AA 549: Estimation and Kalman Filtering
Homework #2
Due: Sunday, Apr 18, 2021 @ 11:59 pm

1. **Congruent Transformations Positive Definiteness**

   Consider a positive definite matrix $R \in \mathbb{R}^{n \times n}$, $R = R^T \succ 0$. Consider a matrix $A \in \mathbb{R}^{m \times n}$.

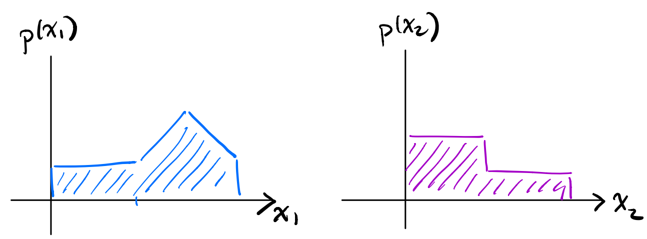
   - **(PTS:0-2)** Show that if A has full row rank, then $ARA^T \succ 0$.
   - **(PTS:0-2)** Show that if A is tall, $m > n$, then ARA^T is no longer positive definite, but rather ARA^T is positive semi-definite, $ARA^T \succeq 0$.
   - **(PTS:0-2)** Now suppose R is no longer symmetric, $R \neq R^T$. Use the symmetric/skew-symmetric decomposition $R = \frac{1}{2}(R + R^T) + \frac{1}{2}(R - R^T)$ to show that

$$x^T R x = \tfrac{1}{2} x^T (R + R^T) x, \qquad \forall x \in \mathbb{R}^n$$

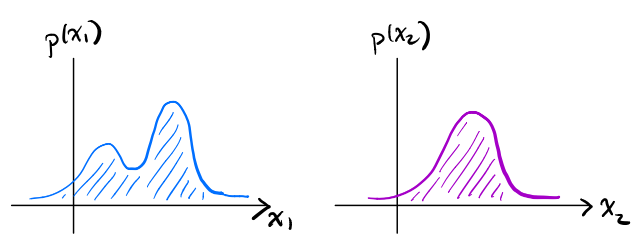
2. **Marginal and Independent Distributions**

   Consider two random variables x_1 and x_2.

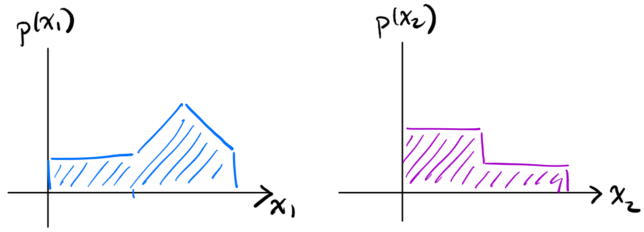
   - Suppose x_1 and x_2 are independently distributed. For each pair of marginal distributions $p(x_1)$ and $p(x_2)$ shown, sketch the joint distribution $p(x_1, x_2)$.
     - **(PTS:0-2)**



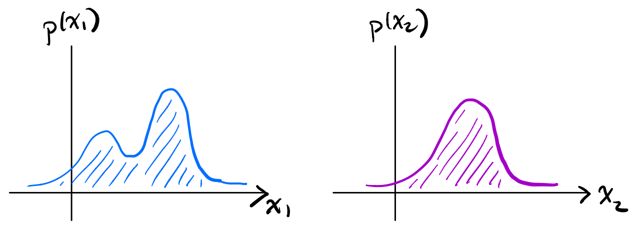
     - **(PTS:0-2)**



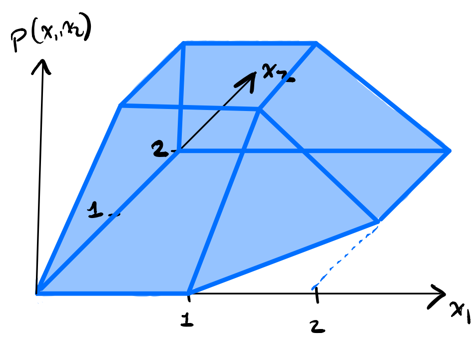
   - Now suppose that x_1, x_2 are not independent. For each pair of marginals $p(x_1)$ and $p(x_2)$ shown, sketch two possible joint distributions $p(x_1, x_2)$ that could have given rise to those marginal distributions.
     - **(PTS:0-4)**



- **(PTS:0-4)**



- Consider the joint distribution $p(x_1, x_2)$ illustrated below.



  - **(PTS:0-2)** Sketch the conditional distribution $p(x_1 | x_2 = 0.5)$.
  - **(PTS:0-2)** Sketch the conditional distribution $p(x_1 | x_1 = x_2)$.

3. **Expected Values**

   Consider a jointly Gaussian distribution $x \sim \mathcal{N}(\mu, R)$ with $x, \mu \in \mathbb{R}^n$ and $R = R^T \in \mathbb{R}^{n \times n}$. $R \succ 0$. Let $y = Ax$ for $A \in \mathbb{R}^{m \times n}$. Note that since x is jointly Gaussian, $y \in \mathbb{R}^m$ is jointly Gaussian as well. Let $y \sim \mathcal{N}(\nu, Q)$. Use the definition of mean and variance to show that

   - **(PTS:0-2)** $\quad \nu = A\mu$
   - **(PTS:0-2)** $\quad Q = ARA^T$

4. **Independence**

   (PTS:0-2) Consider a joint distribution for $x \in \mathbb{R}^n$ with pdf $p(x_1, \cdots x_n)$. Let the mean $E[x] = \mu \in \mathbb{R}^n$. Show that if x_i and x_j are independent from each other then $E\big[(x_i - \mu_i)(x_j - \mu_j)\big] = 0$.

5. **Conditional Probabilities and Bayes Rule**

   (PTS:0-2) Write a homework problem using conditional probabilities.

   (PTS:0-2) Write a homework problem that involves Bayes rule.

6. **Systems of Equations**

   Consider the system of equations $y = Ax$ and particularly the matrix $A \in \mathbb{R}^{4\times 5}$. Notice that each column of A is a linear combination of the first two columns

$$A = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ -1 & 1 & 0 & -2 & -1 \\ 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 3 \end{bmatrix}$$

   - **(PTS:0-2)** Compute a sequence of elementary matrices $E_K \cdots E_1$ (where $E_k \in \mathbb{R}^{4\times 4}$) such that

$$\big(E_k \cdots E_1\big) A = \begin{bmatrix} I_{2\times 2} & B \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 3} \end{bmatrix}$$

     where $B \in \mathbb{R}^{2\times 3}$. Let $E = E_K \cdots E_1$. Tip: perform row reduction operations (and compute the corresponding elementary matrices) that reduce the first two columns as shown. The zeros in the bottom right corner will appear automatically since the last three columns are linearly dependent on the first two.

   - **(PTS:0-2)** Compute E^{-1}. Note that now, we can write

$$A = E^{-1} E \begin{bmatrix} I_{2\times 2} & B \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 3} \end{bmatrix}$$

     Let

$$E^{-1} = \begin{bmatrix} C & D \end{bmatrix}, \qquad E = \begin{bmatrix} F \\ M \end{bmatrix}$$

     with $C, D \in \mathbb{R}^{4\times 2}$ and $F, M \in \mathbb{R}^{2\times 4}$. What are C, D, F, M? How do C and M relate to A_1 and A_2? Use the columns of E^{-1} and the rows of E to write bases for the range of A and the nullspace of A^T. **Note: there was a mistake in lecture; the rows of M form a basis for the nullspace of A^T (as opposed to the columns of D).**

   - **(PTS:0-2)** Even without knowing the specific values of of C, D, F, M what are FC, FD, MC, MD? Use the fact that $EE^{-1} = I$.

   - **(PTS:0-2)** Write bases for the range of A^T and the nullspace of A in terms of B.