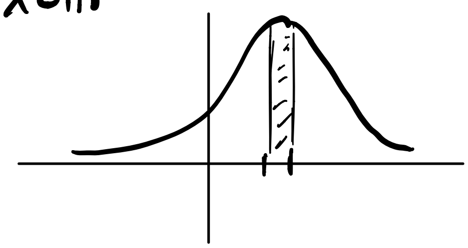
PROBABILITY CONCEPTS:

$x \in \mathbb{R}^n$ — should be measure

$p(x)$: density function

$x \in \mathbb{R}$



$$\int_x p(x)\, dx = 1.$$

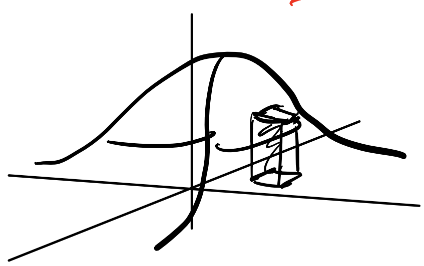
Ex. Normal/Gaussian densities

Multivaraite Densities:

continuous

$x \in \mathbb{R}^2$

$p(x) = p(x_1, x_2)$



$$\int_x p(x) dx = \int_{x_1} \int_{x_2} p(x_1, x_2)\, dx_1\, dx_2 = 1$$

Marginal Density:

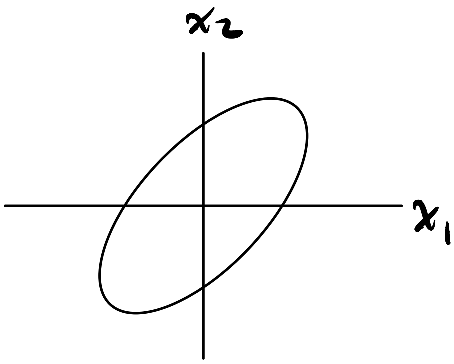
Ex. Discrete Distribution

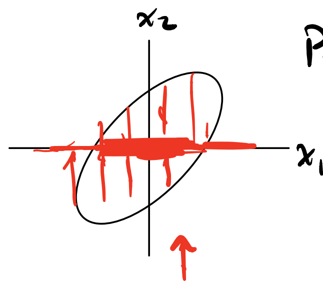
	$\leftarrow x_1 \rightarrow$		$P_{x_2}(x_2)$
$P_{x_1}(x_1)$	0.3	0.7	
$\uparrow x_2 \downarrow$	0.2	0.4	0.6
	0.1	0.3	0.4

$$P_{x_1}(x_1) = \sum_{x_2} p(x_1, x_2)$$

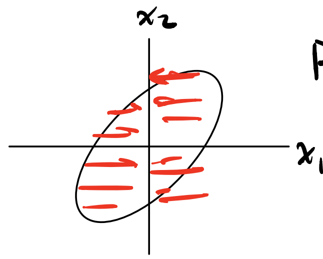
$$P_{x_2}(x_2) = \sum_{x_1} p(x_1, x_2)$$

Continuous Distribution





$$P_{x_1}(x_1) = \int_{x_2} p(x_1, x_2)\, dx_2$$

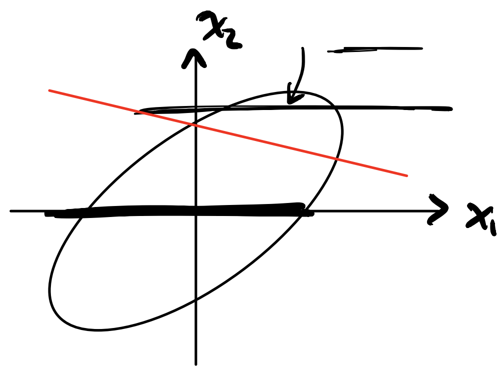


$$P_{x_2}(x_2) = \int_{x_1} p(x_1, x_2)\, dx_1$$

Conditional Distribution

$x \in \mathbb{R}^2 \qquad p(x_1, x_2)$

ex. Condition : $\boxed{x_2 = c}$



$$p(x_1 \mid x_2 = c) = \frac{1}{\int_{x_1} p(x_1, c)\, dx_1}\, p(x_1, c)$$

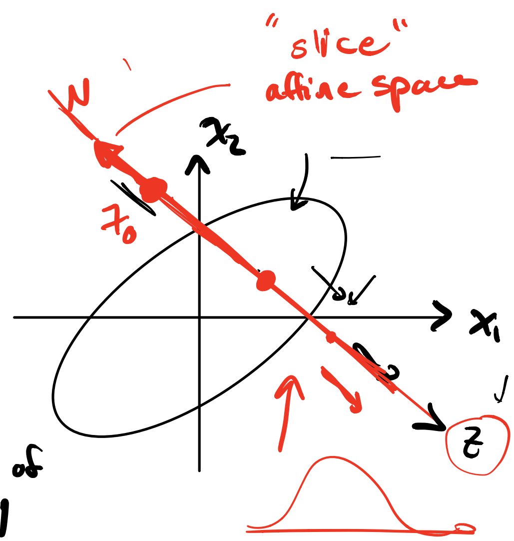
ex. Condition :

Affine space : $\{x \mid y = Ax\}$ A fat matrix.

or explicitly enumerate nullspace of A.

basis for $\mathcal{N}(A) \rightarrow$ cols of N

$\Rightarrow \{x \mid x = Nz + x_0\}$ where $y = Ax_0$



Multivariate Normal: $x \in \mathbb{R}^n$ $\quad x \sim N(\mu, \Sigma)$

$$p(x) = \frac{1}{((2\pi)^n \det(\Sigma))^{1/2}}\, e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

solve for

$$p(z) = \frac{1}{((2\pi)^n |\Sigma|)^{1/2}}\, e^{-\frac{1}{2}\left(z^T N^T + x_0^T - \mu^T\right) \Sigma^{-1} \left(x_0 + Nz - \mu\right)}$$

$$= \frac{e^{-\frac{1}{2} const}}{((2\pi)^n |\Sigma|)^{1/2}}\, e^{-\frac{1}{2}\left(z^T N^T \Sigma^{-1} N z + 2(x_0^T - \mu^T)\Sigma^{-1} N z + (x_0^T - \mu^T)\Sigma^{-1}(x_0 - \mu)\right)}$$

$$\to (z - \mu_z)^T (N^T \Sigma^{-1} N)(z - \mu_z) \boxed{+ const}$$

renormalize...

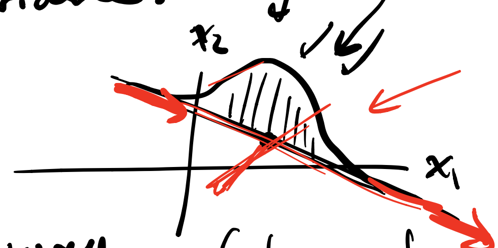
$$p(z) \sim e^{-\frac{1}{2}\left(z^T N^T \Sigma^{-1} N z + 2(x_0^T - \mu^T)\Sigma^{-1} N z + (x_0^T - \mu^T)\Sigma^{-1}(x_0 - \mu)\right)}$$

$$z \sim N\left(\mu_z, (N^T \Sigma^{-1} N)^{-1}\right)$$

covariance.

$\mapsto \Sigma^{-1} [\;] = [\;]$ full rank

$p(x \mid y = Ax)$ $\quad$ N basis for $N(A)$



write x in a different way... (change of coords)

$$x = [A^T N][A^T N]^{-1} x$$

$$= \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} (AA^T)^{-1} A \\ (N^TN)^{-1} N^T \end{bmatrix} x$$

Trick.

$$\begin{bmatrix} AA^T & 0 \\ 0 & N^TN \end{bmatrix}^{-1} \begin{bmatrix} A \\ N^T \end{bmatrix} \begin{bmatrix} A^T & N \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

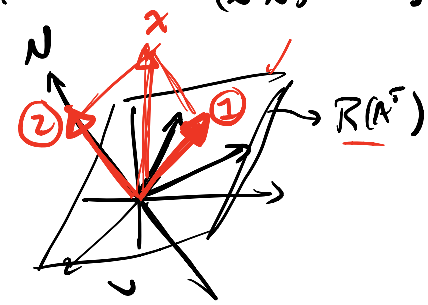
$$\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} (AA^T)^{-1} A \\ (N^TN)^{-1} N^T \end{bmatrix}$$

$$x = A^T(AA^T)^{-1}A\,x \quad ①$$

projection of x onto $R(A^T)$

$$+\ N(N^TN)^{-1}N^T x \quad ②$$

projection onto $N(A)$



applying condition $y = Ax$...

$$x = \left(A^T(AA^T)^{-1}A\,x + N(N^TN)^{-1}N^T x \right)$$

$$\boxed{x = A^T(AA^T)^{-1}y + N(N^TN)^{-1}N^T x}$$

plug into the density func.

this component of x doesn't affect y.

$$P(x \mid y = Ax) \sim e^{-\frac{1}{2}\left(N(N^TN)^{-1}N^Tx + A^T(AA^T)^{-1}y - \mu\right)^T \Sigma^{-1} \left(N(N^TN)^{-1}N^Tx + A^T(AA^T)^{-1}y - \mu\right)}$$

new covariance of x conditioned on $y = Ax$

$$N(N^TN)^{-1}N^T\Sigma^{-1}N(N^TN)^{-1}N^T \quad x$$

low rank

Covariance is degenerate : has 0 eigenvalues

cond. density function
is a lower dim
slice in a higher dim space