

# Block Matrix Multiplication

**Linear Algebra**

**Winter 2022 - Dan Calderone**

# Block Matrix Multiplication

**Matrix  
Multiplication**

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

**Block Matrix  
Multiplication**

$$AB = \begin{matrix} & \begin{matrix} \overbrace{\hspace{1.5cm}}^{n_1} & & \overbrace{\hspace{1.5cm}}^{n_N} \end{matrix} \\ \begin{matrix} m_1 \\ \vdots \\ m_M \end{matrix} \begin{bmatrix} \boxed{A_{11}} & \cdots & \boxed{A_{1N}} \\ \vdots & & \vdots \\ \boxed{A_{M1}} & \cdots & \boxed{A_{MN}} \end{bmatrix} & \begin{matrix} \overbrace{\hspace{1.5cm}}^{p_1} & & \overbrace{\hspace{1.5cm}}^{p_P} \\ \begin{bmatrix} \boxed{B_{11}} & \cdots & \boxed{B_{1P}} \\ \vdots & & \vdots \\ \boxed{B_{N1}} & \cdots & \boxed{B_{NP}} \end{bmatrix} \end{matrix} \end{matrix}$$

*General  
Case*

$$= \begin{matrix} & \overbrace{\hspace{3cm}}^{p_1} & & \overbrace{\hspace{3cm}}^{p_P} \\ \begin{matrix} m_1 \\ \vdots \\ m_M \end{matrix} \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{matrix}$$

# Block Matrix Multiplication

Matrix  
Multiplication

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

Block Matrix  
Multiplication

$$AB = \begin{matrix} m_1 \\ \vdots \\ m_M \end{matrix} \begin{bmatrix} \overbrace{A_{11} \cdots A_{1N}}^{n_1 \quad n_N} \\ \vdots \\ A_{M1} \cdots A_{MN} \end{bmatrix} \begin{matrix} n_1 \\ \vdots \\ n_N \end{matrix} \begin{bmatrix} \overbrace{B_{11} \cdots B_{1P}}^{p_1 \quad p_P} \\ \vdots \\ B_{N1} \cdots B_{NP} \end{bmatrix}$$

All inner  
dimensions  
must match

General  
Case

$$= \begin{matrix} m_1 \\ \vdots \\ m_M \end{matrix} \begin{bmatrix} \overbrace{A_{11}B_{11} + \cdots + A_{1N}B_{N1}}^{p_1} & \cdots & \overbrace{A_{11}B_{1P} + \cdots + A_{1N}B_{NP}}^{p_P} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix}$$

# Block Matrix Multiplication

**Matrix  
Multiplication**

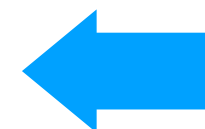
$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

**Block Matrix  
Multiplication**

$$AB = \begin{bmatrix} m_1 \text{I} & & \\ & \vdots & \\ & & m_M \text{I} \end{bmatrix} \begin{bmatrix} \overset{n_1}{A_{11}} & \cdots & \overset{n_N}{A_{1N}} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \begin{bmatrix} \overset{p_1}{B_{11}} & \cdots & \overset{p_P}{B_{1P}} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix}$$

*General  
Case*

$$= \begin{bmatrix} m_1 \text{I} & & \\ & \vdots & \\ & & m_M \text{I} \end{bmatrix} \begin{bmatrix} \overset{p_1}{A_{11}B_{11} + \cdots + A_{1N}B_{N1}} & \cdots & \overset{p_P}{A_{11}B_{1P} + \cdots + A_{1N}B_{NP}} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix}$$



**Outer  
dimensions  
stay the same**

# Block Matrix Multiplication

**Block Matrix Multiplication**

$$AB = \begin{matrix} & \begin{matrix} \overbrace{\hspace{1cm}}^{n_1} & & \overbrace{\hspace{1cm}}^{n_N} \end{matrix} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{matrix} \begin{matrix} \begin{matrix} \overbrace{\hspace{1cm}}^{p_1} & & \overbrace{\hspace{1cm}}^{p_P} \end{matrix} \\ n_1 \mathbf{I} \\ \vdots \\ n_N \mathbf{I} \end{matrix} \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} = \begin{matrix} & \overbrace{\hspace{2cm}}^{p_1} & & \overbrace{\hspace{2cm}}^{p_P} \\ m_1 \mathbf{I} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{matrix}$$

## Case 1a

“Linear  
Combination  
of Columns”

$$Ax = \begin{bmatrix} \left[ \begin{array}{c|c} A_1 & \cdots & A_n \end{array} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \left[ \begin{array}{c} A_1 \end{array} \right] x_1 + \cdots + \begin{bmatrix} \left[ \begin{array}{c} A_n \end{array} \right] x_n \end{bmatrix}$$

## Case 1b

“Linear  
Combination  
of Rows”

$$y^T A = \begin{bmatrix} \left[ \begin{array}{c} y_1 & \cdots & y_m \end{array} \right] \begin{bmatrix} \left[ \begin{array}{c} - a_1^T - \end{array} \right] \\ \vdots \\ \left[ \begin{array}{c} - a_m^T - \end{array} \right] \end{bmatrix} = \left[ \begin{array}{c} y_1 \left[ \begin{array}{c} - a_1^T - \end{array} \right] \\ \vdots \\ y_m \left[ \begin{array}{c} - a_m^T - \end{array} \right] \end{array} \right]$$

# Block Matrix Multiplication

**Block Matrix Multiplication**

$$AB = \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{A_{11} \cdots A_{1N}}^{n_1} \\ \vdots \\ \overbrace{A_{M1} \cdots A_{MN}}^{n_N} \end{bmatrix} \begin{matrix} n_1 \mathbf{I} & & \\ & \ddots & \\ & & n_N \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{B_{11} \cdots B_{1P}}^{p_1} \\ \vdots \\ \overbrace{B_{N1} \cdots B_{NP}}^{p_P} \end{bmatrix} = \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{A_{11}B_{11} + \cdots + A_{1N}B_{N1}}^{p_1} & \cdots & \overbrace{A_{11}B_{1P} + \cdots + A_{1N}B_{NP}}^{p_P} \\ \vdots & & \vdots \\ \overbrace{A_{M1}B_{11} + \cdots + A_{MN}B_{N1}}^{p_1} & \cdots & \overbrace{A_{M1}B_{1P} + \cdots + A_{MN}B_{NP}}^{p_P} \end{bmatrix}$$

## Case 2a

$$Ax = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ | \\ | \\ x \\ | \\ | \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

“Inner product with rows”

## Case 2b

$$y^T A = \begin{bmatrix} - & y^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} y^T A_1 & \cdots & y^T A_n \end{bmatrix}$$

“Inner product with columns”

# Block Matrix Multiplication

## Block Matrix Multiplication

$$AB = \begin{matrix} & \begin{matrix} \overbrace{\hspace{1cm}}^{n_1} & & \overbrace{\hspace{1cm}}^{n_N} \end{matrix} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{matrix} \begin{matrix} \begin{matrix} \overbrace{\hspace{1cm}}^{p_1} & & \overbrace{\hspace{1cm}}^{p_P} \end{matrix} \\ \mathbf{I} & \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{matrix} = \begin{matrix} & \overbrace{\hspace{2cm}}^{p_1} & & \overbrace{\hspace{2cm}}^{p_P} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{matrix}$$

### Case 3a

“A times each column of B”

$$AB = \begin{bmatrix} \boxed{A} \end{bmatrix} \begin{bmatrix} \boxed{B_1} \\ \vdots \\ \boxed{B_p} \end{bmatrix} = \begin{bmatrix} \boxed{AB_1} \\ \vdots \\ \boxed{AB_p} \end{bmatrix}$$

### Case 3b

“A times each sub-matrix of B (horizontal)”

$$AB = \begin{bmatrix} \boxed{A} \end{bmatrix} \begin{bmatrix} \boxed{-B_1-} \\ \vdots \\ \boxed{-B_p-} \end{bmatrix} = \begin{bmatrix} \boxed{-AB_1-} \\ \vdots \\ \boxed{-AB_p-} \end{bmatrix}$$

# Block Matrix Multiplication

**Block Matrix Multiplication**

$$AB = \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{A_{11}}^{n_1} & \cdots & \overbrace{A_{1N}}^{n_N} \\ \vdots & & \vdots \\ \overbrace{A_{M1}} & \cdots & \overbrace{A_{MN}} \end{bmatrix} \begin{matrix} n_1 \mathbf{I} \\ \vdots \\ n_N \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{B_{11}}^{p_1} & \cdots & \overbrace{B_{1P}}^{p_P} \\ \vdots & & \vdots \\ \overbrace{B_{N1}} & \cdots & \overbrace{B_{NP}} \end{bmatrix} = \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{A_{11}B_{11} + \cdots + A_{1N}B_{N1}}^{p_1} & \cdots & \overbrace{A_{11}B_{1P} + \cdots + A_{1N}B_{NP}}^{p_P} \\ \vdots & & \vdots \\ \overbrace{A_{M1}B_{11} + \cdots + A_{MN}B_{N1}} & \cdots & \overbrace{A_{M1}B_{1P} + \cdots + A_{MN}B_{NP}} \end{bmatrix}$$

## Case 4a

“B times each row of A”

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} - & a_1^T B & - \\ & \vdots & \\ - & a_m^T B & - \end{bmatrix}$$

## Case 4b

“B times each sub-matrix of A (vertical)”

$$AB = \begin{bmatrix} -A_1- \\ \vdots \\ -A_m- \end{bmatrix} \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} -A_1 B- \\ \vdots \\ -A_m B- \end{bmatrix}$$



# Block Matrix Multiplication

## Block Matrix Multiplication

$$AB = \begin{matrix} & \begin{matrix} \overbrace{\hspace{1cm}}^{n_1} & & \overbrace{\hspace{1cm}}^{n_N} \end{matrix} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{matrix} \begin{matrix} \begin{matrix} \overbrace{\hspace{1cm}}^{p_1} & & \overbrace{\hspace{1cm}}^{p_P} \end{matrix} \\ n_1 \mathbf{I} \\ \vdots \\ n_N \mathbf{I} \end{matrix} \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} = \begin{matrix} & \overbrace{\hspace{2cm}}^{p_1} & & \overbrace{\hspace{2cm}}^{p_P} \\ m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix}$$

### Case 5a

“Pairwise inner products of rows of  $A$  & columns of  $B$ ”

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1^T B_1 & \cdots & a_1^T B_p \\ \vdots & & \vdots \\ a_m^T B_1 & \cdots & a_m^T B_p \end{bmatrix}$$

### Case 5b

“Sum of outer products of columns of  $A$  and rows of  $B$ ”

$$AB = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & & \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & & \\ & & \end{bmatrix} + \cdots + \begin{bmatrix} | & & | \\ A_n & & \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_n^T & - \\ & & \\ & & \end{bmatrix}$$

# Block Matrix Multiplication

## Block Matrix Multiplication

$$AB = \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{A_{11}}^{n_1} & \cdots & \overbrace{A_{1N}}^{n_N} \\ \vdots & & \vdots \\ \overbrace{A_{M1}}^{n_1} & \cdots & \overbrace{A_{MN}}^{n_N} \end{bmatrix} \begin{matrix} n_1 \mathbf{I} & \cdots & n_N \mathbf{I} \\ \overbrace{B_{11}}^{p_1} & \cdots & \overbrace{B_{1P}}^{p_P} \\ \vdots & & \vdots \\ \overbrace{B_{N1}}^{p_1} & \cdots & \overbrace{B_{NP}}^{p_P} \end{matrix} = \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} \begin{bmatrix} \overbrace{A_{11}B_{11} + \cdots + A_{1N}B_{N1}}^{p_1} & \cdots & \overbrace{A_{11}B_{1P} + \cdots + A_{1N}B_{NP}}^{p_P} \\ \vdots & & \vdots \\ \overbrace{A_{M1}B_{11} + \cdots + A_{MN}B_{N1}}^{p_1} & \cdots & \overbrace{A_{M1}B_{1P} + \cdots + A_{MN}B_{NP}}^{p_P} \end{bmatrix}$$

### Case 6a

“Pairwise inner products of rows of  $A$  & columns of  $B$  around  $D$ ”

$$AB = \begin{bmatrix} - & a_1^T & - \\ \vdots & \vdots & \vdots \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_1^T DB_1 & \cdots & a_1^T DB_p \\ \vdots & & \vdots \\ a_m^T DB_1 & \cdots & a_m^T DB_p \end{bmatrix}$$

### Case 6b

“Sum of scaled outer products (diagonal)”

$$AB = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ \vdots & \vdots & \vdots \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} \vdots & d_{11} & [- & b_1^T & -] \\ \vdots & \vdots & \vdots \\ \vdots & d_{nn} & [- & b_n^T & -] \end{bmatrix} = \sum_i \begin{bmatrix} \vdots & A_i & d_{ii} & [- & b_i^T & -] \end{bmatrix}$$

### Case 6c

“Sum of scaled pairwise outer products”

$$AB = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ \vdots & \vdots & \vdots \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} \vdots & d_{11} & [- & b_1^T & -] \\ \vdots & \vdots & \vdots \\ \vdots & d_{1n} & [- & b_n^T & -] \\ \vdots & \vdots & \vdots \\ \vdots & d_{n1} & [- & b_1^T & -] \\ \vdots & \vdots & \vdots \\ \vdots & d_{nn} & [- & b_n^T & -] \end{bmatrix} = \sum_i \sum_j \begin{bmatrix} \vdots & A_i & d_{ij} & [- & b_j^T & -] \end{bmatrix}$$