AA 549: Estimation and Kalman Filtering
Homework #2
Due: Sunday, Apr 25, 2021 @ 11:59 pm

Notation Reference:

- **True parameters:** $x \in \mathbb{R}^n$
- **Basis functions:** $\{h_i(t)\}_{i=1}^n$, **examples:** $1, t, t^n, \cos(\omega t), \sin(\omega t),$
- **Data matrix:** $H \in \mathbb{R}^{K \times n}$: Matrix of basis functions. Each column corresponds to a basis function.

$$\big[H\big]_{ki} = h_i(t_k), \qquad H = \begin{bmatrix} h_1(t_1) & \cdots & h_n(t_1) \\ \vdots & & \vdots \\ h_1(t_K) & \cdots & h_n(t_K) \end{bmatrix}$$

- **Output:** $y \in \mathbb{R}^K$: output with no noise.
- **Model:** Assumption about how the parameters relate to the output.

$$y = Hx$$

- **Measurement Noise:** $v \in \mathbb{R}^K$, $v \sim \mathcal{N}(\mathbf{0}, R)$, $R = R^T \succ 0 \in \mathbb{R}^{K \times K}$. If noise is independent in time, R diagonal.
- **Measurements:** $\tilde{y} \in \mathbb{R}^n$ what we get out of our sensors

$$\tilde{y} = Hx + v$$

- **Error (Residual):** $e \in \mathbb{R}^K$ error between what we would expect the output to be with no noise and what we measure.

$$e = \tilde{y} - y = \tilde{y} - Hx$$

- **Squared error (sum):**

$$J = \tfrac{1}{2}\sum_k (e_k)^2 = \tfrac{1}{2}e^T e = \tfrac{1}{2}|\tilde{y} - Hx|^2 = \tfrac{1}{2}(\tilde{y} - Hx)^T(\tilde{y} - Hx) = \tfrac{1}{2}x^T H^T H x - \tilde{y}^T H x + \tfrac{1}{2}\tilde{y}^T\tilde{y}$$

- **Weighted squared error:** weight matrix $W = W^T \succ 0 \in \mathbb{R}^{K \times K}$

$$J = \tfrac{1}{2}e^T W e = \tfrac{1}{2}x^T H^T W H x - \tilde{y}^T W H x + \tfrac{1}{2}\tilde{y}^T W \tilde{y}$$

- **Minimization Problem:** $\min_x \; J(x)$
- **Parameter estimate:** $\hat{x} \in \mathbb{R}^n$, $\hat{x} = \text{argmin}_x \; J$
- **Predicted output:** $\hat{y} = H\hat{x}$

1. **Weighted Least Squares**

   (a) **(PTS:0-2)** Create 101 synthetic measurements $\tilde{y}$ at 0.1 second intervals of the following:

   $$\tilde{y}_j = a \sin t_j - b \cos t_j + v_j$$

   where $a = b = 1$, and v is a zero-mean Gaussian noise process with standard deviation given by 0.01.

   (b) **(PTS:0-2)** Determine the unweighted least squares estimates for a and b.

   (c) **(PTS:0-2)** Using the same measurements, find a value of $\tilde{y}$ that is near zero (near time $\pi/4$), and set that "measurement" value to 1.

   Compute the unweighted least squares solution, and compare it to the original solution. Then, use weighted least squares to "deweight" the measurement.

2. **LTI System Identification**

   Consider the discrete time LTI system given by

   $$x_t^+ = Ax_t + Bu_t + w_t$$

   with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and where $w_t \in \mathbb{R}^n$, $w_i(t) \sim \mathcal{N}(0, q)$ is noise in the dynamics.

   (i) **(PTS: 0-2)** Assuming that x_t and u_t are known, write a least squares problem to solve for the matrices A and B.

   (ii) **(PTS: 0-2)** Consider system matrices

   $$A = \exp\left(\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \Delta t\right), \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \Delta t,$$

   control input and initial condition

   $$u_t = \begin{bmatrix} \cos(t) \\ \sin(2t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

   with $\Delta t = 0.01$ and noise covariance $q = 0.2$. Simulate the system for 5 seconds (500 time steps).

   (iii) **(PTS:0-2)** With the generated trajectory and control input as known data, used least squares to estimate A and B. How good is your estimate?

3. **ARX Model**

   Consider the following dynamic model:

   $$y_k = \sum_{i=1}^{n} \phi_i y_{k-i} + \sum_{i=1}^{p} \gamma_i u_{k-i}$$

where u_i is a known input. This is known as the ARX (AutoRegressive model with eXogenous input) model for system identification.

(PTS:0-4) Given measurements of y_i and the known inputs u_i recast the above model into least squares form and determine estimates for ϕ_i and γ_i. (Your answers can be in terms of variable expressions.)

4. **Picking Basis Functions**

Data for temperature in the arctic over a period of years is given in the file: `hw1-arcticdata.csv`. Find the least squares fit to these data using:

(i) A polynomial curve fit with at least three terms: $h_i(t) = t^{i-1},\ i \in \{1, 2, ...\}$

(ii) A trigonometric curve fit with at least three terms: $h_i(t) = \sin(i\omega t)$. Note: you will need to choose ω.

(iii) A trigonometric curve fit with at least three terms: $h_i(t) = \begin{cases} \sin(\frac{1}{2}i\omega t), & i \text{ even} \\ \cos(\frac{1}{2}(i-1)\omega t), & i \text{ odd} \end{cases}$, $i \in \{1, 2, 3, ...\}$

For each case, produce the following:

(i) **(PTS:0-2)** A plot of the original data along with the least squares curve.

(ii) **(PTS:0-2)** Compute the root mean squared error (RMS).

$$\text{RMS} = \sqrt{(\hat{y} - \tilde{y})^T(\hat{y} - \tilde{y})}$$

where $\hat{y} = H\hat{x}$.

(iii) **(PTS:0-2)** Experiment to determine the number of terms necessary to produce are an RMS error less than 1.5 degrees C.

(iv) **(PTS:0-2)** The prediction of temperature for the next year following the given data.