LINEAR KF

- DT KF
  - fin, inf
- CT KF
  - fin, inf } → limit of DT case
- $\underbrace{\text{CT}}_{\text{dynamics}} / \underbrace{\text{DT}}_{\text{measurement}}$ KF } → practice

Nonlinear

- Extended KF (EKF)

$$\dot{x} = f(x, u, t) + G(t)w(t) \qquad w(t) \sim N(0, Q(t))$$

$$\tilde{y} = h(x, t) + v(t) \qquad v(t) \sim N(0, R(t))$$

GAIN: $K(t) = P(t)H^T(t)R^{-1}(t)$

nonlinear dynamics

PROP: $\dot{\hat{x}} = f(\hat{x}, u, t) + K(t)[\tilde{y} - h(\hat{x}, t)]$

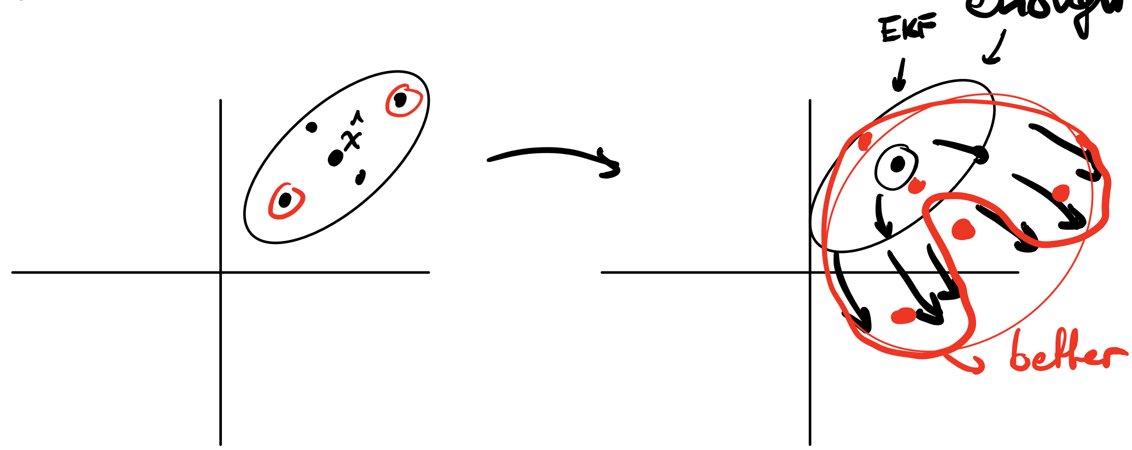
linearize

COV: $\dot{P}(t) = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G(t)^T$

$$F(t) = \frac{\partial f}{\partial x}\bigg|_{\hat{x}, u} \qquad H(t) = \frac{\partial h}{\partial x}\bigg|_{\hat{x}}$$

- CT/DT EXTENDED KF

What if the linearization is not accurate enough?



Unscented KALMAN FILTER (UKF)

$$x_{k+1} = f(x_k, w_k, u_k, k)$$

$$\tilde{y}_k = h(x_k, u_k, v_k, k)$$

Before DT filter

$$\hat{x}_k^+ = \hat{x}_k^- + K_k e_k^- \qquad e_k^- = \tilde{y}_k - H_k \hat{x}_k^-$$

$$P_k^+ = P_k^- - K_k \underbrace{P_k^{e_y e_y}} K_k^T$$

$$\boxed{P_k^{e_y e_y} = E[e_k^- e_k^{-T}]}$$

$$\hat{x}_k^+ - \hat{x}_k^- = K_k e_k^-$$

$$\boxed{P_k^{e_x e_y} = E\left((\hat{x}_k^+ - \hat{x}_k^-) e_k^{-T}\right)}$$

$$P_k^{e_x e_y} = K_k E\left[e_k^- e_k^{-T}\right]$$

$$= K_k P_k^{e_y e_y}$$

$$\Rightarrow \boxed{K_k = P_k^{e_x e_y} \left(P_k^{e_y e_y}\right)^{-1}} \Leftarrow$$

$$\underbrace{\overline{P_k^- H_k^T}}_{\text{num. approx}} \underbrace{\left(H_k P_k^- H_k^T + R_k\right)^{-1}}_{\text{numerically approx}}$$

linear case

non linear case

unscented KF gives nonlin. way of estimating these covariances...

Updates:

1. $\hat{x}_k^+ = \hat{x}_k^- + K_k e_k^-$
2. $P_k^+ = P_k^- - K_k P_k^{e_y e_y} K_k^T$
3. $K_k = P_k^{e_x e_y} \left(P_k^{e_y e_y}\right)^{-1}$

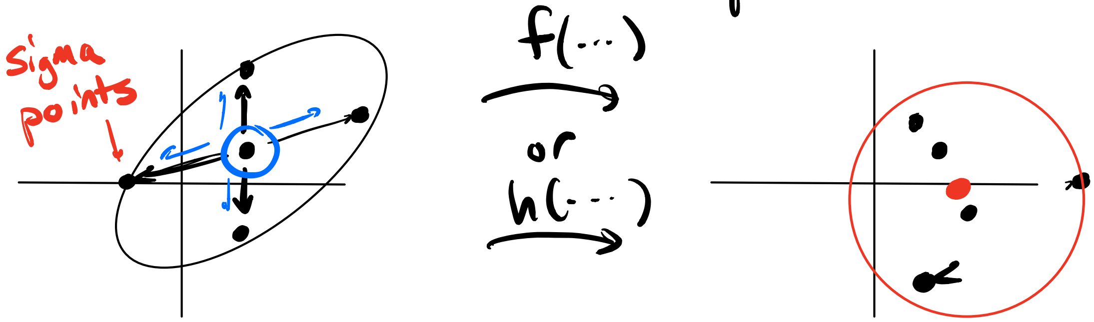
Estimating $P_k^{e_x e_y}$, $P_k^{e_y e_y}$

Augmented state.

$$x_k^a = \begin{bmatrix} x_k \\ w_k \\ v_k \end{bmatrix} \qquad P_k^a = \underbrace{\begin{bmatrix} P_k^+ & P_k^{xw} & P_k^{xv} \\ (P_k^{xw})^T & Q_k & P_k^{wv} \\ (P_k^{xv})^T & (P_k^{wv})^T & R_k \end{bmatrix}}_{\text{sampling points in this augmented space.}}$$

length: L

Choose sigma points



$\sigma_k \leftarrow 2L$ columns from $\pm \gamma \sqrt{P_k^a}$ come back to

$\rightarrow \chi_k^{a(0)} = \hat{x}_k^a = \begin{bmatrix} \hat{x}_k \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \rightarrow w_k \\ \rightarrow v_k \end{matrix}$ zero mean

(center point)

$\rightarrow \chi_k^{a(i)} = \sigma_k^{(i)} + \hat{x}_k^a$

Parameters $\rightarrow$ book wikipedia

L : length of x_k^a

γ : $\sqrt{L+\lambda}$ $\leftarrow$

λ : $\alpha^2(L+\kappa) - L$

$10^{-4} \leq \alpha \leq 1$: spread of σ points

κ :

β : prior knowledge $\beta = 2$

$$W_0^{mean} = \frac{\lambda}{L+\lambda} \qquad W_0^{cov} = \frac{\lambda}{L+\lambda} + (1 - \alpha^2 + \beta)$$

$$W_i^{mean} = W_i^{cov} = \frac{1}{2(L+\lambda)} \quad i = 1, 2, \ldots, 2L$$

PROPAGATE POINTS

$$\mathcal{X}_{k+1}^{x(i)} = f(\underbrace{\mathcal{X}_k^{x(i)}}, \underbrace{\mathcal{X}_k^{w(i)}}, u_k, k)$$ propagating thru dynamics

subvectors

$\mathcal{X}_k^a \in \mathbb{R}^L$

$$\mathcal{Y}_k^{(i)} = h(\mathcal{X}_k^{x(i)}, u_k, \mathcal{X}_k^{v(i)}, k)$$ meas. eqn.

RECONSTRUCT

$$\hat{x}_k^- = \sum_{i=0}^{2L} W_i^{mean} \mathcal{X}_k^{x(i)}$$

$$P_k^- = \sum_{i=0}^{2L} W_i^{cov} [\mathcal{X}_k^{x(i)} - \hat{x}_k^-][\mathcal{X}_k^{x(i)} - \hat{x}_k^-]^T$$

$$\hat{y}_k^- = \sum_{i=0}^{2L} W_i^{mean} \mathcal{Y}_k^{(i)}$$

$$P_k^{e_y e_y} = \sum_{i=0}^{2L} W_i^{cov} [\mathcal{Y}_k^{(i)} - \hat{y}_k^-][\mathcal{Y}_k^{(i)} - \hat{y}_k^-]^T$$

$$P_k^{e_x e_y} = \sum_{i=0}^{2L} W_i^{cov} [\mathcal{X}_k^{x(i)} - \hat{x}_k^-][\mathcal{Y}_k^{(i)} - \hat{y}_k^-]^T$$

GAIN $$K_k = P_k^{e_x e_y} (P_k^{e_y e_y})^{-1}$$

Update

1. $\hat{x}_k^+ = \hat{x}_k^- + K_k e_k^-$ ←

2. $P_k^+ = P_k^- - K_k P_k^{e_y e_y} K_k^T$ ←

SAMPLE

use $\hat{x}_k^+$, P_k^+ } → compute sample points

SAMPLING FROM P_k^a to get $\sigma_k^{(i)}$

cols of $\pm \gamma \sqrt{P_k^a}$ → matrix square root

Matrix Square Root:

2 meanings $A \in \mathbb{R}^{n \times n}$

- $A^{1/2} = B$ s.t. $B^2 = BB = A$ ✓
- $A^{1/2} = B$ s.t. $BB^T = A$]

spectral mapping thm.

$A = PDP^{-1}$ $\quad A^{1/2} = P\underset{\pm}{\underline{D}}^{1/2}P^{-1}$

$(A^{1/2}A^{1/2}) = PD^{1/2}\underset{+}{\underline{P^{-1}P}}D^{1/2}P^{-1}$

$A = A^T \succ 0$ $\quad \exists\, R$ s.t. $R^T = R^{-1}$

$A = RDR^T$ $\quad A^{1/2} = RD^{1/2}R^T$

orthogonal eigenvectors

$A^{1/2} = (A^{1/2})^T \succ 0$

$A^{1/2}$ sym, PD, unique

if instead want $BB^T = A$ only makes sense if $A = A^T \succeq 0$

$B = U \Sigma V^T$ } in general.

$BB^T = U \Sigma V^T V \Sigma U^T$

$= U \Sigma^2 U^T$

$R \quad \Sigma = D^{1/2}$

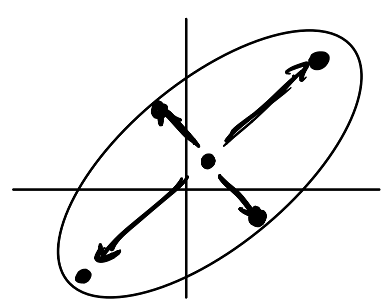
V is a free parameter can be anything.

many B's s.t. $BB^T = A$

2 choices

1. $B = RD^{1/2}$ ← length

cols axes of ellipsoid



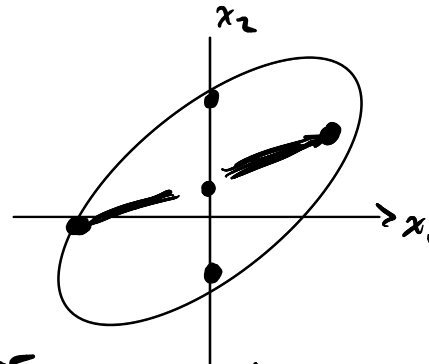
$$BB^T = RD^{1/2}D^{1/2}R^T = RDR^T = A$$

2. $A = LL^T$ Cholesky Decomp.

$L \in \mathbb{R}^{n \times n}$ L : lower triangular

$$B = L$$

$$L = \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$$



(in case it wasn't clear here $A = P_k^a$ $B = \sqrt{P_k^a}$)