LINEAR ALGEBRA REVIEW – PART 2.

POLAR DECOMPOSITION:

$$z = \underbrace{|z|}_{\text{pos def}} \underbrace{e^{i\theta}}_{\text{rot}}$$

$$A = \underbrace{A(A^TA)^{-1/2}}_{\text{orthonormal columns}} \cdot \underbrace{(A^TA)^{1/2}}_{\text{positive definite}}$$

$$= \underbrace{(AA^T)^{1/2}}_{\text{pos def.}} \underbrace{(AA^T)^{-1/2} A}_{\text{orthonormal}}$$

SINGULAR VALUE DECOMPOSITION:

spec. – set of evals

singular values : $\sigma_1 \ldots \sigma_k > 0$

$\sigma_1^2 \ldots \sigma_k^2 \in \operatorname{spec}(A^TA)$ $\quad$ $\sigma_1^2, \ldots, \sigma_k^2 \in \operatorname{spec}(AA^T)$

$\sigma_1 \ldots \sigma_k \in \operatorname{spec}(A^TA)^{1/2}$ $\quad$ $\sigma_1 \ldots \sigma_k \in \operatorname{spec}(AA^T)^{1/2}$

$A \in \mathbb{R}^{m \times n}$

$$\underline{A} = \underbrace{U}_{\substack{m\times m \\ \text{orthonormal}}} \underbrace{\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}}_{\text{diagonal, pos.}} \underbrace{V^T}_{\substack{n\times n \\ \text{orthonormal}}} \qquad S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

$$= \begin{bmatrix} \underline{U_1} & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ \hline 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} x$$

U_1 : cols orthonormal basis for $R(A)$

U_2 : cols orthonormal basis for $N(A^T)$

V_1^T : rows " " " $R(A^T)$

V_2^T : rows " " " $N(A)$

Moore-Penrose Pseudo Inverse $A \in \mathbb{R}^{m \times n}$

$$A^{\dagger} = V \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$AA^{+} = U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V^T V \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= U_1 U_1^T \leftarrow$$

other pseudo inverses

Least squares $A^{-L} = (A^TA)^{-1}A^T$ left inverse   A tall

for a left inverse to exist A^TA invertible

- Multiply on left $(A^TA)^{-1}A^T \times A = \boxed{I} \leftarrow$ identity.
- multiply on right $A(A^TA)^{-1}A^T = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} A^T \end{bmatrix} \leftarrow$ projection matrix onto the range (A)

projection : Identity on a particular subspace

deletes components in an orthogonal subspace

Minimum Norm $A^{-R} = A^T(AA^T)^{-1}$ right inverse

- Multi. on right $AA^T(AA^T)^{-1} = I$
- Multi on left $A^T(AA^T)^{-1}A \leftarrow$ proj onto range (A^T)

Connections

(A^TA) invertible... plug in SVD $\quad A = U\begin{bmatrix} S \\ 0 \end{bmatrix} V^T$

$$A^TA = V\begin{bmatrix} S & \cancel{0} \end{bmatrix} U^T U \begin{bmatrix} S \\ \cancel{0} \end{bmatrix} V^T = VS^2V^T \leftarrow$$

$$A(A^TA)^{-1} = U\begin{bmatrix} S \\ 0 \end{bmatrix} V^T V S^{-2} V^T = U\begin{bmatrix} S \\ 0 \end{bmatrix} S^{-2} V^T$$

$$= U\begin{bmatrix} S^{-1} \\ 0 \end{bmatrix} V^T = A^+$$

$y = Ax$

A not full row or col rank

$x = A^+ y \implies$ smallest norm x that makes Ax as close as possible to y

(min norm & least squares soln)

Connections

w/ polar decomposition: $A \in \mathbb{R}^{n\times n}$ invertible

SVD: $A = USV^T$

POLAR

$$A = \underbrace{A(A^TA)^{-1/2}}_{USV^TVS^{-1}V^T \,=\, UV^T}\underbrace{(A^TA)^{1/2}}_{VSV^T}$$

$$A = \underbrace{(AA^T)^{1/2}}_{USU^T}\underbrace{(AA^T)^{-1/2}A}_{US^{-1}U^TUSV^T \,=\, UV^T}$$

Positive Definite: $Q \in \mathbb{R}^{n\times n}$ $Q = Q^T$

$x^T Q x > 0 \quad \forall x$ defn of pos def

[Q is PD iff spec(Q) > 0] only true for $Q = Q^T$

red ← Q is sym.

Q PD ⇒ select x to be evec of Q...

$x^T Q x = \lambda x^T x$ ← if not > 0 contradiction

spec $Q > 0$ ⇒ $Q = R\underset{\text{all pos diagonals}}{D}R^T$

$$\forall x, \quad x^T Q x = \underbrace{x^T R}_{z^T} D \underbrace{R^T x}_{z}$$

$$= \sum_i \lambda_i z_i^2 > 0$$

Do similarity transforms preserve pos. def. ness.?

→ PQP^{-1} → eigenvalues are still pos.

is this symmetric? probably not

Congruent Transform: P is invertible...

$PQP^T \longrightarrow$ eigenvalues the same? probably not

if $x^TQx > 0 \ \forall x$

then $x^TPQP^Tx > 0 \ \forall x$ $\Big\} \rightarrow$ PD is preserved

$z = P^T x$ $\qquad z^TQz > 0 \ \forall z$

$\Rightarrow x^TPQP^Tx > 0 \ \forall x$

When are congruent transforms the same as similarity transforms?

if P is a rotation... $P^{-1} = P^T$.

suppose Q is not sym...

$$x^TQx = x^T\tfrac{1}{2}(Q+Q^T)x + x^T\tfrac{1}{2}(Q-Q^T)x$$

$$= x^T\tfrac{1}{2}(Q+Q^T)x \qquad \tfrac{1}{2}(x^TQx - x^TQ^Tx)$$

0

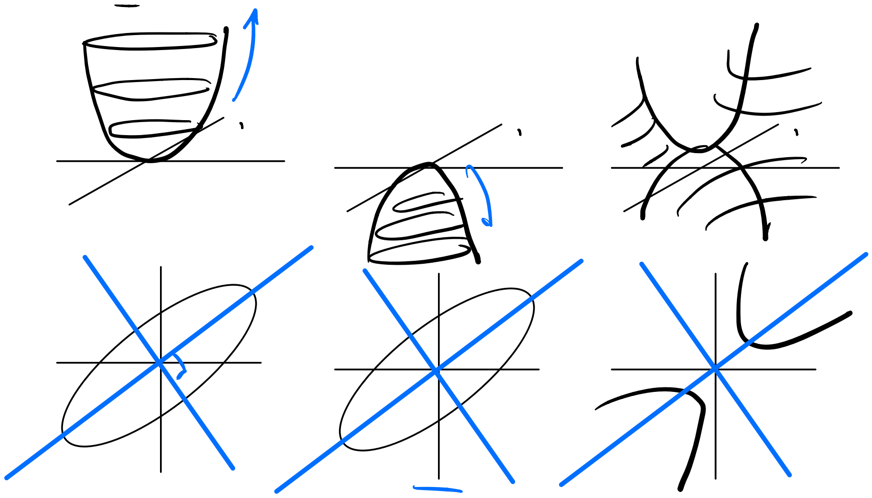
$K = -K^T$

$x^TKx = 0$

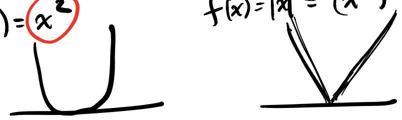
Quadratic Forms: $Q = Q^T$

$f(x) = x^T Q x$

$Q \succeq 0 \qquad Q \prec 0 \qquad Q$ indef.



$f(x) = x^2 \qquad f(x) = |x| = (x^2)^{1/2}$



$f(x) = (x^T Q x)^{1/2} \Leftarrow$

$x_1^2 + x_1 x_2 (\,) + x_2^2$

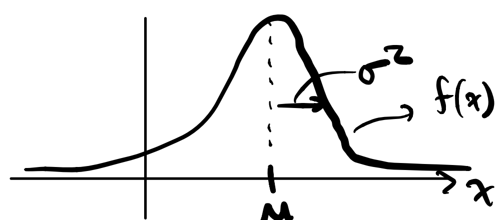
Gaussian or Normal Distributions

mean μ

covariance Σ

$$N(\mu, \Sigma)$$

Scalar: $N(\mu, \sigma^2)$

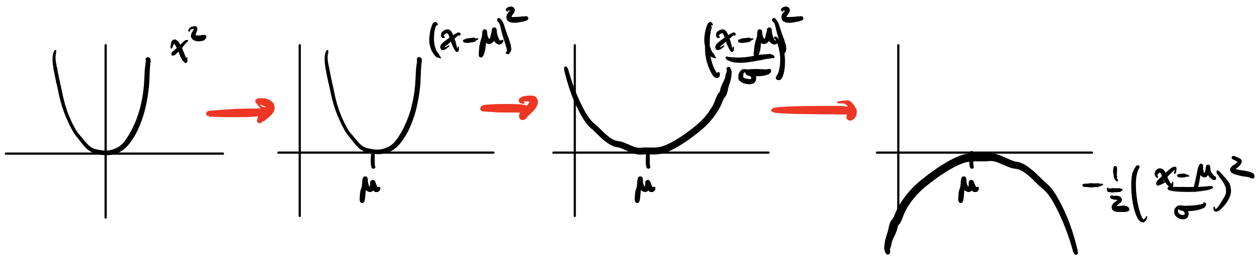


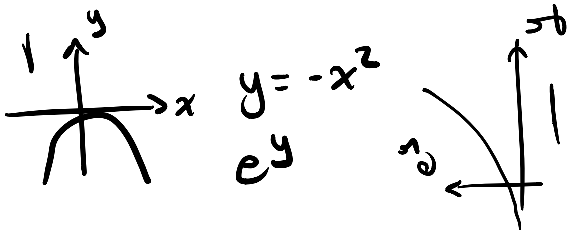
$\mu \in \mathbb{R}$

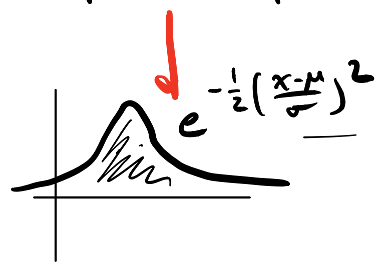
$\Sigma = \sigma^2$

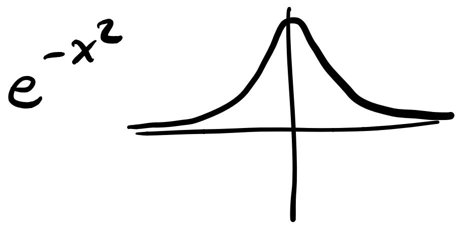
$$f(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

density function









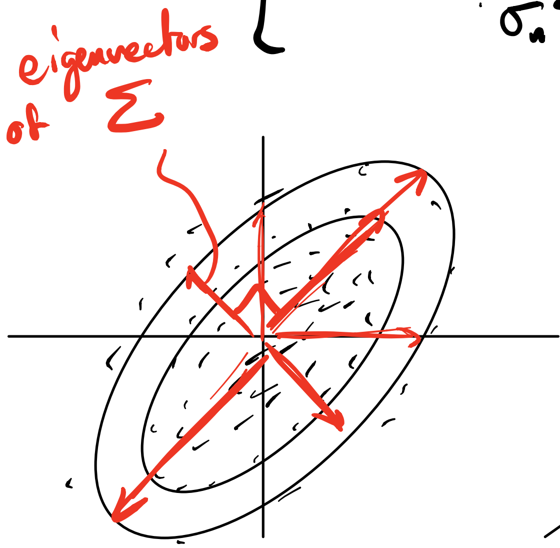
Multivariable case $\quad N(\mu, \Sigma)$

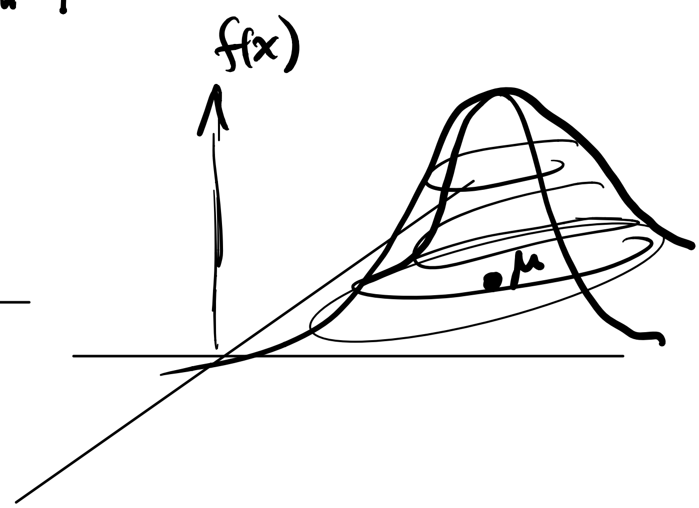
$\mu \in \mathbb{R}^n$

$\Sigma \in \mathbb{R}^{n \times n}$

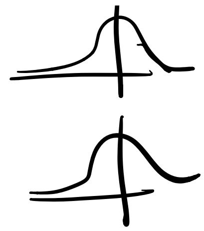
$\Sigma = \Sigma^T$

$$\Sigma = \begin{bmatrix} \boxed{\sigma_1^2} & \sigma_1\sigma_2 & & \\ \sigma_1\sigma_2 & \boxed{\sigma_2^2} & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$





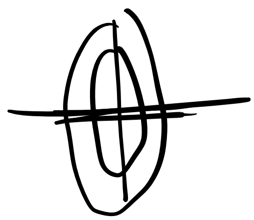
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} \rightarrow N(\mu_1, \sigma_1^2) \\ \rightarrow N(\mu_2, \sigma_2^2) \end{matrix}$



x_1, x_2 are **ind.** distributed

↓

$$N\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2\end{bmatrix}\right)$$



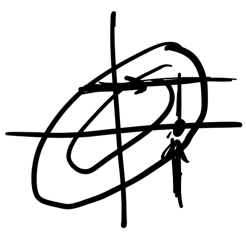
ind. x_1 & x_2

Σ covariance diagonal

Not independent

$$N\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \Sigma\right)$$

Σ ↓ full (not diagonal)



pdf: multivariate Gaussian

$$f(x) = \frac{1}{\left((2\pi)^k \det(\Sigma)\right)^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$