AA 549: Estimation and Kalman Filtering
Homework #4
Due: Sunday, May 2, 2021 @ 11:59 pm

1. **Batch/Sequential Least Squares**

   **(PTS:0-6)** Derive the sequential least squares update given in lecture.

2. **Parallel Axis Theorem**

   **(PTS: 0-2)**. Prove the parallel axis theorem for covariance matrices

$$E\Big[\big(x - E[x]\big)\big(x - E[x]\big)^T\Big] = E[xx^T] - E[x]E[x]^T$$

3. **Bias and Variance Insights**

   (a) **(PTS:0-2)**. Show that if $\hat{x}$ is an unbiased estimate of $x$ and var$(\hat{x})$ does not equal $0$, then $\hat{x}^2$ is not an unbiased estimate of $x^2$.

   (b) **(PTS:0-2)**. If $\hat{x}$ is an estimate of $x$ and its bias is $b = E[\hat{x}] - x$. Show that $E\big[(\hat{x} - x)^2\big] =$ var$(\hat{x}) + b^2$.

4. **Biased Estimator Variance Bound**

   **(PTS:0-4)** Suppose that an estimator of a non-random scalar $x$ is biased, with bias denoted by $b(x)$. Show that a lower bound on the variance of the estimate $\hat{x}$ is given by

$$\text{var}(\hat{x} - x) \geq \left(1 = \frac{db}{dx}\right)^2 J^{-1}$$

   where

$$J = E\left[\left(\frac{\partial}{\partial x}\ln\big(p(\tilde{y}|x)\big)\right)^2\right], \qquad b(x) = \int_{-\infty}^{\infty}(x - \hat{x})p(\tilde{y}|x)d\tilde{y}$$

5. **Nonlinear Least Squares**

   **(PTS:0-6)** Problem 1.17 in the textbook (reproduced here).

   A measurement process used in three-axis magnetometers for low-Earth attitude determination involves the following measurement model:

$$b_j = A_j r_j + c + \epsilon_j$$

   where $b_j$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time $t_j$, $r_j$ is the corresponding value of the geomagnetic field with respect to some reference coordinate system, $A_j$ is the orthogonal attitude matrix (see Section A.7.1 in the book), $c$ is the magnetometer bias, and $\epsilon_j$ is the measurement error. We can eliminate the dependence on the attitude by transposing terms and computing the square, and can define an effective measurement by

$$\tilde{y}_j = b_j^T b_j - r_j^T r_j$$

which can be rewritten to form the following measurement model:

$$\tilde{y}_j = 2b_j^T c - c^T c + v_j$$

where $v_j$ is the effective measurement error, whose closed-form expression is not required for this problem. For this exercise assume that

$$Ar = \begin{bmatrix} 10\sin(0.001t) \\ 5\sin(0.002t) \\ 10cos(0.001t) \end{bmatrix}, \qquad c = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.6 \end{bmatrix}$$

Also, assume that $\epsilon$ is given by a zero-mean Gaussian noise process with standard deviation given by 0.05 in each component. Using the above values, create 1001 synthetic measurements of $b$ and $\tilde{y}$ at 5-second intervals. The estimated output is computed from

$$\hat{y}_j = 2b^T j\hat{c} - \hat{c}^T \hat{c}$$

where $\hat{c}$ is the estimated solution from the nonlinear least square iterations. Use nonlinear least squares to determine $\hat{c}$ for a starting value of $x_c = [0\ 0\ 0]^T$,. Also, try various starting values to check convergence. Note: $r^T r = r^T A^T A r$, since $A^T A = I$.

6. **Minimum Variance Estimation with Prior**

Write a simple computer program to simulate measurements of some discretely measured process

$$\tilde{y}_j = x_1 + x_2 \sin(10t_j) + x_3 e^{2t_j^2} + v_j, \qquad j = 1, 2, \ldots, 11$$

with $t_j$ sampled every 0.1 seconds. The true values $(x1, x2, x3)$ are $(1, 1, 1)$ and the measurement errors are synthetic Gaussian random variables with zero mean. The measurement-error covariance matrix is diagonal where

$$R = E[vv^T] = \text{diag}[\sigma_1^2\ \sigma_2^2\ \cdots\ \sigma_{11}^2]$$

with

$$(\sigma_1,\ \cdots,\ \sigma_{11}) = \big(0.001,\ 0.002,\ 0.005,\ 0.010,\ 0.008,\ 0.002,\ 0.010,\ 0.007,\ 0.020,\ 0.006,\ 0.001\big)$$

You are also given the a priori x-estimates

$$x_a^T = (1.01, 0.98, 0.99)$$

and associated a priori covariance matrix

$$Q = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix}$$

(a) **(PTS:0-2)** Use the minimal variance estimation version of the least squares equations

$$\hat{x} = P\big(H^T R^{-1} \tilde{y} + Q^{-1} \hat{x}_a\big)$$

to compute the parameter estimates and estimate covariance matrix

$$P = \left(H^T R^{-1} H + Q^{-1}\right)^{-1}$$

with the $j$th row of $H$ given by $[1\ \sin(10t_j)\ e^{2t_j^2}]$. Calculate the mean and standard deviation of the residual

$$r_j = \tilde{y}_j - \left(\hat{x}_1 + \hat{x}_2 \sin(10t_j) + \hat{x}_3 e^{2t_j^2}\right)$$

as

$$r = \frac{1}{11}\sum_{j=1}^{11} r_j, \qquad \sigma_r = \left[\frac{1}{10}\sum_{j=1}^{11} r_j^2\right]^{\frac{1}{2}}$$

(b) **(PTS:0-2)** Do a parametric study in which you hold the a priori estimate covariance $Q$ fixed, but vary the measurement-error covariance according to

$$R' = \alpha R$$

with $\alpha = 10^{-3}, 10^{-2}, 10^{-1}, 10, 10^2, 10^3$. Study the behavior of the calculated results for the estimates of $\hat{x}$, the estimate covariance matrix $P$, and mean $r$ and standard deviation $\sigma_r$ of the residual.

(c) **(PTS:0-2)** Do a parametric study in which $R$ is held fixed, but $Q$ is varied according to

$$Q' = \alpha Q$$

with $\alpha$ taking the values as in part $(b)$. Compare the results for the estimates $\hat{x}$, the estimate covariance matrix $P$, the mean $r$ and standard deviation $\sigma_r$ of the residual with those of part $(b)$.

7. **Project topic: (PTS: 0-4)**

Write up a description of the estimation or identification problem you will be studying for your project. You do not need to indicate what techniques you will be using to solve the problem (although if you know you may include that information). The points that you must address are:

- What system are you studying?
- Will your data be generated directly from experiment, provided from available sources, or constructed from simulation?
- What are the measurements y and the data to be determined x? (note that your data may be time-varying states in the case of estimation or parameters as in identification)
- What type of model will you be using (linear, nonlinear, continuous time, etc)? You should write up this project description using the first two steps of the problem solving guideline (available on the course web page).