{e_y e_y}_{k+1} K^T_{k+1}$$

Goal:

New form for optimal gain: $\boxed{K_{k+1}} = P^{e_x e_y}_{k+1}\left(P^{e_y e_y}_{k+1}\right)^{-1}$

$$P^-_{k+1} H^T_{k+1}\left(H_{k+1} P^-_{k+1} H^T_{k+1} + R_{k+1}\right)^{-1}$$

this form extends better to

$\hat{x}_{k+1} = f(x, u, k)$

$\hat{y}_k = h(x_k, k)$

$$K = PH^T\left[HPH^T + R\right]^{-1}$$

Unscented KF:

$\hat{x}_k^+ , P_k^+ \xrightarrow[\text{dynamics}]{\hat{x}_{k+1}^- = f(\hat{x}_k^+, w_k, u_k, k)}$

$\hat{x}_{k+1}^- , P_{k+1}^- \xrightarrow[\text{measurement}]{\hat{y}_{k+1} = h(\hat{x}_{k+1}^-, u_{k+1}, v_{k+1}, k+1)}$

$\hat{x}_{k+1}^- = ?$

$P_{k+1}^- = ?$

$\hat{y}_{k+1} = ?$

$P_{k+1}^{e_y e_y} = ?$

$P_{k+1}^{e_x e_y} = ?$

EKF

$\hat{x}_{k+1}^- = f(\hat{x}_k^+, 0, u_k, k)$

$P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

$\Phi_k = \frac{\partial f}{\partial x} \quad \Upsilon_k = \frac{\partial f}{\partial w}$

$\hat{y}_{k+1} = h(\hat{x}_{k+1}^-, u_{k+1}, 0, k+1)$

$P_{k+1}^{e_y e_y} = H_{k+1} P_{k+1}^- H_{k+1}^T + R_{k+1}$

$H_{k+1} = \frac{\partial h}{\partial x}$

$\vdots$

linearize for the covariance update equations

$$\Longrightarrow K_{k+1} = P_{k+1}^{e_x e_y} (P_{k+1}^{e_y e_y})^{-1}$$

$$\hat{x}_{k+1}^+ = \hat{x}_{k+1}^- + K_{k+1} (e_y)_{k+1}$$

$$P_{k+1}^+ = P_{k+1}^- - K_{k+1} P_{k+1}^{e_y e_y} K_{k+1}^T$$

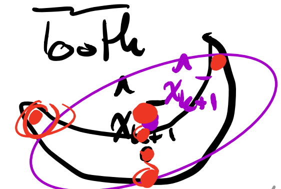
? what if linearization doesn't work?

(when f & h are very nonlinear)

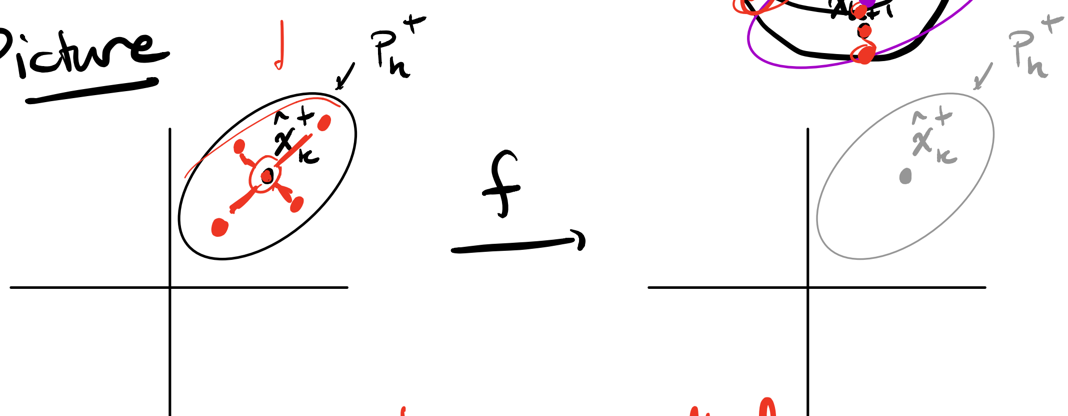
Unscented KF

Need to compute

$\underbrace{\hat{x}_{k+1}^- \quad P_{k+1}^-}_{\text{dynamics } f}$ $\quad \underbrace{\hat{y}_{k+1} \quad P_{k+1}^{e_y e_y}}_{\text{meas, } h}$ $\quad \underbrace{P_{k+1}^{e_x e_y}}_{\text{both}}$



Picture



red sample points are called sigma points.

treat noise & state as one big vector

$$x_k^a = \begin{bmatrix} x_k \\ w_k \\ v_k \end{bmatrix} \Big\} L \qquad \hat{x}_k^a = \begin{bmatrix} \hat{x}_k \\ 0 \\ 0 \end{bmatrix} \qquad P_k^a = \begin{bmatrix} P_k^+ & P_k^{xw} & * \\ * & Q_k & * \\ P_k^{vx} & * & R_k \end{bmatrix}$$

$$x_k^a \in \mathbb{R}^L \qquad P_k^a \in \mathbb{R}^{L \times L}$$

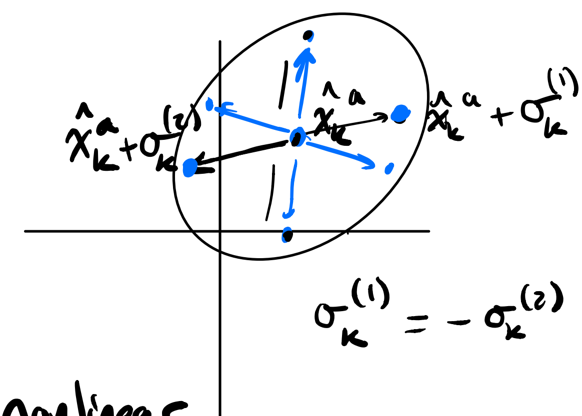
0 is noise & state not correlated

Sigma Points

$$\mathcal{X}_k^{a(0)} = \hat{x}_k^a$$

$$\mathcal{X}_k^{a(i)} = \sigma_k^{(i)} + \hat{x}_k^a$$

where $\sigma_k \leftarrow 2L$ columns from $\pm \gamma \sqrt{P_k^a}$



$$\sigma_k^{(1)} = -\sigma_k^{(2)}$$

push sigma points thru nonlinear function ...

w component of $\mathcal{X}_k^{a(i)}$

$$\rightarrow \mathcal{X}_{k+1}^{x(i)} = f\left(\mathcal{X}_k^{x(i)}, \mathcal{X}_k^{w(i)}, u_k, k\right)$$

use sigma points to reconstruct $\hat{x}_{k+1}^-, P_{k+1}^-$

$$\boxed{\hat{x}_{k+1}^- = \sum_{i=0}^{2L} W_i^{mean} \mathcal{X}_{k+1}^{x(i)}} \leftarrow \text{weighted average of mean}$$

$$\boxed{P_{k+1}^- = \sum_{i=0}^{2L} W_i^{cov} \left[\mathcal{X}_{k+1}^{x(i)} - \hat{x}_{k+1}^-\right]\left[\mathcal{X}_{k+1}^{x(i)} - \hat{x}_{k+1}^-\right]^T} \leftarrow \text{weighted average differences}$$

What are these weights?

Many options...

Conditions: $\sum_{i=0}^{2L} W_i^{mean} = 1$ $\qquad \hat{x}_k^a = \sum_{i=0}^{2L} W_i^{mean} \mathcal{X}_k^{a(i)}$

(wikipedia Kalman Filter unscented) $\qquad \sum_{i=0}^{2L} W_i^{cov} = 1$ $\qquad \boxed{E[\hat{x}_k^a \hat{x}_k^{aT}] = \sum_{i=0}^{2L} W_i^{cov} \mathcal{X}_k^{a(i)} \mathcal{X}_k^{a(i)T}}$

In book particular weights...

L dim of $\hat{x}_k^a$

$$W_0^{mean} = \frac{\lambda}{L+\lambda}$$

$$W_0^{cov} = \frac{\lambda}{L+\lambda} + (1-\alpha^2+\beta)$$

$$W_i^{mean} = W_i^{cov} = \frac{1}{2(L+\lambda)} \quad i = 1, \dots, 2L$$

parameters: $\gamma, \lambda, \alpha, \beta, \kappa$

$\gamma = \sqrt{L+\lambda}$ → distance of sigma points from center

$\lambda = \alpha^2 (L+\kappa) - L$

α: something we pick $\quad$ usually $1 \times 10^{-4} \le \alpha \le 1$

β: used for prior knowledge $\quad$ often $\beta = 2$

κ: prior knowledge about higher order moments $\quad$ scalar systems: $\kappa = 2$

higher dim... $\kappa = 3 - L$

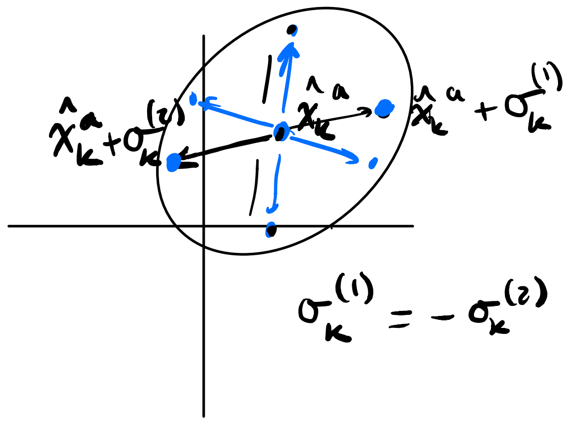
what is $\sqrt{P_k^a}$?

if β negative have to be careful that covariance stays pos def.

$$\boxed{E[\hat{x}_k^a \hat{x}_k^{a\,T}] = \sum_{i=0}^{2L} w_i^{cov} \chi_k^{a(i)} \chi_k^{a(i)T}}$$

$$P_k^a = \begin{bmatrix} | & | & | & | & | & | \end{bmatrix} \begin{bmatrix} w & & 0 \\ & \ddots & \\ 0 & & w \end{bmatrix} \begin{bmatrix} — \\ — \\ — \\ — \end{bmatrix}$$

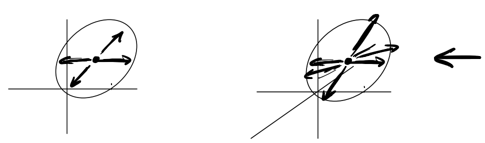
$$(P_k^a)^{1/2} (P_k^{a})^{T/2}$$



$$P_k^a = MM^T \qquad M = \sqrt{P_k^a}$$

lots of options for M...

- Cholesky decomposition of $P_k^a = MM^T$ M triangular

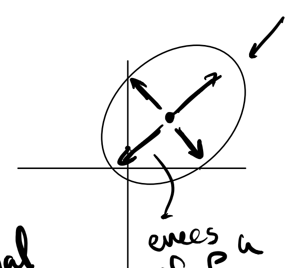


- eigenvalue or SVD of P_k^a

$$P_k^a = SDS^T = \underbrace{(SD^{1/2})}_{M}(SD^{1/2})^T$$

S ← evecs (orthogonal)

M → cols are orthogonal



pushing through h ...

$$\mathcal{Y}_{k+1}^{(i)} = h(\mathcal{X}_{k+1}^{x(i)}, u_{k+1}, \mathcal{X}_{k+1}^{v(i)}, k) \quad i = 0, \ldots, 2L$$

$$\hat{y}_{k+1}^{-} = \sum_{i=0}^{2L} w_i^{mean} \mathcal{Y}_{k+1}^{(i)}$$

$$P_{k+1}^{e_y e_y} = \sum_{i=0}^{2L} w_i^{cov} \left[\mathcal{Y}_{k+1}^{(i)} - \hat{y}_{k+1}^{-}\right]\left[\mathcal{Y}_{k+1}^{(i)} - \hat{y}_{k+1}^{-}\right]^T$$

$$P_{k+1}^{e_x e_y} = \sum_{i=0}^{2L} w_i^{cov} \left[\underline{\mathcal{X}_{k+1}^{x(i)} - \hat{x}_{k+1}^{-}}\right]\left[\mathcal{Y}_{k+1}^{(i)} - \hat{y}_{k+1}^{-}\right]^T$$

$$K_{k+1} = P_{k+1}^{e_x e_y}\left(P_{k+1}^{e_y e_y}\right)^{-1}$$

$$P_{k+1}^{+} = P_{k+1}^{-} - K_{k+1} P_{k+1}^{e_y e_y} K_{k+1}^T$$

$$\hat{x}_{k+1}^{+} = \hat{x}_{k+1}^{-} + K_{k+1}(e_y)_{k+1}$$