Block Matrix Multiplication

Linear Algebra

Winter 2022 - Dan Calderone

Block Matrix Multiplication

Matrix Multiplication

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

Block Matrix Multiplication

General Case

$$AB = \begin{array}{c c} & \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c c} & \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{array}{c} n_1 \\ \\ n_N \end{array} & \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Block Matrix Multiplication

Matrix Multiplication

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

Block Matrix Multiplication

General Case

$$AB = \begin{array}{c} \\ m_1 \\ \\ m_M \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c} \\ n_1 \\ \\ n_N \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array}$$

All inner dimensions must match

$$= \begin{array}{c} \\ m_1 \\ \\ m_M \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Block Matrix Multiplication

Matrix Multiplication

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

Block Matrix Multiplication

General Case

$$AB = \begin{array}{c} \\ m_1 \\ \\ m_M \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & \quad & n_N \end{array} \\ \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c} \\ n_1 \\ \\ n_N \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & \quad & p_P \end{array} \\ \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ m_1 \\ \\ m_M \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & \qquad\qquad & p_P \end{array} \\ \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Outer dimensions stay the same

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{array}{c} \\ m_1 \\ \vdots \\ m_M \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c} \\ n_1 \\ \vdots \\ n_N \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array} = \begin{array}{c} \\ m_1 \\ \vdots \\ m_M \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Case 1a

"Linear Combination of Columns"

$$Ax = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

Case 1b

"Linear Combination of Rows"

$$y^T A = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} = y_1 \begin{bmatrix} - & a_1^T & - \end{bmatrix} + \cdots + y_m \begin{bmatrix} - & a_m^T & - \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{array}{c c} & \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c c} & \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{array}{c} n_1 \\ \\ n_N \end{array} & \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Case 2a

"Inner product with rows"

$$Ax = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

Case 2b

"Inner product with columns"

$$y^T A = \begin{bmatrix} - & y^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} y^T A_1 & \cdots & y^T A_n \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{array}{cc} & \begin{array}{ccc} n_1 & \quad & n_N \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccc} p_1 & \quad & p_P \end{array} \\ \begin{array}{c} n_1 \\ \\ n_N \end{array} & \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array} = \begin{array}{cc} & \begin{array}{ccc} p_1 & \qquad & p_P \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Case 3a

"A times each column of B"

$$AB = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ AB_1 & \cdots & AB_p \\ | & & | \end{bmatrix}$$

Case 3b

"A times each sub-matrix of B (horizontal)"

$$AB = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | & & | \\ -B_1- & \cdots & -B_p- \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ -AB_1- & \cdots & -AB_p- \\ | & & | \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{array}{c c} & \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c c} & \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{array}{c} n_1 \\ \\ n_N \end{array} & \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{array}{c} m_1 \\ \\ m_M \end{array} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Case 4a

"B times each row of A"

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} - & a_1^T B & - \\ & \vdots & \\ - & a_m^T B & - \end{bmatrix}$$

Case 4b

"B times each sub-matrix of A (vertical)"

$$AB = \begin{bmatrix} | \\ -A_1- \\ | \\ \vdots \\ | \\ -A_m- \\ | \end{bmatrix} \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} | \\ -A_1B- \\ | \\ \vdots \\ | \\ -A_mB- \\ | \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{array}{c} \\ m_1 \\ \\ m_M \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c} \\ n_1 \\ \\ n_N \end{array}\!\!\begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array} = \begin{array}{c} \\ m_1 \\ \\ m_M \end{array}\!\!\begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Case 5a

"Pairwise inner products of rows of A & columns of B"

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1^T B_1 & \cdots & a_1^T B_p \\ \vdots & & \vdots \\ a_m^T B_1 & \cdots & a_m^T B_p \end{bmatrix}$$

Case 5b

"Sum of outer products of columns of A and rows of B"

$$AB = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} \begin{bmatrix} - & b_n^T & - \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{array}{c} \\ m_1 \\ \\ m_M \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n_1 & & n_N \end{array} \\ \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \end{array} \begin{array}{c} \\ n_1 \\ \\ n_N \end{array}\!\!\begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{array} = \begin{array}{c} \\ m_1 \\ \\ m_M \end{array}\!\!\begin{array}{c} \begin{array}{ccc} p_1 & & p_P \end{array} \\ \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{array}$$

Case 6a

"Pairwise inner products of rows of A & columns of B around D"

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} | & & & | \\ B_1 & & \cdots & B_p \\ | & & & | \end{bmatrix} = \begin{bmatrix} a_1^T D B_1 & \cdots & a_1^T D B_p \\ \vdots & & \vdots \\ a_m^T D B_1 & \cdots & a_m^T D B_p \end{bmatrix}$$

Case 6b

"Sum of scaled outer products (diagonal)"

$$AB = \begin{bmatrix} | & & & | \\ A_1 & & \cdots & A_n \\ | & & & | \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} d_{11} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} d_{nn} \begin{bmatrix} - & b_n^T & - \end{bmatrix} = \sum_i \begin{bmatrix} | \\ A_i \\ | \end{bmatrix} d_{ii} \begin{bmatrix} - & b_i^T & - \end{bmatrix}$$

Case 6c

"Sum of scaled pairwise outer products"

$$AB = \begin{bmatrix} | & & & | \\ A_1 & & \cdots & A_n \\ | & & & | \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} d_{11} \begin{bmatrix} - & b_1^T & - \end{bmatrix} \ddots + \cdots + \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} d_{1n} \begin{bmatrix} - & b_n^T & - \end{bmatrix} \ddots$$
$$+ \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} d_{n1} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} d_{nn} \begin{bmatrix} - & b_n^T & - \end{bmatrix} = \sum_i \sum_j \begin{bmatrix} | \\ A_i \\ | \end{bmatrix} d_{ij} \begin{bmatrix} - & b_j^T & - \end{bmatrix}$$