System ID:

State space:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

TF: $G(s) = C(sI - A)^{-1}B + D$

$$u(t) \longrightarrow \boxed{\text{Sys}} \longrightarrow y(t)$$

$[u_0 \cdots u_t]$ $\qquad\qquad$ $[y_0 \cdots y_t]$

Discrete frequency domain $\rightarrow$ z-transform

s-transform

(Laplace transform)

2 representations of signals:

<u>time</u> $\Longleftrightarrow$ (dual) frequency

$u(t), y(t)$ $\qquad\qquad$ $U(s), Y(s)$

$\cos(\omega t) \rightarrow$ quite odd.

- transfer funcs
- Laplace / z-transforms / Fourier transform / DFT
- wave/particle duality, quantum mechanics
- Heisenberg uncertainty $\Leftarrow$ Fourier transform

Circulant Matrices $\Leftarrow$

$$c = \begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_{t-1} \end{bmatrix} \updownarrow \text{discrete time vector} \qquad C = \begin{bmatrix} c_0 & c_{t-1} & \cdots & c_1 \\ \vdots & c_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_{t-1} \\ c_{t-1} & c_{t-2} & \cdots & c_0 \end{bmatrix}$$

Discrete convolution: - Banded
- Toeplitz

$$x = \begin{bmatrix} x_0 \\ \vdots \\ \vdots \\ x_{t-1} \end{bmatrix} \qquad \underbrace{c * x}_{\text{convolution}} = Cx$$

Any $n \times n$ circulant has the same eigenvectors no matter what vector c

Reason: $C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{t-1} S^{t-1}$

$$S = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ 0 & 1 & \ddots & \\ 0 & \vdots & 1 & 0 \end{bmatrix}$$

S is a shift matrix

if c & x are signals in time

$\Rightarrow$ S just shifts signal 1 step.

eigenvectors of S are eigenvectors of C

$\hookrightarrow$ Discrete Fourier Basis vectors

F : cols of F $\quad F \in \mathbb{C}^{n \times n}$

F is a unitary matrix (orthogonal) $\rightarrow$ $FF^* = I$

$F^* = F^{-1}$

DFT: on x $\quad \frac{1}{n} F^* \underline{x}$

Cx

Diagonalize C:

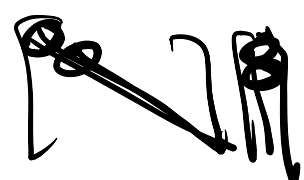
$$C = \underline{F}\, \mathrm{diag}(F^* c) \underbrace{F^*}_{F^{-1}} \underline{x}$$

The eigenvectors of taking a step in time are the discrete Fourier basis vectors

Eigenfunctions of $\frac{d}{dt}$ $\Rightarrow$ Fourier basis functions

$y = c * x$

$y = Cx = F\, \mathrm{diag}(F^* c) F^* x$



$$\underbrace{F^* y}_{\text{DFT } y} = \mathrm{diag}(\underbrace{F^* c}_{\text{DFT of } c}) \underbrace{F^* x}_{\text{DFT } x} \Bigg\} \rightarrow \text{elementwise multiply of } F^* c \;\&\; F^* x$$

Convolution in the time domain $\Longrightarrow$ multiplication in frequency

Parseval's Theorem:

$$\underbrace{(F^* x)^* F^* x}_{|F^* x|^2} = x^* F F^* x = \underbrace{x^* x}_{|x|^2}$$

Laplace Transform: (continuous time)

$$y(t) : \xrightarrow{\mathcal{L}} Y(s) = \int_0^\infty e^{-st} \underbrace{y(t)}\, dt$$

think of as a coord. transform.

$$\mathcal{L}(\dot{y}(t)) = sY(s) - y(0)$$

taking a derivative

y represented in the eigenbasis of derivation

s is all of the eigenvalues

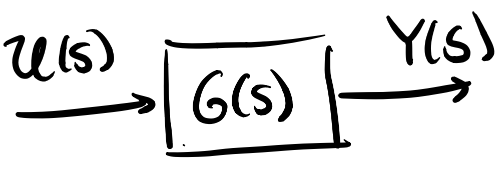
s is a "continuous diagonal matrix"

Convolution becomes multiplication:

$$y(t) = \int_{-\infty}^{t} g(t-\tau)u(\tau)\, d\tau = \int_0^\infty g(\tau)u(t-\tau)\, d\tau$$

reparametrization of time

$$Y(s) = G(s)\, U(s)$$



$g(t)$: impulse response of a system

if $u(t) = \delta(t) \longrightarrow y(t) = g(t)$

$u(t) = \delta(t-t')$ $\qquad$ $y(t) = g(t-t')$

$$y(t) = \int_{-\infty}^{t} g(t-\tau)\,\underbrace{\delta(\tau-t')}_{\text{zero everywhere except at } t'}\,d\tau$$

$$= g(t-t')$$

convolution gives system response to $u(t)$

$$u(t) = \sum_{t'} u_{t'}\,\delta(t-t')$$

$\mathcal{L}(\delta(t)) = 1$ $\qquad$ $u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$

$u(t) = \delta(t)$ $\qquad$ $Y(s) = G(s)\,\underbrace{U(s)}_{1}$

$$Y(s) = G(s)$$

Intuitively:

impulse excites all freq of system

$$\mathcal{L}(e^{At}) = (sI-A)^{-1}$$

Geometric Series:

$\underline{0 < \gamma < 1}$ $\qquad \underline{\sum_{t=0}^{\infty} \left(\frac{1}{\gamma}\right)^t} = \frac{1}{1-\gamma}$

$$\mathcal{L} \mid e^{At} = \underline{\sum_{t=0}^{\infty} \frac{(At)^t}{t!}} = (sI - A)^{-1}$$

$$\dot{x} = Ax + Bu \Rightarrow x(t) = e^{At} x(0) + \int_0^t \underline{e^{A(t-\tau)} B} u(\tau) d\tau$$

$$y = Cx \qquad y(t) = Cx(t)$$

$$y(t) = Ce^{At} x(0) + \int_0^t \underline{Ce^{A(t-\tau)} B} u(\tau) d\tau$$

$$g(t) = Ce^{At}B$$

How to do this in discrete time.

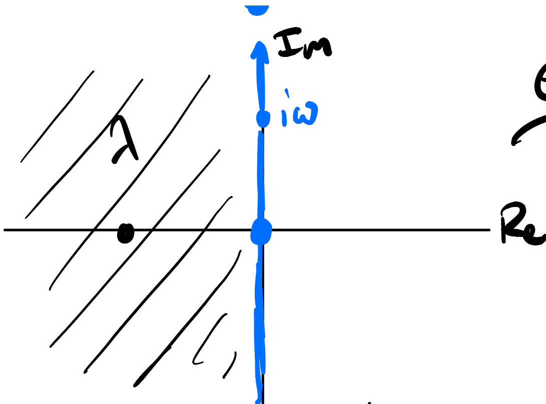
Ex. CT: $\dot{x} = Ax + Bu \Rightarrow$ DT: w/ time step Δt $x^+ = \bar{A}x + \bar{B}u$

A, eigenvals $\lambda \Rightarrow \bar{A} = e^{A\underline{\Delta t}}$, eigenvals $\mu = e^{\lambda \Delta t}$

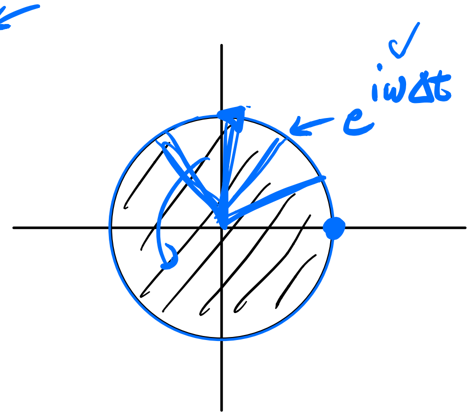
$\lambda \longrightarrow e^{\lambda \Delta t}$

Laplace transform $\updownarrow$ $\qquad$ $\underline{\text{z-transform}}$

$s \in \mathbb{C} \longrightarrow z = e^{s\Delta t} \in \mathbb{C}$



$e^{\lambda \Delta t}$

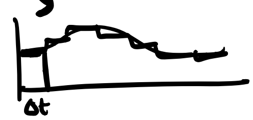


CT: stability

$$\underset{\text{Laplace transform}}{Y(s)} \Longrightarrow \underset{\text{Fourier transform}}{\underline{Y(i\omega)}}$$

CT: integration:

$$y(t) = C\underline{e^{At}}x(0) + \int_0^t C\underline{e^{A(t-\tau)}}\,\underline{Bu(\tau)}\,d\tau$$



DT: integration

$$y_{k+1} = C\underline{\bar{A}^{k+1}}x(0) + \sum_{k'=0}^{k} C\underline{\bar{A}^{(k-k')}}Bu_{k'}$$

t is at $k+1$ time step $\Rightarrow$ $t = (k+1)\Delta t$

$$\bar{A}^{k+1} = \left(e^{A\Delta t}\right)^{k+1}$$
$$= e^{A(\Delta t(k+1))}$$
$$= e^{At}$$

Z-transform:

$$y_k \longrightarrow Y(z) = \sum_{k=0}^{\infty} z^{-k} y_k$$

compare w/

$$Y(s) = \int_0^{\infty} e^{-st} y(t)\, dt$$

$$z \approx e^{s\Delta t}$$ where $t = k\Delta t$

Back to Circulant matrices

infinite version of $F^* y \longrightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$F = \begin{bmatrix} F_1 & \cdots & F_n \end{bmatrix} \qquad \begin{bmatrix} F_1^* \\ \vdots \\ F_n^* \end{bmatrix} y = \begin{bmatrix} F_1^* y \\ \vdots \\ F_n^* y \end{bmatrix}$$

for periodic signals $\longrightarrow$ finite # of Fourier basis vectors

Discrete Convolution:

$$y(t) = \int_{-\infty}^{t} g(t-\tau) u(\tau)\, d\tau = \int_0^{\infty} g(\tau) u(t-\tau)\, d\tau$$

$$y_k = \sum_{k'=0}^{k} \underline{g_{k-k'}}\, \underline{u_{k'}} = \sum_{k'=0}^{\infty} g_{k'} u_{k-k'}$$

DT: impulse response

Cx

$$g = \begin{bmatrix} g_0 \\ \vdots \\ g_k \end{bmatrix} u_0$$

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = \underbrace{\begin{bmatrix} g_0 & 0 & 0 & \cdots & 0 \\ g_1 & g_0 & & & \\ \vdots & g_1 & & & 0 \\ g_k & g_{k-1} & & & g_0 \end{bmatrix}}_{} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_k \end{bmatrix}$$

$$y = G\, u$$

Can think about z-transforms
DT- convolutions

as infinite matrices
(Heisenberg quantum mechanics)
inf. dim line algebra. $\langle \psi | x \rangle$

inf. matrix.

wave func. continuous vec

Z-transform:

$\delta_{k-k'} \longrightarrow z^{-k'}$

a shift back in time of k'

shift matrices are diagonalized the z-transforms.

related to shift matrices

$y_0 \cdots y_k \leftarrow u_0 \cdots u_k$

outputs inputs

Sys ID:

$y_k = a_{k-1} y_{k-1} + \cdots + a_{k-k''} y_{k-k''} + b_{k-1} u_{k-1} + \cdots + b_{k-k'} u_{k-k'}$

$=$

$$y_k = \sum_{j=1}^{k''} a_{k-j} y_{k-j} + \sum_{j=1}^{k'} b_{k-j} u_{k-j}$$

Discrete transfer functions

apply z-transform:

$$Y(z) = \sum_{j=1}^{k''} a_{k-j} z^{-j} Y(z) + \sum_{j=1}^{k'} b_{k-j} z^{-j} U(z)$$

$$Y(z) = \underbrace{\frac{\sum_{j=1}^{k'} b_{k-j} z^{-j}}{1 - \sum_{j=1}^{k''} a_{k-j} z^{-j}}}_{\text{discrete transfer function}} U(z)$$

$$Y(z) = \underbrace{\frac{b_{k-1} z^{-1} + \cdots + b_{k-k'} z^{-k'}}{1 - a_{k-1} z^{-1} - \cdots - a_{k-k''} z^{-k''}}}_{\text{rational expression in terms of } z^{-1}} U(z)$$

TF: rational expressions of s