Particle Filtering

Model: $x_{k+1} = f(x_k, u_k, w_k)$ $\quad w_k \sim p(w_k)$ white noise

$x_0 \sim p(x_0)$

$v_k \sim p(v_k)$ white noise

$\tilde{y}_k = h(x_k, v_k)$

Bayesian Filter:

$\tilde{Y}_{k+1} = \{\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_{k+1}\}$ measurement trajectory

WANT $p(x_{k+1} \mid \tilde{Y}_{k+1}) \rightarrow$ density over the state space

$$\left.\begin{array}{ll} p(x_k \mid \tilde{Y}_k) & \text{prev estimate} \\ p(x_{k+1} \mid x_k) & \text{dynamics} \\ p(\tilde{y}_{k+1} \mid x_{k+1}) & \text{meas} \end{array}\right\} \Rightarrow p(x_{k+1} \mid \tilde{Y}_{k+1})$$

new estimate

Bayes Rule:

$$p(x_{k+1}|\tilde{y}_{k+1},\tilde{Y}_k)\,p(\tilde{y}_{k+1}|\tilde{Y}_k) = p(x_{k+1},\tilde{y}_{k+1}|\tilde{Y}_k) = p(\tilde{y}_{k+1}|x_{k+1},\tilde{Y}_k)\,p(x_{k+1}|\tilde{Y}_k)$$

$$\boxed{p(x_{k+1}|\tilde{Y}_{k+1}) = \frac{\overbrace{p(\tilde{y}_{k+1}|x_{k+1},\tilde{Y}_k)}^{\text{meas}}\,\overbrace{p(x_{k+1}|\tilde{Y}_k)}^{\text{dynamics}}}{\underbrace{p(\tilde{y}_{k+1}|\tilde{Y}_k)}_{\text{normalize}}}}$$

$\hookrightarrow$ $p(x_{k+1}|\tilde{Y}_k) = \int p(x_{k+1}|x_k)\,p(x_k|\tilde{Y}_k)\,dx_k$

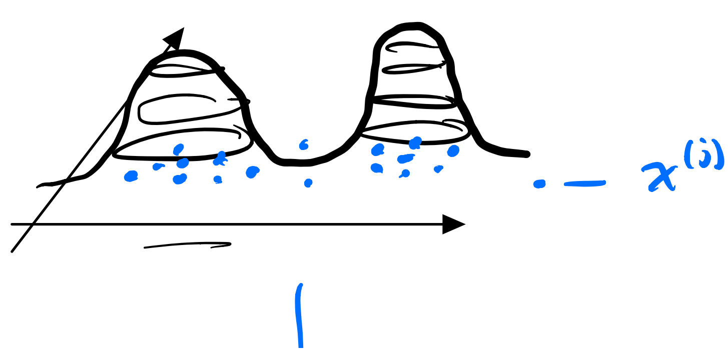
$$x_{k+1} - f(x_k, u_k, w_k) = 0$$

$\hookrightarrow$ $p(x_{k+1}|x_k) = \int \delta(x_{k+1} - f(x_k,u_k,w_k))\,p(w_k)\,dw_k$

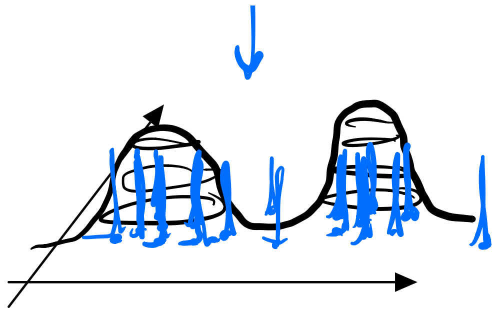
$\hookrightarrow$ $p(\tilde{y}_{k+1}|x_{k+1}) = \int \delta(\tilde{y}_{k+1} - h_{k+1}(x_{k+1},v_{k+1}))\,p(v_{k+1})\,dv_{k+1}$

Estimate Distributions w Particles

$p(x)$



$$\frac{1}{N}\sum_{j=1}^{N}\delta(x - x^{(j)})$$



$$p(x) = \frac{1}{N}\sum_{j=1}^{N}\delta(x - x^{(j)})$$

$$\int f(x)\,p(x)\,dx \approx \frac{1}{N}\sum_{j=1}^{N} f(x^{(j)}) \qquad x^{(j)} \sim p(x)$$

Importance Sampling:

Problem: don't have access to $p(x)$ ...

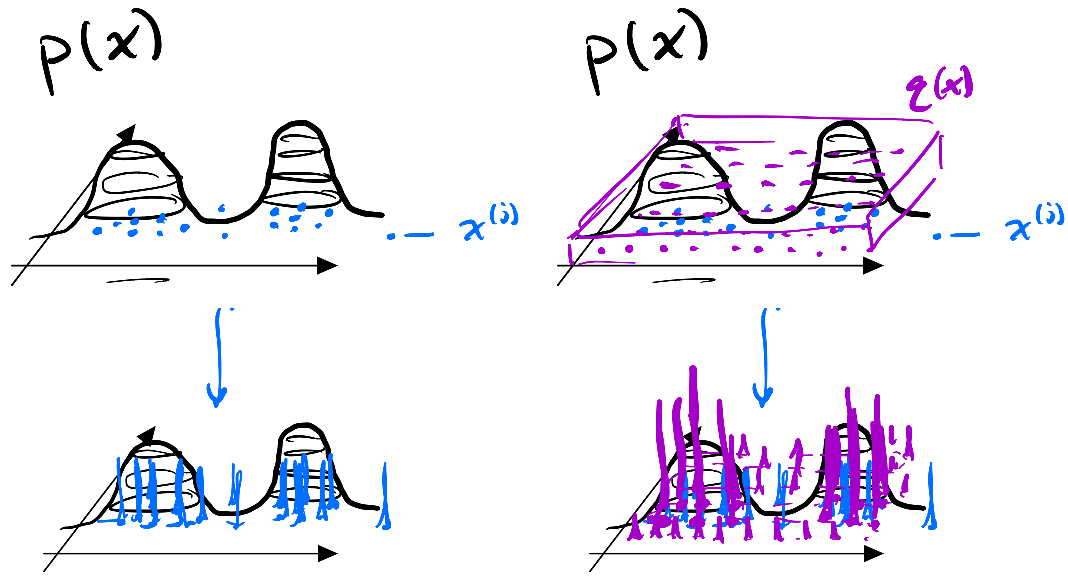
solution: sample from a different dist.

$q(x)$: importance function (another dist.)

$$\int f(x)\,p(x)\,dx = \sum_{j=1}^{N} w^{(j)} f(x^{(j)}) \qquad x^{(j)} \sim q(x)$$

$$w^{(j)} \propto \frac{p(x^{(j)})}{q(x^{(j)})} \qquad \sum_{j=1}^{N} w^{(j)} = 1$$

Ex:
$q(x)$: uniform



Come up with an iterative scheme
to update

$x^{(i)}$: particle positions

$w^{(i)}$: weights applied to ea particle

→ discrete representation of the distribution $p(x)$

Particle Trajectory:

$$X_{k+1}^{(i)} = \{x_0^{(i)}, \ldots, x_{k+1}^{(i)}\}$$

$$w_{k+1}^{(j)} = \frac{p\left(X_{k+1}^{(j)} \mid \tilde{Y}_{k+1}\right)}{q\left(X_{k+1}^{(j)} \mid \tilde{Y}_{k+1}\right)} \rightarrow$$

captures relationship between p & q

Assume $q\left(X_{k+1}^{(j)} \mid \tilde{Y}_{k+1}\right) = q\left(X_k^{(j)} \mid \tilde{Y}_k\right) \, q\left(x_{k+1}^{(j)} \mid X_k^{(j)}, \tilde{Y}_{k+1}\right)$

$$\boxed{w_{k+1}^{(j)} = \frac{\overbrace{p\left(\tilde{y}_{k+1} \mid x_{k+1}^{(j)}\right)}^{\text{meas. particle } j} \, \overbrace{p\left(x_{k+1}^{(j)} \mid x_k^{(j)}\right)}^{\text{dynamics of particle } j}}{\boxed{q\left(x_{k+1}^{(j)} \mid X_k^{(j)}, \tilde{Y}_{k+1}\right)}} \, w_k^{(j)}}$$

How the weights should evolve.

need to pick.

System is Markov:

$$q\left(x_{k+1}^{(j)} \mid X_k^{(j)}, \tilde{Y}_{k+1}\right) = q\left(x_{k+1}^{(j)} \mid x_k^{(j)}, \tilde{y}_{k+1}\right)$$

Optimal choice for q:

$$q\left(x_{k+1}^{(j)} \mid x_k^{(j)}, \tilde{y}_{k+1}\right)_{opt} = \frac{p\left(\tilde{y}_{k+1} \mid x_{k+1}^{(j)}, x_k^{(j)}\right) p\left(x_{k+1}^{(j)} \mid x_k^{(j)}\right)}{p\left(\tilde{y}_{k+1} \mid x_k^{(j)}\right)}$$

$$w_{k+1}^{(j)} \propto w_k^{(j)} \, p\left(\tilde{y}_{k+1} \mid x_k^{(j)}\right)$$

$$p(\tilde{y}_{k+1} | x_k^{(j)}) = \int p(\tilde{y}_{k+1} | x_{k+1}) \, p(x_{k+1} | x_k^{(j)}) \, dx_k$$

can be computed for a model of the form.

$$\left.\begin{aligned} x_{k+1} &= f(x_k, u_k) + \Upsilon_k \bar{w}_k \,, \quad \bar{w}_k \sim N(0, Q_k) \\ \tilde{y}_{k+1} &= H_k x_k + v_k \quad\quad v_k \sim N(0, R_k) \end{aligned}\right]$$

soln: in book 278 (Eqn 4.179)

BOOTSTRAP FILTER:

$$q(x_{k+1} | X_k^{(j)}, \tilde{Y}_{k+1}) = p(x_{k+1} | x_k^{(j)})$$

- $w_{k+1}^{(j)} \propto w_k^{(j)} \, p(\tilde{y}_{k+1} | x_{k+1}^{(j)})$ simple form for weight update
- draw samples from

  $p(x_{k+1} | x_k^{(j)}) \longrightarrow$ based on dynamics.

Steps for Particle Filter:

initially draw $x_0^{(j)}$ from $p(x_0) \rightarrow$ uniform

$x_k^{(j)}$ $j = 1, \ldots, N$ $\longleftarrow$ particles

Propagation Step:

$$\underline{x_{k+1}^{(j)}} = f(\underline{x_k^{(j)}, u_k, \bar{w}_k^{(j)}}) \quad \bar{w}_k^{(j)} \sim p(\bar{w}_k)$$

Update Weights

$$\underline{w_{k+1}^{(j)}} = w_k^{(j)} \, p(\underline{\tilde{y}_{k+1} | x_{k+1}^{(j)}})$$

$$w_{k+1}^{(j)} \longleftarrow \frac{w_{k+1}^{(j)}}{\sum_j w_{k+1}^{(j)}}$$

Resample resample w/ replacement.

$$\{ \underbrace{x_{k+1}^{(j)}}_{\substack{\text{values} \\ \text{discrete}}} \mid \underbrace{w_{k+1}^{(j)}}_{\substack{\text{prob.} \\ \text{dist.}}} \} \rightarrow \begin{matrix}\text{resample} \\ \text{to get} \\ \text{new} \\ x_{k+1}^{(j)}\end{matrix} \rightarrow \{ x_{k+1}^{(j)} \mid \tfrac{1}{N} \}$$

Roughening $x_{k+1}^{(j)} \longleftarrow x_{k+1}^{(j)} + \underline{c_{k+1}^{(j)}}$ $c_{k+1} \sim N(0, R)$

Problem