LINEAR ALGEBRA REVIEW:

Coordinates & Bases $\quad P \in \mathbb{R}^{n \times n}$

$$\underline{x} = P\underline{x}'$$

$\underline{x}'$ → coords of x w.r.t the basis P

$$P = \begin{bmatrix} \underbrace{p_1 \cdots p_n}_{\text{lin ind cols}} \end{bmatrix}$$

cols of P are a basis for $\mathbb{R}^n$

basis → lin ind vectors that span a space

$$= \begin{bmatrix} p_1 \cdots p_n \end{bmatrix} \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

$$x = p_1 x_1' + \cdots + p_n x_n'$$

$$x = P\underline{x}' \qquad x = Q\underline{x}'' \qquad Px' = Qx''$$

$$x' = \underline{P^{-1}Q}\,x''$$

$$x'' = \underline{Q^{-1}P}\,x'$$

Similarity Transform: $A \in \mathbb{R}^{n \times n}$

$$y = Ax \qquad x = \underline{P}x', \quad y = \underline{P}y'$$

$$Py' = APx'$$

$$y' = \underbrace{P^{-1}AP}_{A'}x'$$

A' performs same action as A but w.r.t the P coords.

A is similar to A'

Similar matrices A & A'

- same eigenvalues
- $Tr(A) = Tr(A') = \sum_i A_{ii} = \sum_i \lambda_i$
- $det(A) = det(A') = \prod_i \lambda_i$

P is transform on eigenvectors

$A' = P^{-1}AP$

Diagonalization (A is diagonalizable)

P: cols are right eigenvectors

D: diagonal $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ ← eigenvalues

$AP = PD \Rightarrow$

$A\begin{bmatrix} P_1 \cdots P_n \end{bmatrix} = \begin{bmatrix} P_1 \cdots P_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\begin{bmatrix} AP_1 \cdots AP_n \end{bmatrix} = \begin{bmatrix} \lambda_1 P_1 \cdots \lambda_n P_n \end{bmatrix}$

$\boxed{A = PDP^{-1}}$

Q: rows are left evecs

$QA = DQ \Rightarrow \boxed{A = Q^{-1}DQ}$ $\quad Q = P^{-1}$

$A = \underset{\text{cols are right evecs}}{P}\ \underset{\text{diag eigenvalues}}{D}\ \underset{\text{rows are left evecs}}{\underline{P^{-1}}} \Leftarrow$

$spec(A) = \{\lambda_1 \cdots \lambda_n\}$ ← set of evals

Right evecs $P = \begin{bmatrix} p_1 \cdots p_n \end{bmatrix}$ $\quad P^{-1}P = I$

left evecs $P^{-1} = \begin{bmatrix} -\varepsilon_1^T- \\ \vdots \\ -\varepsilon_n^T- \end{bmatrix}$ $\quad \begin{cases} \varepsilon_j^T p_i = 0 & i \neq j \\ \varepsilon_i^T p_i = 1 \end{cases}$

$$Ap_i = PDP^{-1}p_i$$

$$= \begin{bmatrix} p_1 \cdots p_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \underline{\begin{bmatrix} \varepsilon_1^T \\ \vdots \\ \varepsilon_n^T \end{bmatrix}} p_i = \lambda_i p_i$$

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\lambda_i p_i = \begin{bmatrix} p_1 \cdots p_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 \cdots p_n \end{bmatrix} \qquad Q = \begin{bmatrix} \varepsilon_1^T \\ \vdots \\ \varepsilon_n^T \end{bmatrix}$$

$$Q = P^{-1}$$

$$QP = \begin{bmatrix} \varepsilon_1^T \\ \vdots \\ \varepsilon_n^T \end{bmatrix} \begin{bmatrix} p_1 \cdots p_n \end{bmatrix} = \begin{bmatrix} \varepsilon_1^T p_1 & \cdots & \cancel{\varepsilon_1^T p_n} \\ \cancel{\varepsilon_n^T} p_1 & \cdots & \varepsilon_n^T p_n \end{bmatrix}$$

for normal ε_i, p_i, $p_i^T p_i = 1$

$$\varepsilon_i^T p_i = \cos\theta_i$$

$$Q = \underline{P^T = P^{-1}}$$

cols are $\perp$

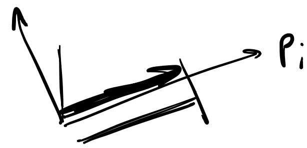
(P is orthonormal or a rotation.)

in general

$$P^{-1}P = I \quad \Rightarrow \quad q_j^T p_i = 0 \quad j \neq i$$

$$q_i^T p_i = 1$$

$$\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \cdots p_n \end{bmatrix}}_{P}$$



Diagonalizable:

sufficient: $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

no repeat evals $\Rightarrow$ diagonalizable

if repeat evals, but still full basis of evecs $\Rightarrow$ diagonalizable

General case: Jordan Canonical Form.

$$A = \underbrace{\begin{bmatrix} P^1 \,|\, \cdots \,|\, P^k \end{bmatrix}}_{P} \begin{bmatrix} \lambda_1 I + N^1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k I + N^k \end{bmatrix} P^{-1}$$

$$P^i = \begin{bmatrix} \underset{\text{evec}}{p_1^i} & \underbrace{\cdots \; p_x^i}_{\text{gen evecs}} \end{bmatrix} \qquad N^i = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

Spectral Mapping Thm: $f(\cdot)$: polynomial

$A \in \mathbb{R}^{n \times n}$ $\quad A = PDP^{-1}$ $\quad \lambda \in \text{spec}(A)$

$f(A) \in \mathbb{R}^{n \times n}$ $\quad f(A) = P f(D) P^{-1} = P \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} P^{-1}$ $\quad f(\lambda) \in \text{spec}(f(A))$

Matrix Exponential:

$$e^{At} := I + At + \frac{(At)^2}{2} + \frac{1}{3!}(At)^3 + \cdots$$

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$\dot{x} = Ax \quad x(0) = x_0 \implies \underline{x(t)} = \underline{e^{At} x_0}$$

$$x = Px'$$

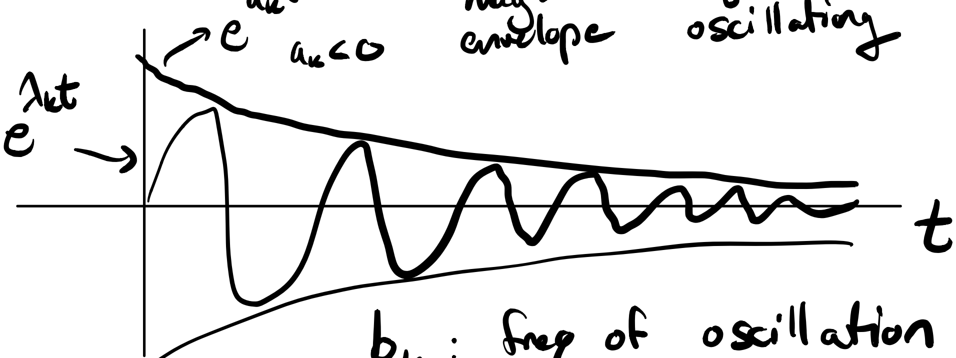
$$\dot{x}' = P^{-1}APx' = Dx' \implies \dot{x}_i' = \lambda_i x_i'$$

$$x_i'(t) = e^{\lambda_i t} x_i'(0)$$

$$x_k'(t) = \underline{e^{\lambda_k t}} \underbrace{x_k'(0)}_{\text{init. const}}$$

$$\lambda_k = a_k + b_k j \Longleftarrow \text{complex \#}$$

$$e^{\lambda_k t} = e^{(a_k + b_k j)t} = \underbrace{e^{a_k t}}_{\text{mag. envelope}} \underbrace{e^{b_k j t}}_{\text{mag 1 oscillating}}$$



b_k: freq of oscillation

Symmetric: $S = S^T$ $\quad S \in \mathbb{R}^{n \times n}$

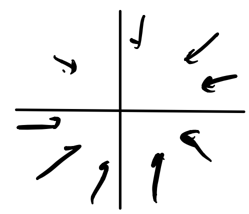
- purely real eigenvalues
- orthogonal eigenvectors $R = [R_1 \cdots R_n]$ right evecs

$S = RDR^T \qquad R^T R = I$

- $\dot{x} = Sx \longrightarrow \exists\, f(x) = \frac{1}{2} x^T S x$ s.t. $Sx = \frac{\partial f}{\partial x}^T$

gradient flow fields

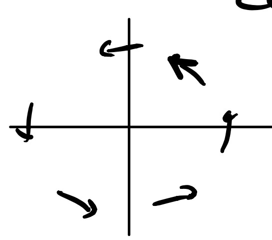
ODE: descending a surface -



Skew Symmetric: $K = -K^T$

- purely imag eigenvalues $\lambda = \pm bi$
- $K = R' \begin{bmatrix} 0 & -b & \\ b & 0 & \\ & & \ddots \end{bmatrix} R'^T \longrightarrow$ real matrices w/ orthonormal cols.
- $\dot{x} = Kx$

rotational vector field:



e^{Kt}: rotation matrix if $K = -K^T$

$$A = S + K$$

$$= \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{sym}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{skew sym}} =$$

$$\dot{x} = Ax = \underbrace{Sx}_{\substack{\text{grad}\\ \text{flow}}} + \underbrace{Kx}_{\substack{\text{rot.}\\ \text{flow}}} \leftarrow \underline{\text{Helmholtz}} \text{ decomp.}$$

$$\underline{\mathbb{R}^{n \times n}} = \underset{\substack{\downarrow \\ \text{set of} \\ n\times n \text{ sym}}}{\text{Sym}} \overset{\perp}{\oplus} \underset{\substack{\downarrow \\ \text{set of} \\ n\times n \text{ skew sym}}}{\text{Skew}}$$

Matrix inner product: $\langle S, K \rangle = \sum_{ij} S_{ij} K_{ij} = 0$

$$= \mathrm{Tr}(S^T K)$$

$$\overset{\checkmark}{} \quad \cancel{\downarrow}$$

$$A = S + K$$

$$\downarrow \qquad \searrow$$

$$\text{spec}(S) \qquad \text{spec}(K)$$

$$\cap \qquad\qquad \cap$$

$$\text{Reals} \qquad \text{Imag.}$$

$a \in \text{spec}(S)$ $\quad bi \in \text{spec}(K)$

is $\lambda = \underline{a} + \underline{b}i \in \underline{\text{spec}}(A)$?

<u>NO</u>. N<u>OT</u> TR<u>UE</u>

$A = S + K$ analogous $z = a + bi$

Polar Decomposition:

Grammians: A^TA, AA^T

$$A^TA = \begin{bmatrix} A_1^T \\ \vdots \\ A_n^T \end{bmatrix} \begin{bmatrix} A_1 \cdots A_n \end{bmatrix} = \begin{bmatrix} A_1^TA_1 & \cdots & A_1^TA_n \\ A_n^TA_1 & \cdots & A_n^TA_n \end{bmatrix}$$

$$A^TR^T(RA) = A^T\underbrace{R^TR}_{I}A = A^TA$$

AA^T: $\Leftarrow$ relative orient. & mag of rows.

$(A^TA)^{1/2}$, $(AA^T)^{1/2}$

$z = a + bi$

$|z| = \sqrt{a^2 + b^2}$

$= \sqrt{z^* z}$

mag of complex #

$$A = \underbrace{A(A^TA)^{-1/2}}_{\text{rotation/reflection}} \underbrace{(A^TA)^{1/2}}_{\text{positive definite}}$$

$$z = \underbrace{|z|}_{\text{mag}}\,\underbrace{e^{i\theta}}_{\text{rot.}} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$(A^TA)^{-1/2} A^TA (A^TA)^{-1/2} = I$$

columns of $A(A^TA)^{-1/2}$ are orthonormal

Question: set of vectors $A = [\underbrace{A_1 \cdots A_k}_{\text{lin ind.}}]$

find a set of orthonormal vectors that span the same subspace.

- Gram Schmidt...
- $A(A^TA)^{-1/2} \Leftarrow$ cols span $\mathcal{R}(A)$ also orthonormal.

Projection Matrix:

$$\underline{A(A^TA)^{-1}A^T} = \underbrace{A(A^TA)^{-1/2}}_{\text{orthonormal basis for a subspace}} \underbrace{(A^TA)^{-1/2}A^T}_{\text{projects onto subspace}}$$

to return.

$$A = \underbrace{A(A^TA)^{-1/2}}_{\text{rotation}} \underbrace{(A^TA)^{1/2}}_{\text{pos def.}}$$

analogous to $z = |z| e^{i\theta}$

$|z|$: mag, pos

$e^{i\theta}$: rot

$$A = \underbrace{(AA^T)^{1/2}}_{\text{pos def.}} \underbrace{(AA^T)^{-1/2}A}_{\text{rotation}}$$

both valid polar decompositions