AA 549: Estimation and Kalman Filtering
Kalman Filters

- **Discrete-Time Kalman Filter**

Model: $x_{k+1} = \Phi_k x_k + \Gamma_k u_k + \Upsilon_k w_k, \qquad w_k \sim \mathcal{N}(0, R_k)$

$\tilde{y}_k = H_k x_k + v_k, \qquad v_k \sim \mathcal{N}(0, R_k)$

Initialize: $\hat{x}(t_0) = \hat{x}_0, \qquad P_0 = E\big[(\hat{x}_0 - x_0)(\hat{x}_0 - x_0)^T\big]$

Gain: $K_k = P_k^- H_k^T \big[H_k P_k^- H_k^T + R_k\big]^{-1}$

Update: $\hat{x}_k^+ = \hat{x}_k^- + K_k[\tilde{y}_k - H_k \hat{x}_k^-]$

$P_k^+ = \big[I - K_k H_k\big] P_k^-$

Propagation: $\hat{x}_{k+1}^- = \Phi_k \hat{x}_k^+ + \Gamma_k u_k$

$P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

Alternative Forms:

$$P_k^+ = \big[I - K_k H_k\big] P_k^- \big[I - K_k H_k\big]^T + K_k R_k K_k^T$$
$$P_k^+ = P_k^- - P_k^- H_k^T \big[H_k P_k^- H_k^T + R_k\big]^{-1} H_k P_k^-$$
$$P_k^+ = \big[(P_k^-)^{-1} + H_k^T R_k^{-1} H_k\big]^{-1}$$

$$K_k = P_k^+ H_k^T R_k^{-1}, \qquad \big[I - K_k H_k\big] = P_k^+ (P_k^{-1})^{-1}, \qquad \hat{x}_k^+ = P_k^+ \big[(P_k^-)^{-1} \hat{x}_k^- + H_k^T R_k^{-1} \tilde{y}_k\big]$$

Combined Form

$$\hat{x}_{k+1} = \Phi_k \hat{x}_k + \Gamma_k u_k + \Phi_k K_k \big[\tilde{y}_k - H_k \hat{x}_k\big]$$
$$K_k = P_k H_k^T \big[H_k P_k H_k^T + R_k\big]^{-1}$$

Discrete Riccati Eqn. $P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k P_k H_k^T \big[H_k P_k H_k^T + R_k\big]^{-1} H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

Other forms $P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k \big[H_k P_k H_k^T + R_k\big] K_k^T \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

$= \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

Infinite Horizon

$$K_\infty = P_\infty H^T \big[H P_\infty H^T + R\big]^{-1}$$

DT Alg. Riccati Eqn. (DARE) $P_\infty = \Phi P_\infty \Phi - \Phi P_\infty H^T \big[H P_\infty H^T + R\big]^{-1} H P_\infty \Phi^T + \Upsilon Q \Upsilon^T$

DARE other forms $P_\infty = \Phi P_\infty \Phi - \Phi K_\infty \big[H P_\infty H^T + R\big] K_\infty^T \Phi^T + \Upsilon Q \Upsilon^T$

$= \Phi P_\infty \Phi - \Phi K_\infty H P_\infty \Phi^T + \Upsilon Q \Upsilon^T$

- **Continuous-Time Kalman Filter**

$$
\begin{aligned}
\textbf{Model:} \quad & \dot{x}(t) = F(t)x(t) + B(t)u(t) + G(t)w(t), \qquad w(t) \sim \mathcal{N}\big(0, Q(t)\big) \\
& \tilde{y}(t) = H(t)x(t) + v(t), \qquad v(t) \sim \mathcal{N}\big(0, R(t)\big) \\
& E[w(t)w(\tau)^T] = Q(t)\delta(t-\tau) \\
& E[v(t)v(\tau)^T] = R(t)\delta(t-\tau) \\
& E[v(t)w(\tau)^T] = 0 \\
\textbf{Initialize:} \quad & \hat{x}(t_0) = \hat{x}_0, \qquad P_0 = E\big[(\hat{x}(0) - x(0))(\hat{x}(0) - x(0))^T\big] \\
\textbf{Gain:} \quad & K(t) = P(t)H^T(t)R^{-1}(t) \\
\textbf{Covariance:} \quad & \dot{P}(t) = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t) \\
\textbf{Estimate:} \quad & \dot{\hat{x}} = F(t)\hat{x}(t) + B(t)u(t) + K(t)\big[\tilde{y}(t) - H(t)\hat{x}(t)\big]
\end{aligned}
$$

Alternative Forms:

$$
\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + K(t)R(t)K(t)^T + G(t)Q(t)G(t)^T
$$

Infinite Horizon

$$
\begin{aligned}
& K_\infty = P_\infty H^T R^{-1} \\
\textbf{CT Alg. Riccati Eqn. (CARE)} \quad & 0 = FP_\infty + P_\infty F^T - P_\infty H^T R^{-1} H P_\infty + GQG^T \\
& 0 = FP_\infty + P_\infty F^T + K_\infty R K_\infty^T + GQG^T
\end{aligned}
$$

- **Continuous/Discrete Kalman Filter**

$$
\begin{aligned}
\textbf{Model:} \quad & \dot{x}(t) = F(t)x(t) + B(t)u(t) + G(t)w(t), \qquad w(t) \sim \mathcal{N}\big(0, Q(t)\big) \\
& \tilde{y}_k = H_k x_k + v_k, \qquad v_k \sim \mathcal{N}\big(0, R_k\big) \\
\textbf{Initialize:} \quad & \hat{x}(t_0) = \hat{x}_0, \qquad P_0 = E\big[(\hat{x}(0) - x(0))(\hat{x}(0) - x(0))^T\big] \\
\textbf{Gain:} \quad & K_k = P_k^- H_k^T \big[H_k P_k^- H_k^T + R_k\big]^{-1} \\
\textbf{Update:} \quad & \hat{x}_k^+ = \hat{x}_k^- + K_k\big[\tilde{y}_k - H_k \hat{x}_k^-\big] \\
& P_k^+ = \big[I - K_k H_k\big] P_k^- \\
\textbf{Propagation:} \quad & \dot{\hat{x}}(t) = F(t)\hat{x}(t) + B(t)u(t) \\
& \dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G(t)^T
\end{aligned}
$$

- **Continuous-Time Extended Kalman Filter**

Model: $\dot{x}(t) = f(x,u,t) + G(t)w(t), \qquad w(t) \sim \mathcal{N}(0, Q(t))$

$\tilde{y}(t) = h(x,t) + v(t), \qquad v(t) \sim \mathcal{N}\big(0, R(t)\big)$

Initialize: $\hat{x}(t_0) = \hat{x}_0, \qquad P_0 = E\big[(\hat{x}(0) - x(0))(\hat{x}(0) - x(0))^T\big]$

Gain: $K(t) = P(t)H(t)^T R(t)^{-1}$

Covariance: $\dot{P}(t) = F(t)P(t) + P(t)F(t)^T - P(t)H(t)^T R(t)^{-1} H(t)P(t) + G(t)Q(t)G(t)^T$

$$F(t) = \frac{\partial f}{\partial x}\Big|_{\hat{x}(t),u(t)}, \qquad H(t) = \frac{\partial h}{\partial x}\Big|_{\hat{x}(t)}$$

Estimate: $\dot{\hat{x}}(t) = f(\hat{x},u,t) + K(t)\big[\tilde{y}(t) - h(\hat{x},t)\big]$

- **Continuous-Discrete Extended Kalman Filter**

Model: $\dot{x}(t) = f(x,u,t) + G(t)w(t), \qquad w(t) \sim \mathcal{N}(0, Q(t))$

$\tilde{y}_k = h(x_k) + v_k, \qquad v_k \sim \mathcal{N}\big(0, R_k\big)$

Initialize: $\hat{x}(t_0) = \hat{x}_0, \qquad P_0 = E\big[(\hat{x}(0) - x(0))(\hat{x}(0) - x(0))^T\big]$

Gain: $K_k = P_k^- H_k^T \big[H_k P_k^- H_k^T + R_k\big]^{-1}, \qquad H_k = \frac{\partial h}{\partial x}\Big|_{\hat{x}_k^-}$

Update: $\hat{x}_k^+ = \hat{x}_k^- + K_k\big[\tilde{y}_k - h(\hat{x}_k^-)\big]$

$P_k^+ = \big[I - K_k H_k\big]P_k$

Propagation: $\dot{\hat{x}}(t) = f(\hat{x},u,t)$

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + G(t)Q(t)G(t)^T, \qquad F(t) = \frac{\partial f}{\partial x}\Big|_{\hat{x}(t),u(t)}$$

- **Unscented Kalman Filter**

Model:

$$x_{k+1} = f(x_k, u_k, w_k, k), \qquad w(t) \sim \mathcal{N}(0, Q_k)$$

$$\tilde{y}_k = h(x_k, u_k, v_k, k), \qquad v_k \sim \mathcal{N}\big(0, R_k\big)$$

Sample:

$$x_k^a = \begin{bmatrix} x_k \\ w_k \\ v_k \end{bmatrix}, \quad \hat{x}_k^a = \begin{bmatrix} \hat{x}_k \\ 0 \\ 0 \end{bmatrix}, \qquad P_k^a = \begin{bmatrix} P_k^+ & P_k^{xw} & P_k^{xv} \\ (P_k^{xw})^T & Q_k & P_k^{wv} \\ (P_k^{xv})^T & (P_k^{wv})^T & R_k \end{bmatrix}$$

$P = MM^T, \quad M = \sqrt{P_k^a},$ **using Cholesky, SVD, etc...**

$$\sigma_k^{(i)} = \text{cols of } \pm\gamma\sqrt{P_k^a}$$

$$\chi_k^{a(0)} = \hat{x}_k^a, \quad \chi_k^{a(i)} = \hat{x}_k^a + \sigma_k^{(i)}, \quad i = 1, 2, \ldots, 2L, \quad \chi_k^{a(i)} = \begin{bmatrix} \chi_k^{x(i)} \\ \chi_k^{w(i)} \\ \chi_k^{v(i)} \end{bmatrix}$$

Propagate:

$$\chi_{k+1}^{x(i)} = f(\chi_k^{x(i)}, \chi_k^{w(i)}, u_k, k)$$

$$\gamma_k^{(i)} = h(\chi_k^{x(i)}, u_k, \chi_k^{v(i)}, k))$$

Reconstruct:

$$\hat{x}_k^- = \sum_{i=0}^{2L} W_i^{\text{mean}} \chi_k^{x(i)}, \qquad P_k^- = \sum_{i=0}^{2L} W_i^{\text{cov}} \big[\chi_k^{x(i)} - \hat{x}_k^-\big]\big[\chi_k^{x(i)} - \hat{x}_k^-\big]^T$$

$$\hat{y}_k^- = \sum_{i=0}^{2L} W_i^{\text{mean}} \gamma_k^{(i)}, \qquad P_k^{e_y e_y} = \sum_{i=0}^{2L} W_i^{\text{cov}} \big[\gamma_k^{(i)} - \hat{y}_k^-\big]\big[\gamma_k^{(i)} - \hat{y}_k^-\big]^T$$

$$P_k^{e_x e_y} = \sum_{i=0}^{2L} W_i^{\text{cov}} \big[\chi_k^{x(i)} - \hat{x}_k^-\big]\big[\gamma_k^{(i)} - \hat{y}_k^-\big]^T$$

Gain:

$$K_k = P_k^{e_x e_y} \big(P_k^{e_y e_y}\big)^{-1}$$

Update:

$$\hat{x}_k^+ = \hat{x}_k^- + K_k e_k^- = \hat{x}_k^- + K_k\big(\tilde{y}_k - \hat{y}_k^-\big)$$

$$P_k^+ = P_k^- - K_k P_k^{e_y e_y} K_k^T$$

Params/weights:

$L:$ length of $x_k^a, \quad \gamma = \sqrt{L + \lambda}, \quad \lambda = \alpha^2(L + \kappa) - L$

$$10^{-4} \le \alpha \le 1, \quad \beta = 2$$

$$W_0^{\text{mean}} = \frac{\lambda}{L + \lambda}, \qquad W_0^{\text{cov}} = \frac{\lambda}{L + \lambda} + \big(1 - \alpha^2 + \beta\big)$$

$$W_i^{\text{mean}} = W_i^{\text{cov}} = \frac{1}{2(L + \lambda)}, \quad i = 1, 2, \ldots, 2L$$