Row & Column Geometry

Linear Algebra

Winter 2022 - Dan Calderone

Block Matrix Multiplication

Column Geometry

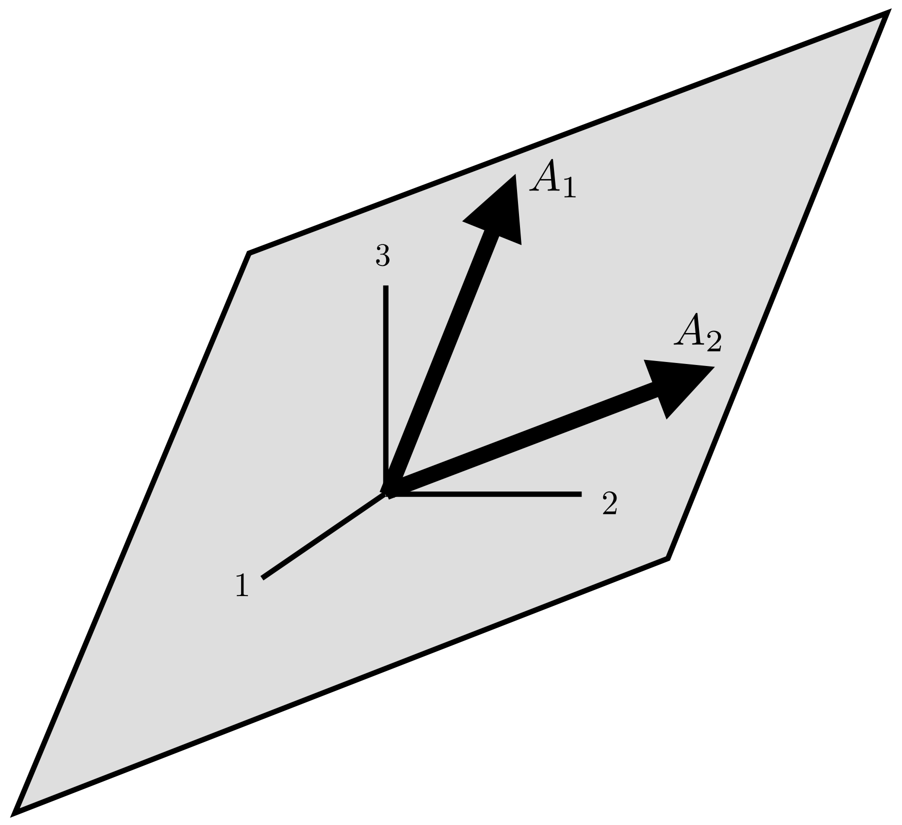
Row Geometry

Range Space

$\mathcal{R}(A)$

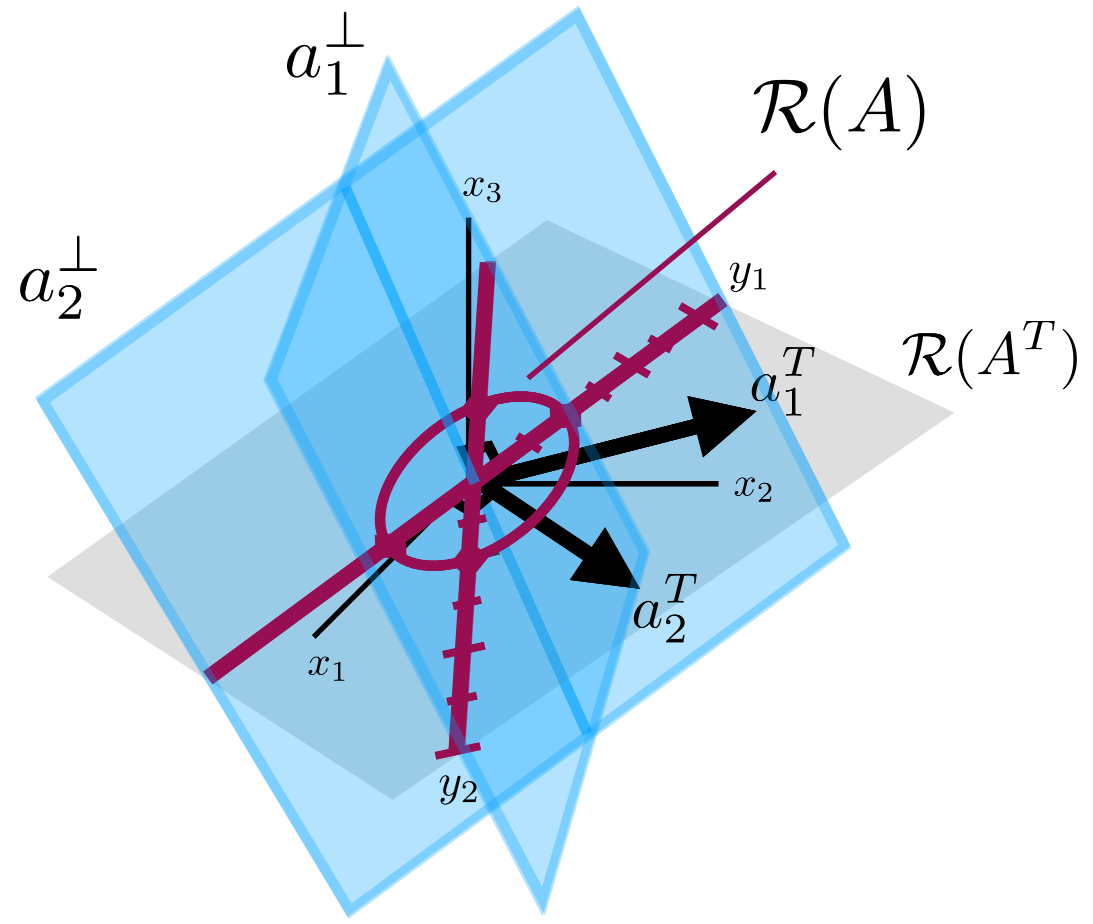
$\mathcal{R}(A)$

"span of columns"



$\mathcal{R}(A)$

"projection orthogonal to other rows"

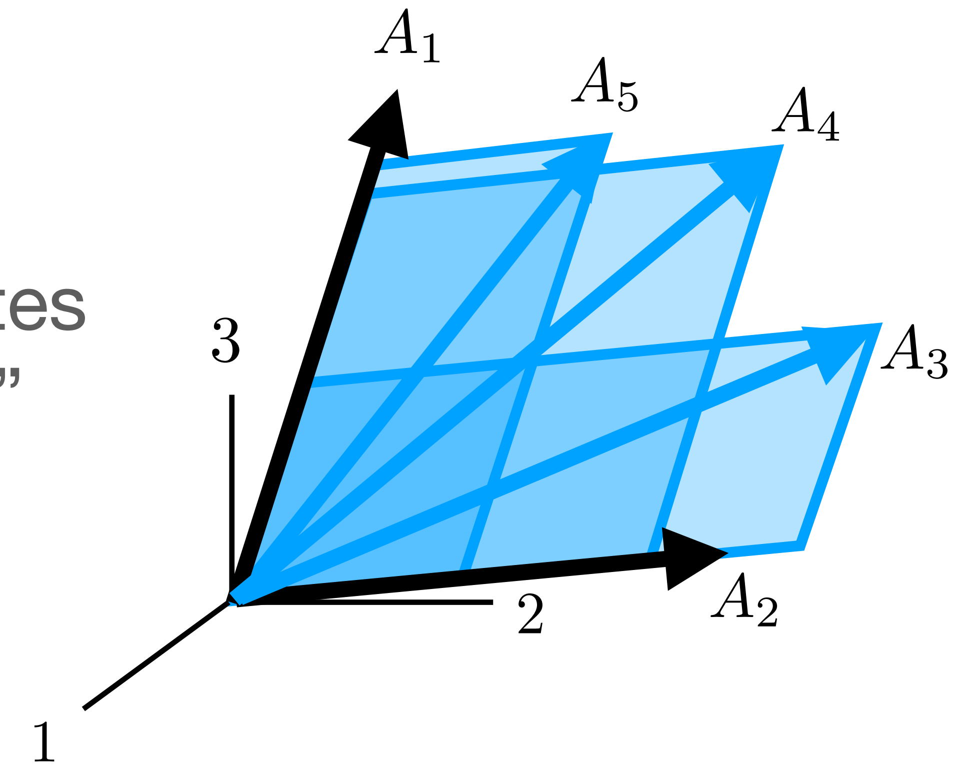


Nullspace

$\mathcal{N}(A)$

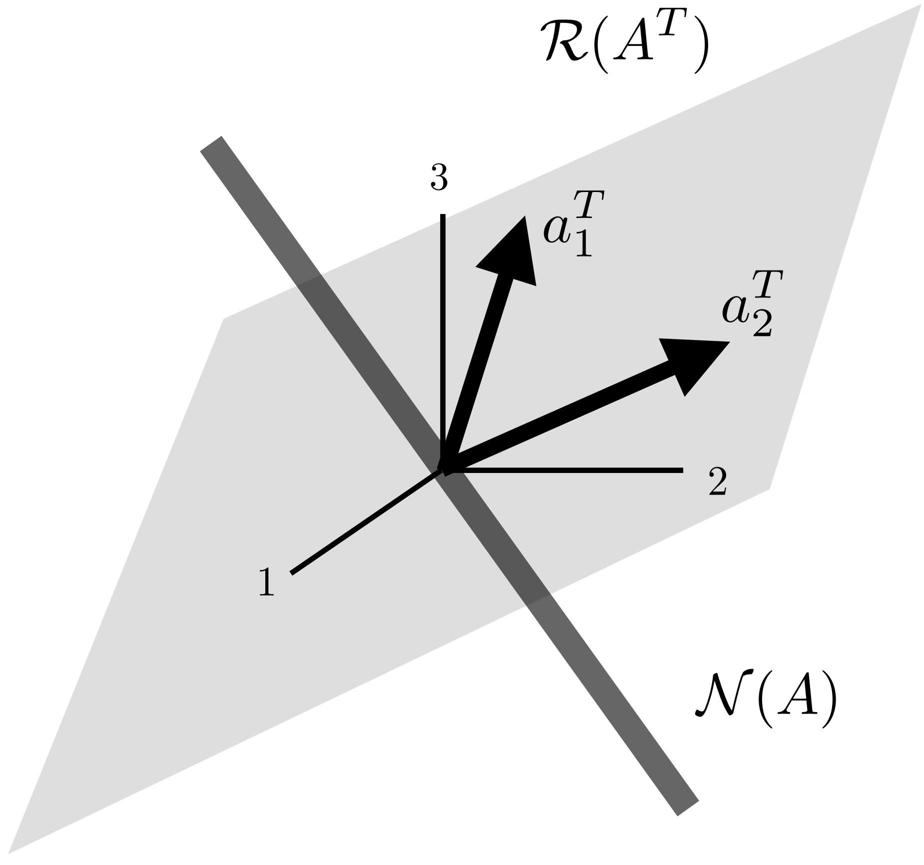
$\mathcal{N}(A)$

"coordinates of origin"



$\mathcal{N}(A)$

"orthogonal to all rows"



Column & Row Geometry

Column Geometry

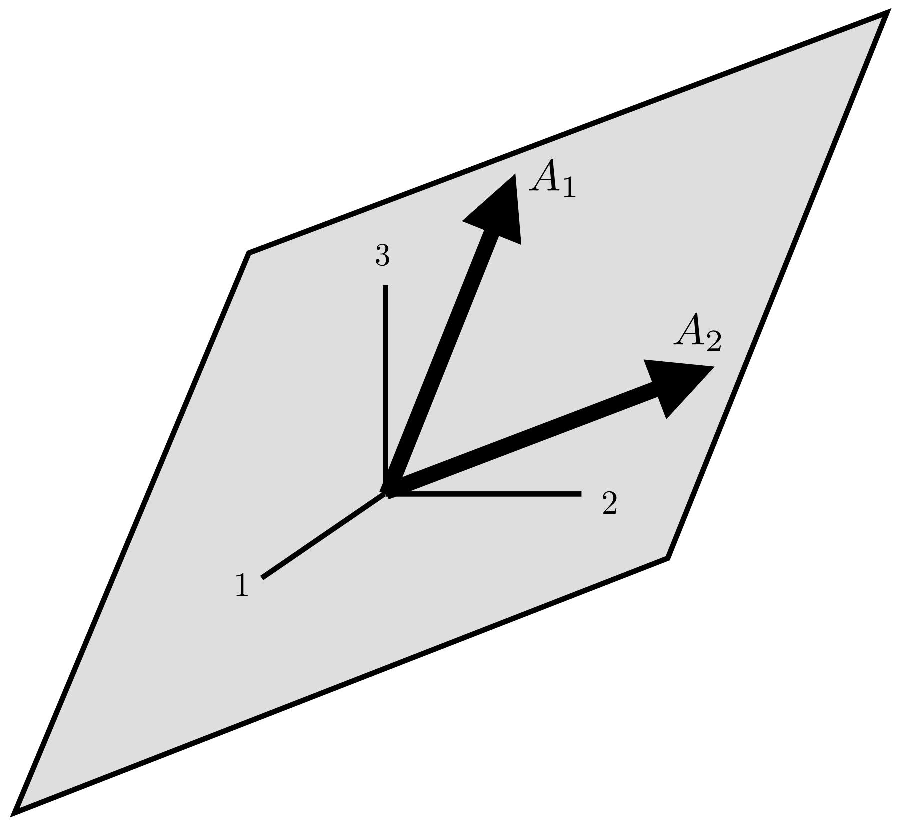
Row Geometry

Range Space

$\mathcal{R}(A)$

$\mathcal{R}(A)$

"span of columns"



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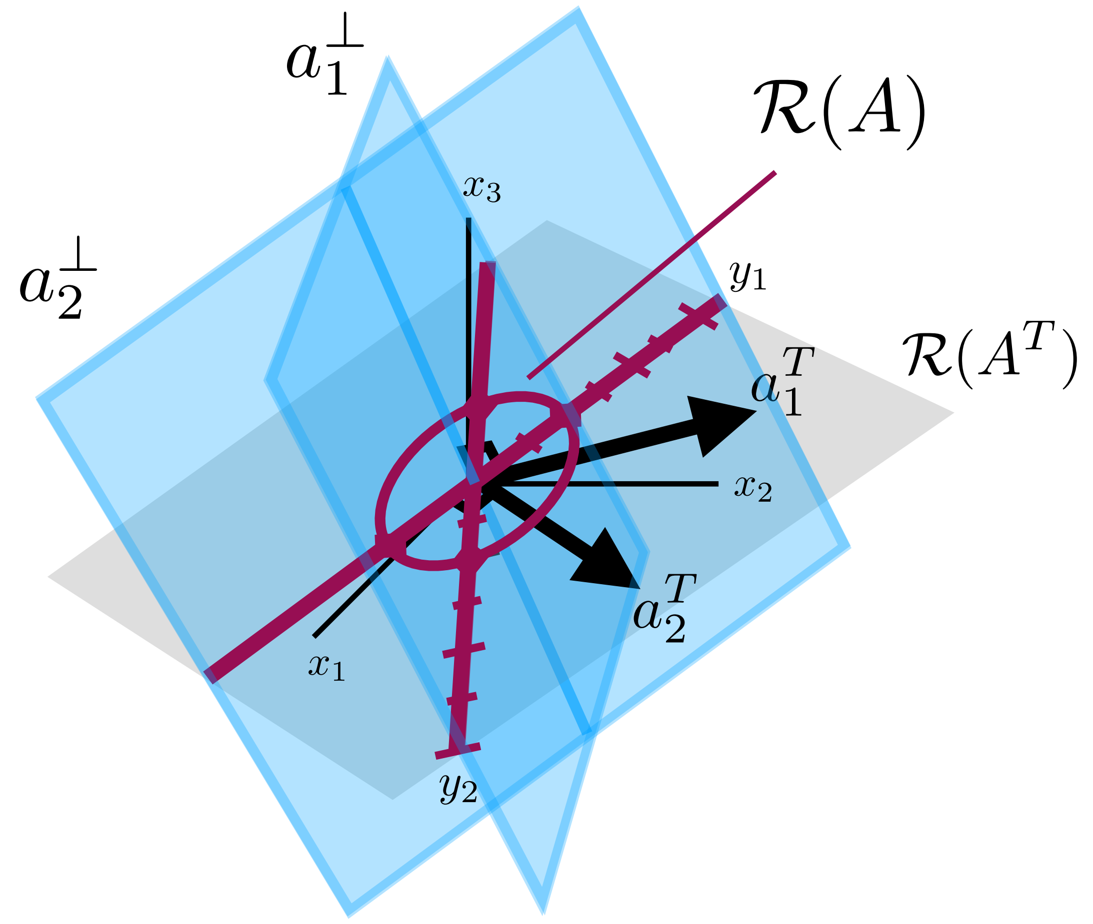

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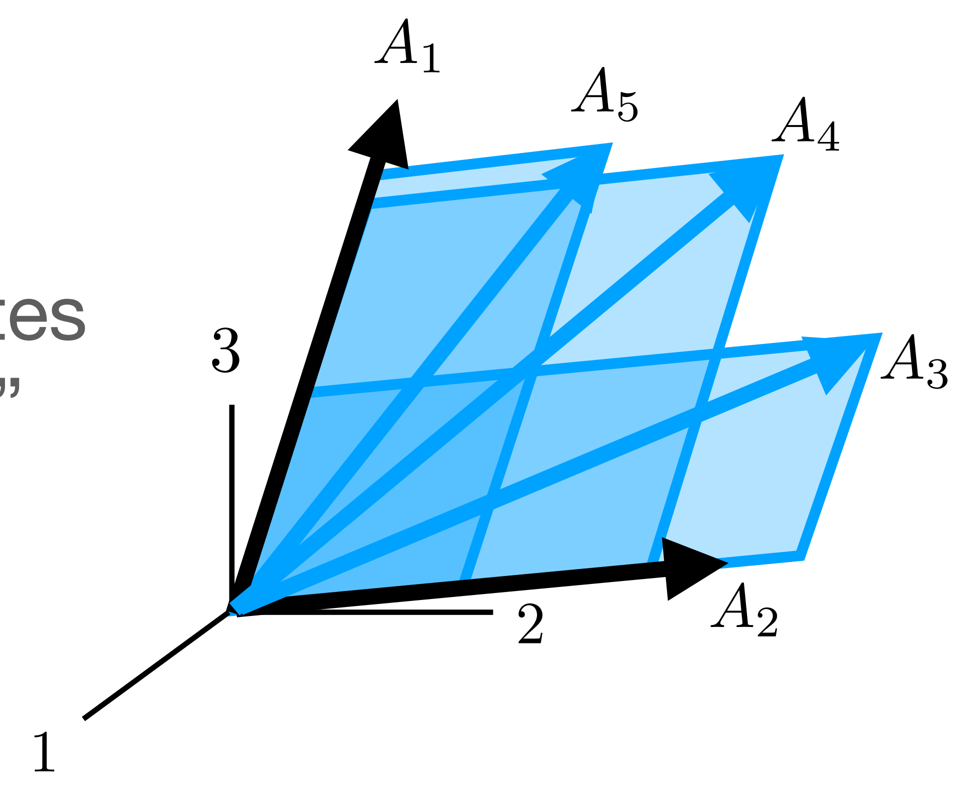
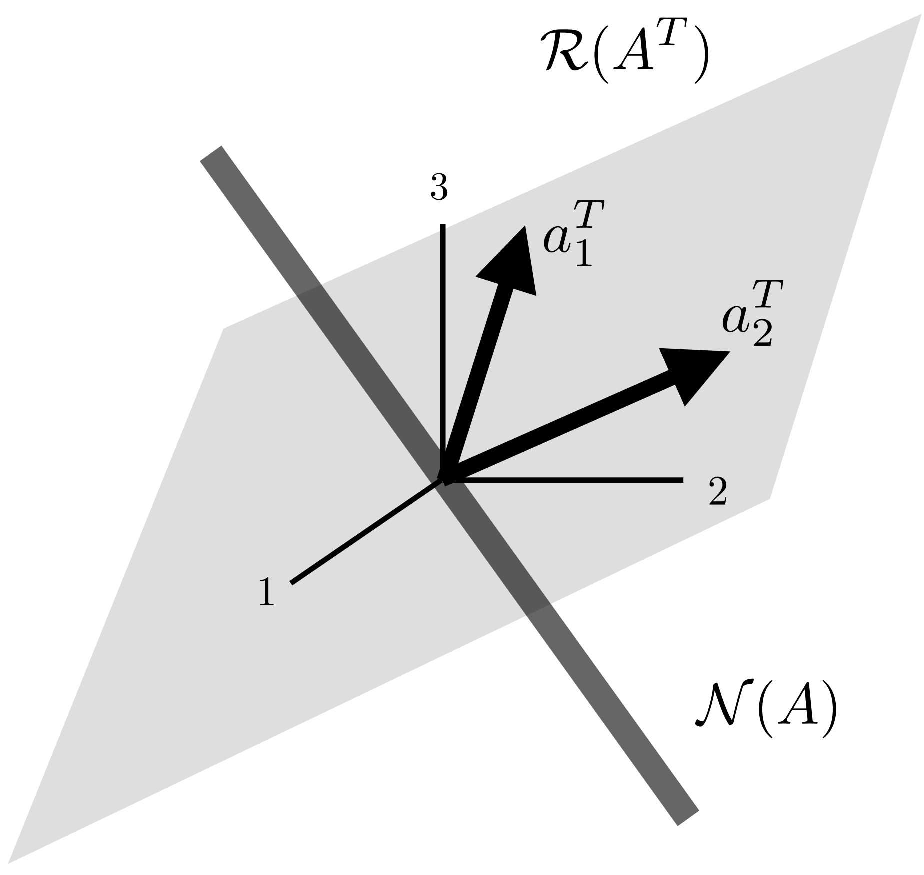


$\mathcal{N}(A)$

"orthogonal to all rows"



Column & Row Geometry

Column Geometry

Row Geometry

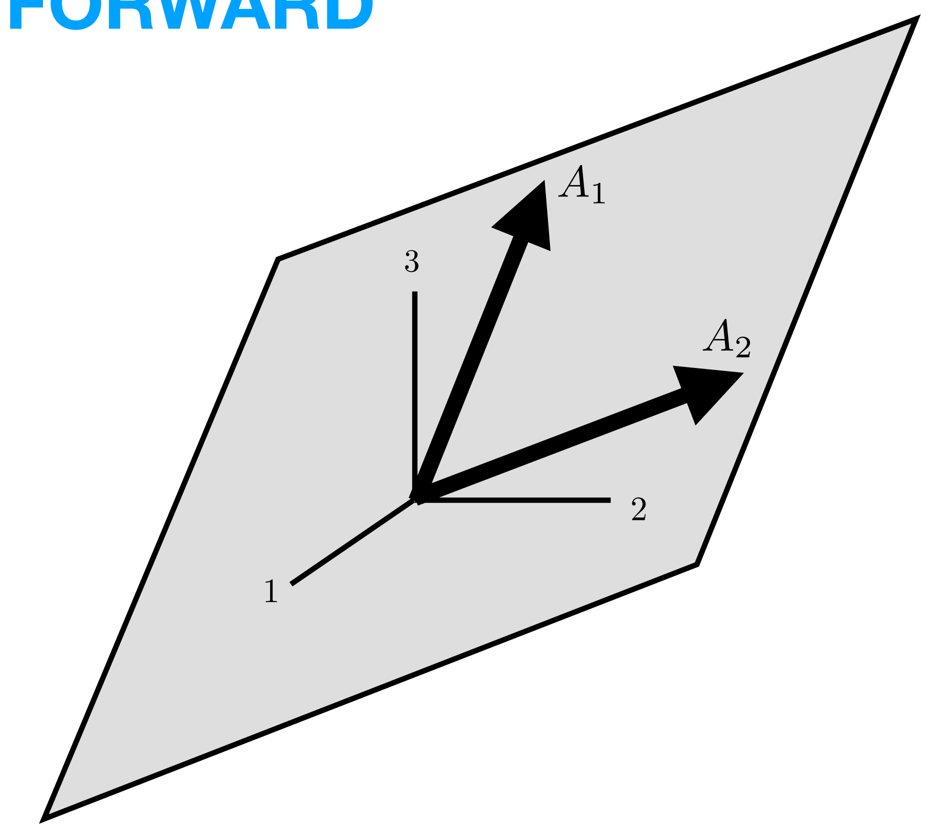
Range Space

$\mathcal{R}(A)$

Column Geometry — STRAIGHTFORWARD

$\mathcal{R}(A)$

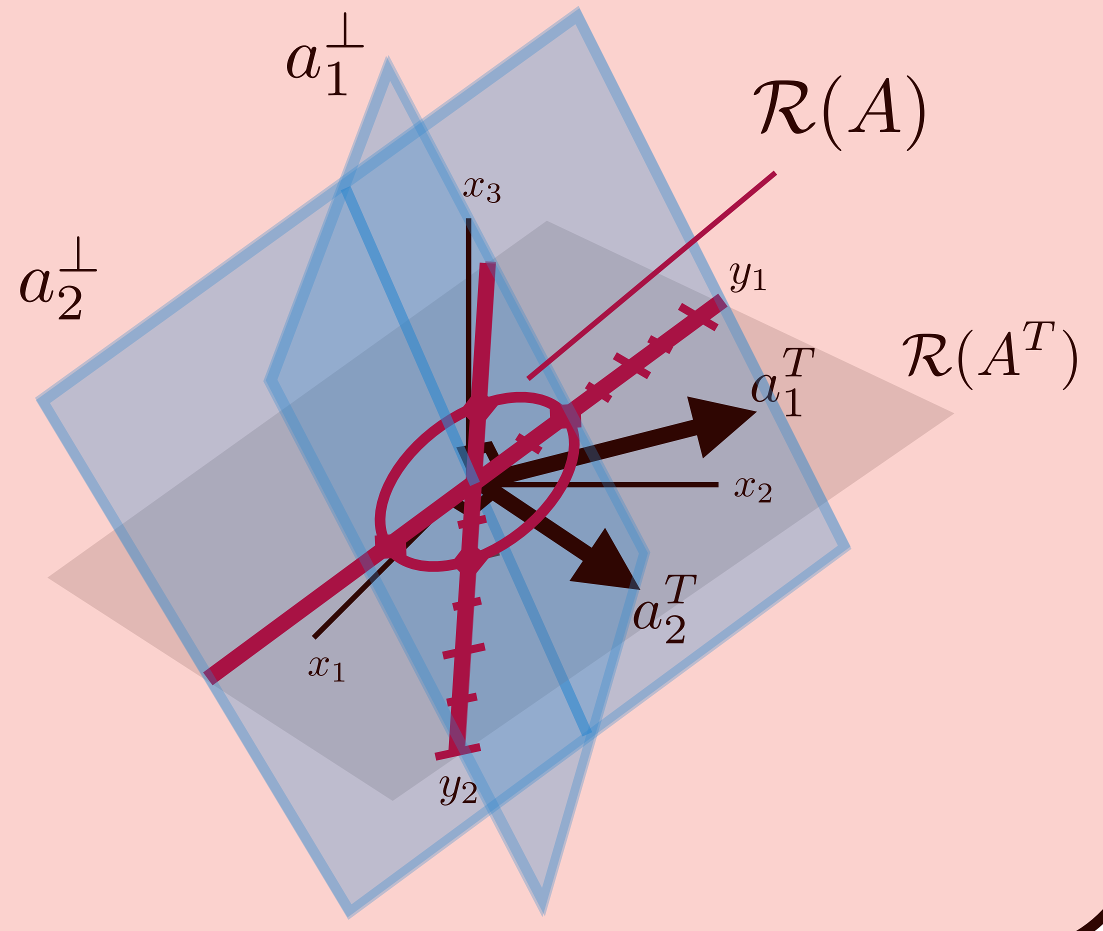
"span of columns"



Row Geometry — HARDEST

$\mathcal{R}(A)$

"projection orthogonal to other rows"



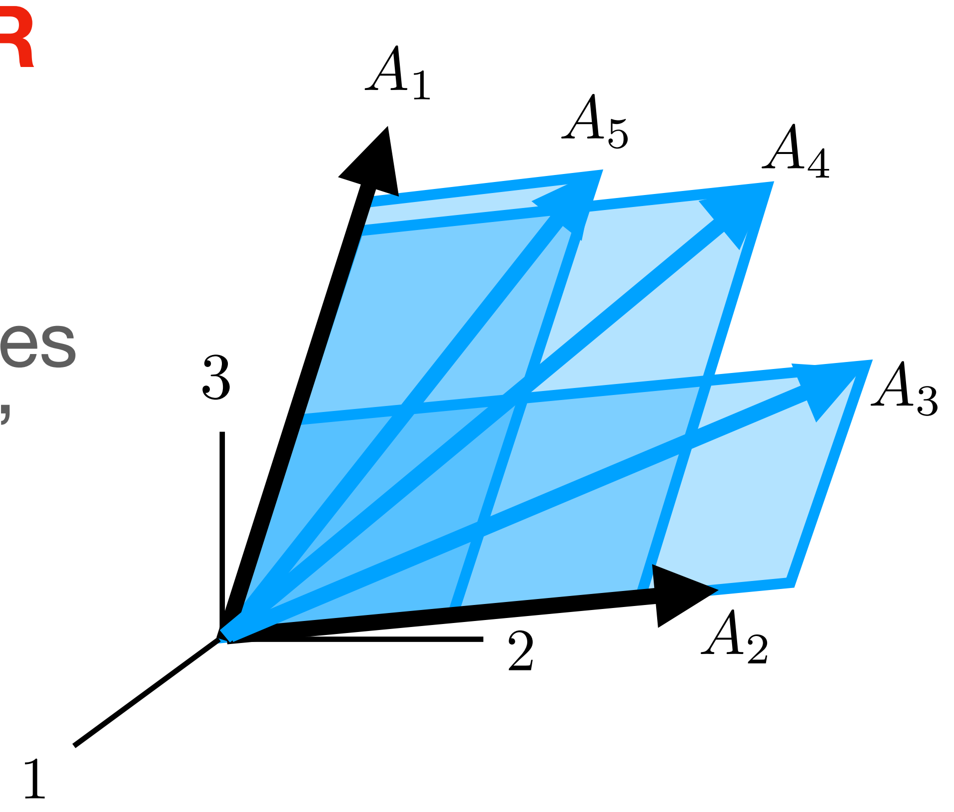
Nullspace

$\mathcal{N}(A)$

Column Geometry — HARDER

$\mathcal{N}(A)$

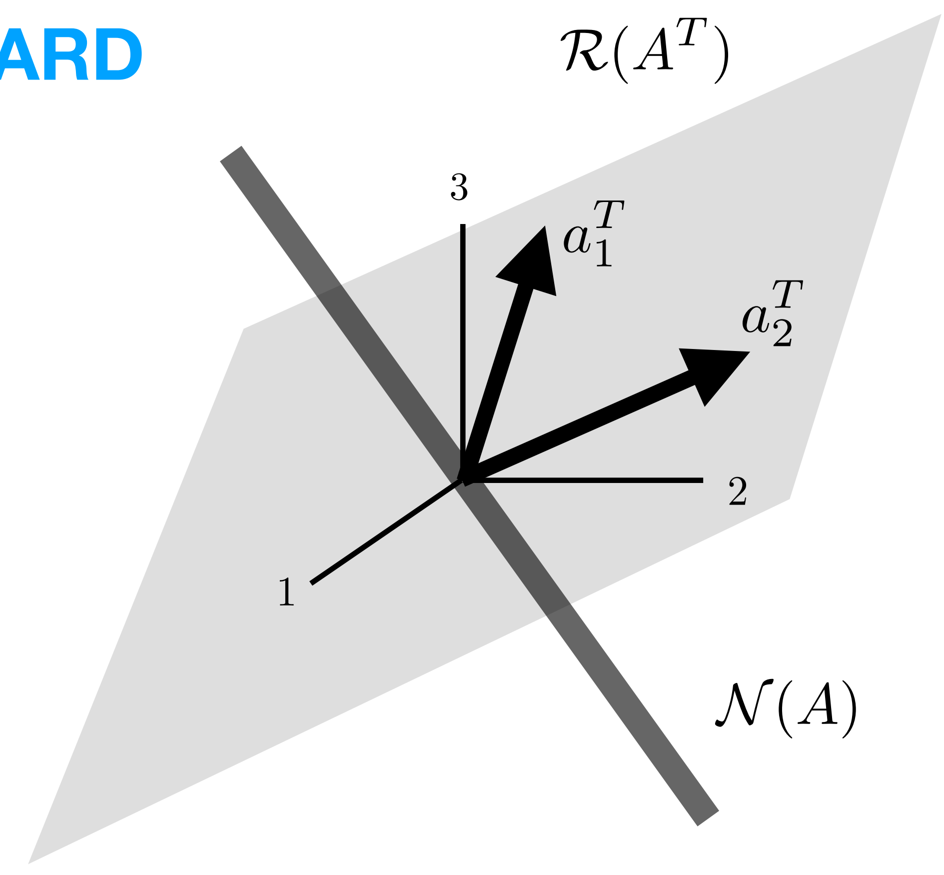
"coordinates of origin"



Row Geometry — STRAIGHTFORWARD

$\mathcal{N}(A)$

"orthogonal to all rows"



Column & Row Geometry

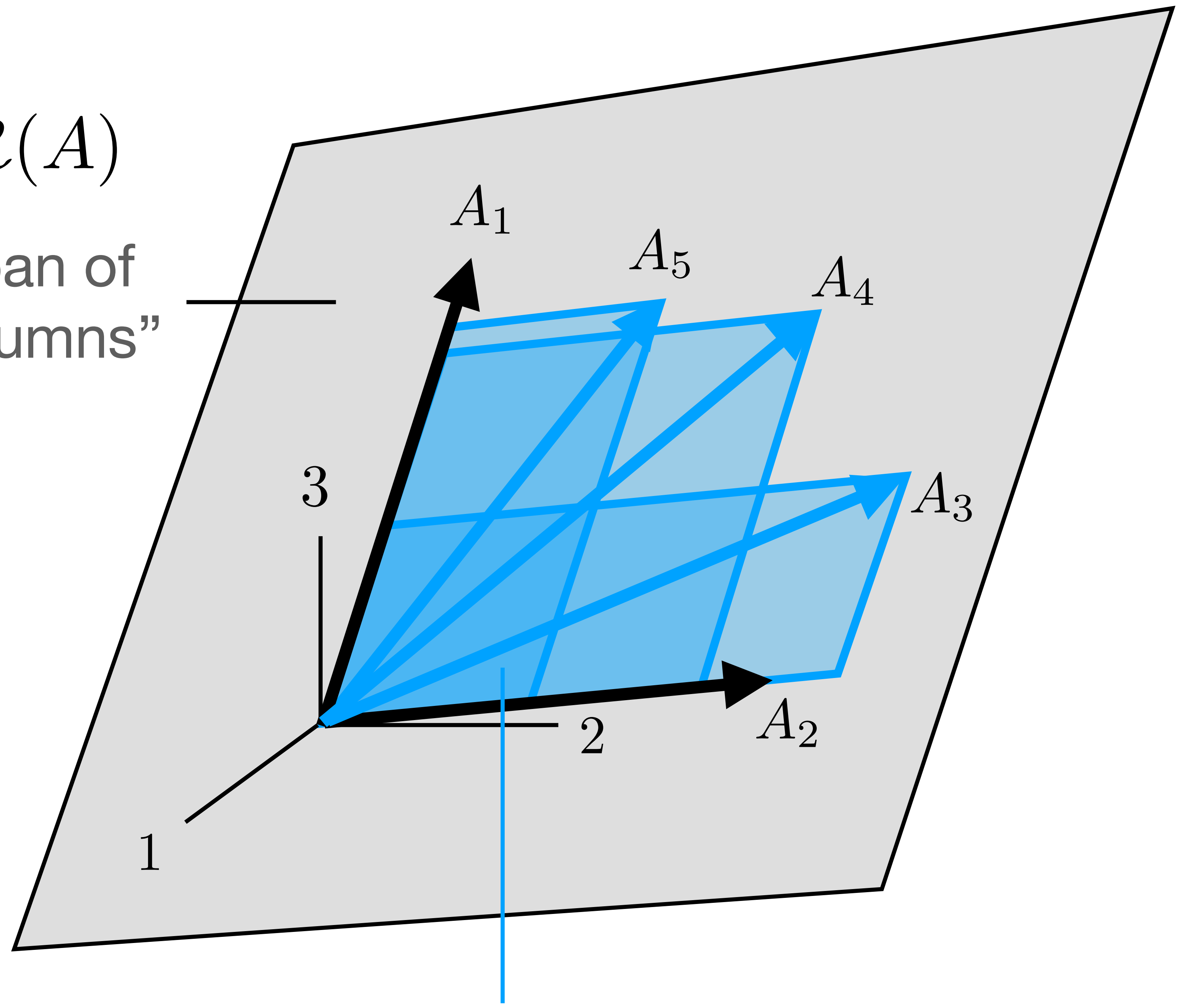
Range Space

$\mathcal{R}(A)$

Nullspace

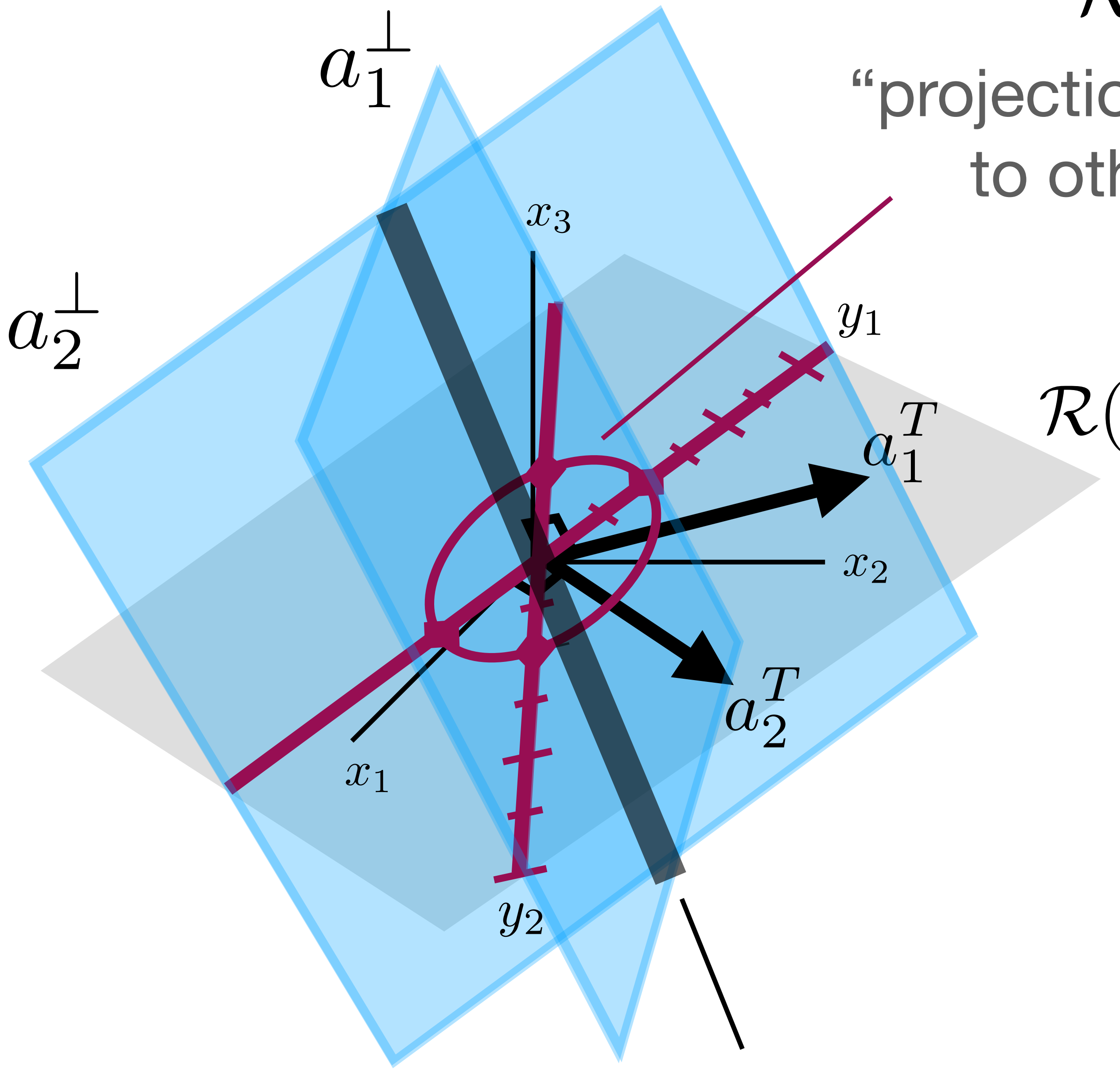
$\mathcal{N}(A)$

Column Geometry



$$A = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix} \qquad \mathrm{rk}(A) = 2$$

Row Geometry



$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \end{bmatrix} \qquad \mathrm{rk}(A) = 2$$

Range - Column Geometry

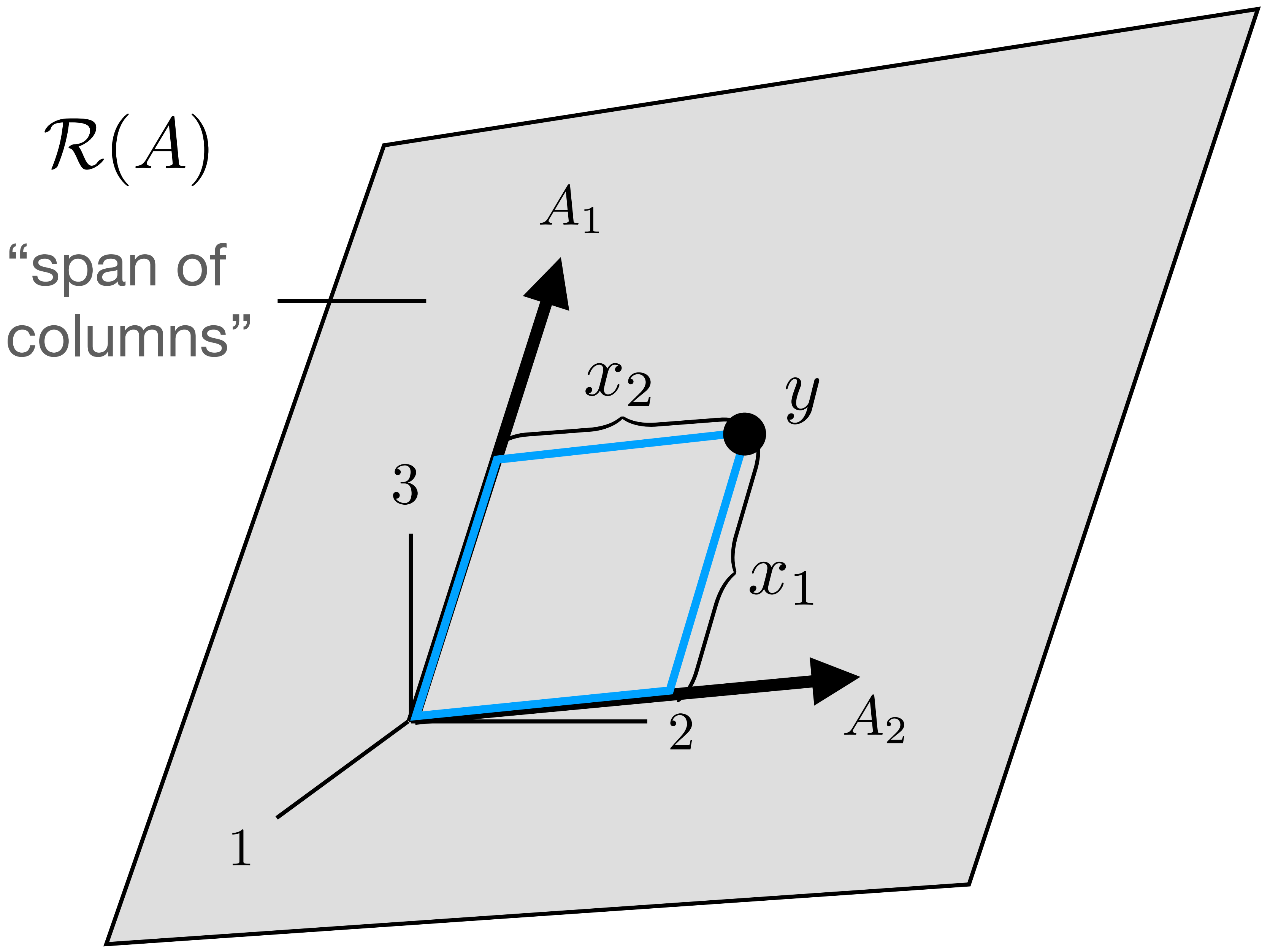
$y \in \mathcal{R}(A)$ $\quad y = Ax$

$$y = \sum_i A_i x_i$$

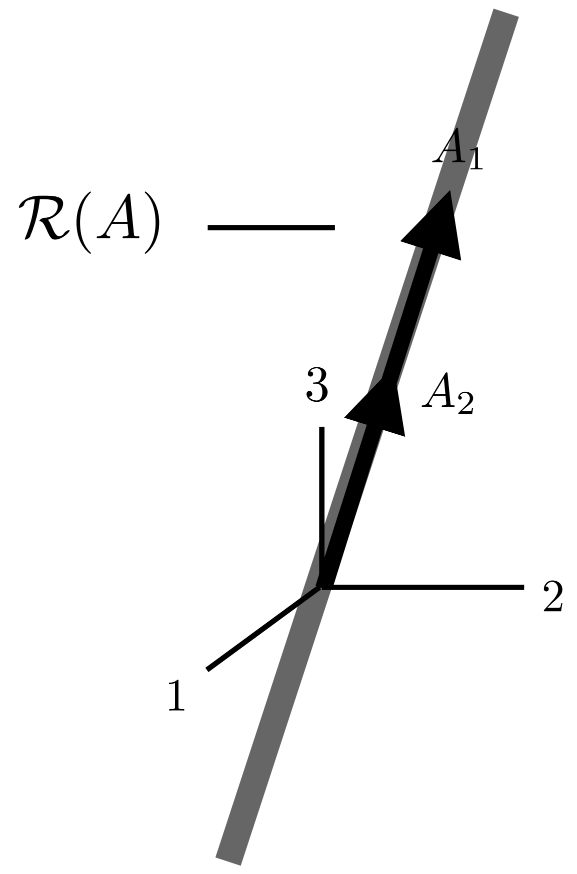
$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

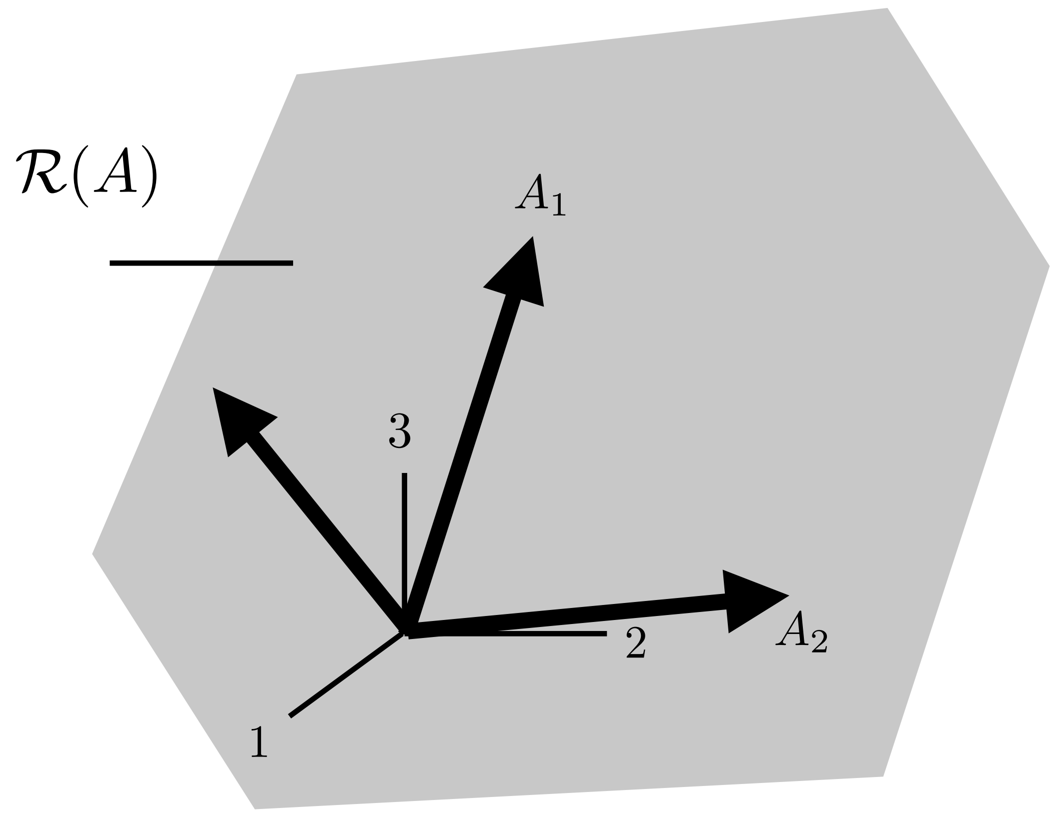
y are the coordinates of x w.r.t the columns of A



…1D span



…3D span



Range - Column Geometry

$y \in \mathcal{R}(A)$ $\quad y = Ax$

$$y = \sum_i A_i x_i$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

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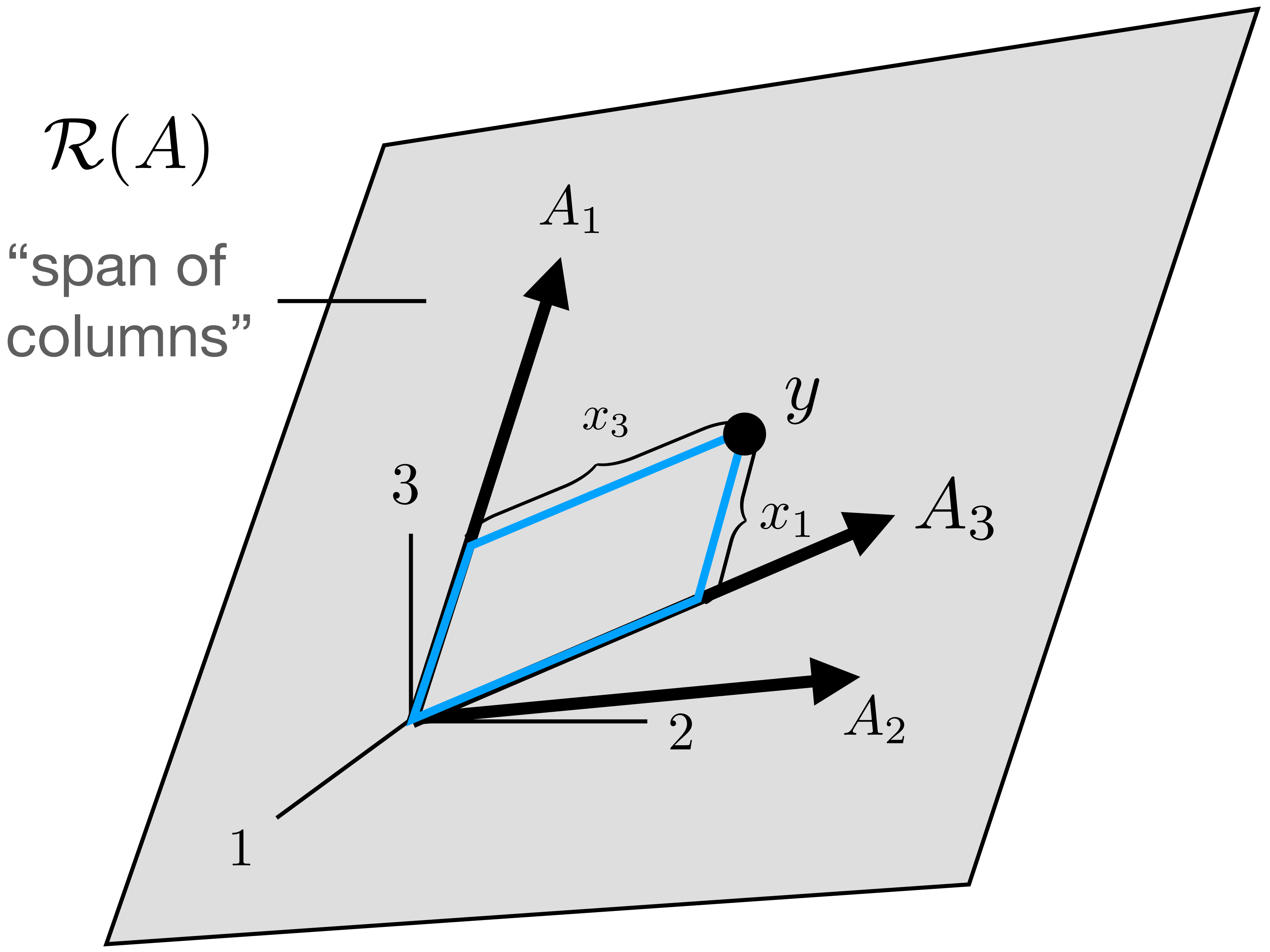
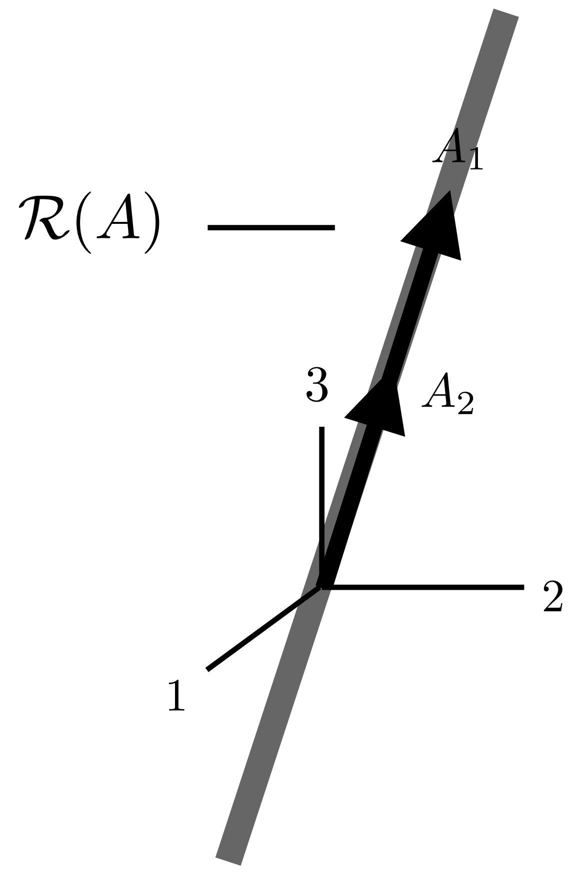
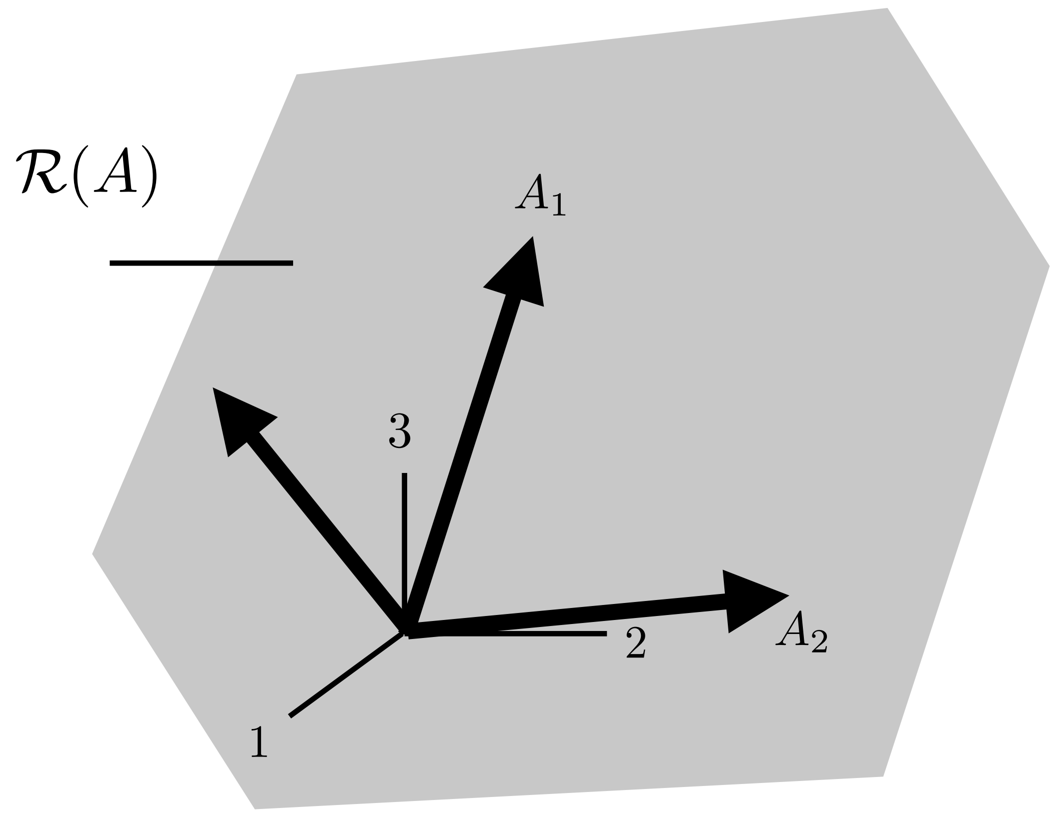


…1D span



…3D span



Range - Column Geometry

$y \in \mathcal{R}(A)$ $\quad y = Ax$

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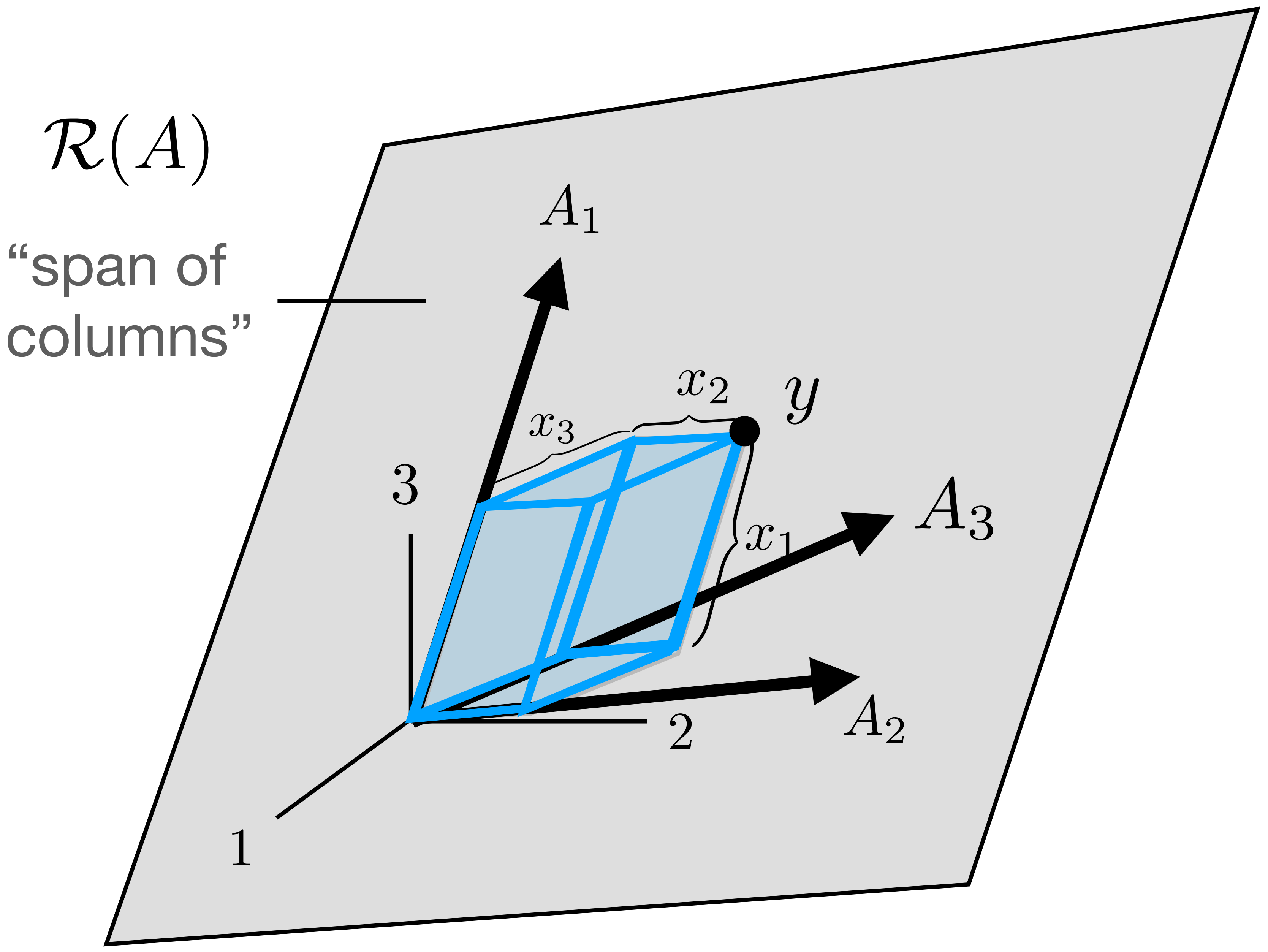
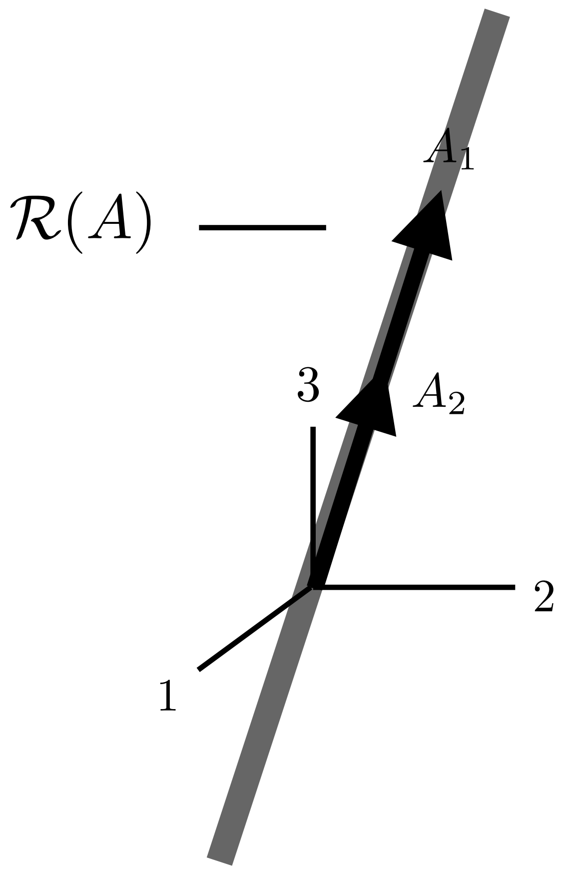
$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

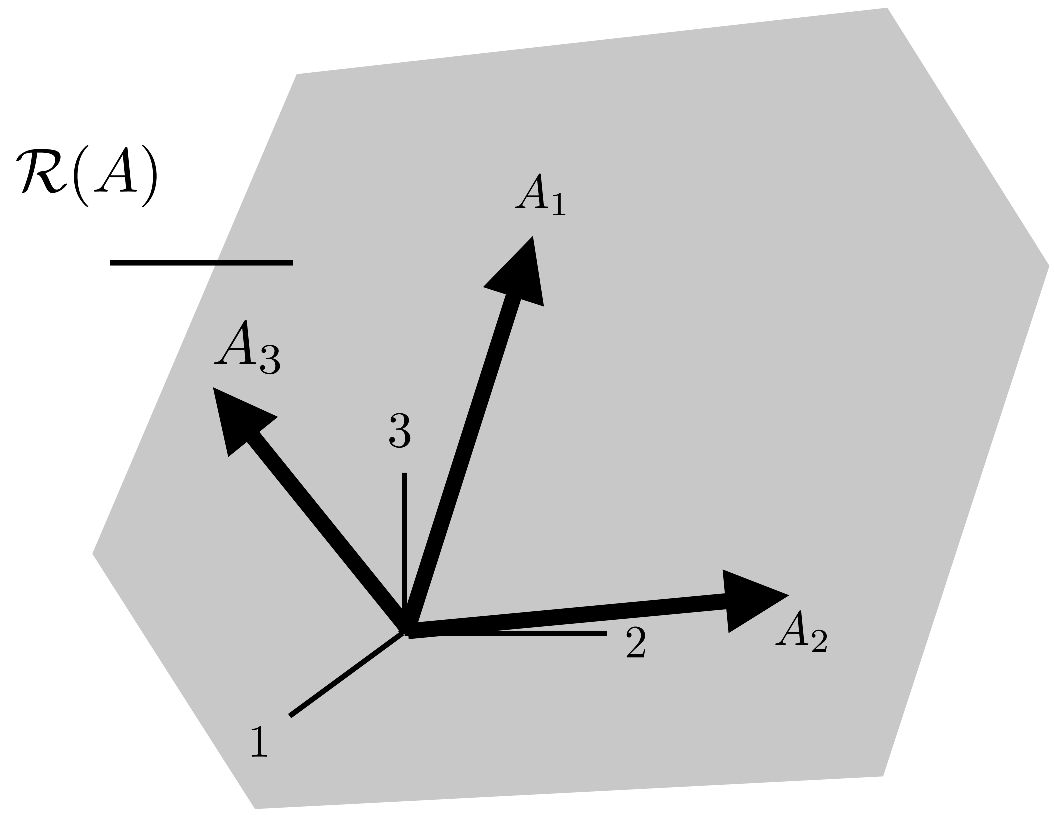
y are the coordinates of x w.r.t the columns of A



…1D span



…3D span

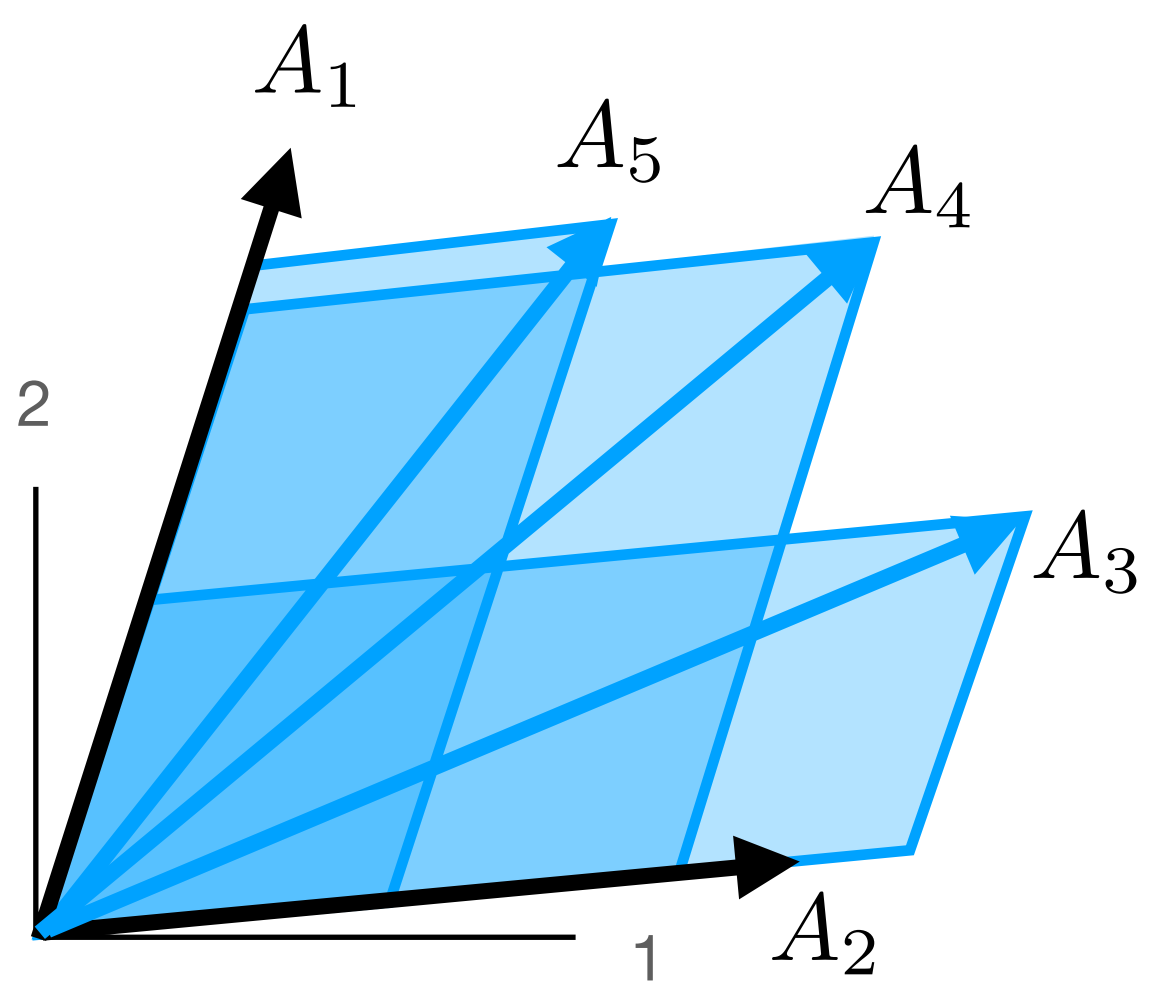


Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns



Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} = A' \begin{bmatrix} I & B \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ A_3 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1B_{13} + A_2B_{23} & A_1B_{14} + A_2B_{24} & A_1B_{15} + A_2B_{25} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{33} & B_{34} & B_{35} \end{bmatrix}$$

$$A'' = A'B$$

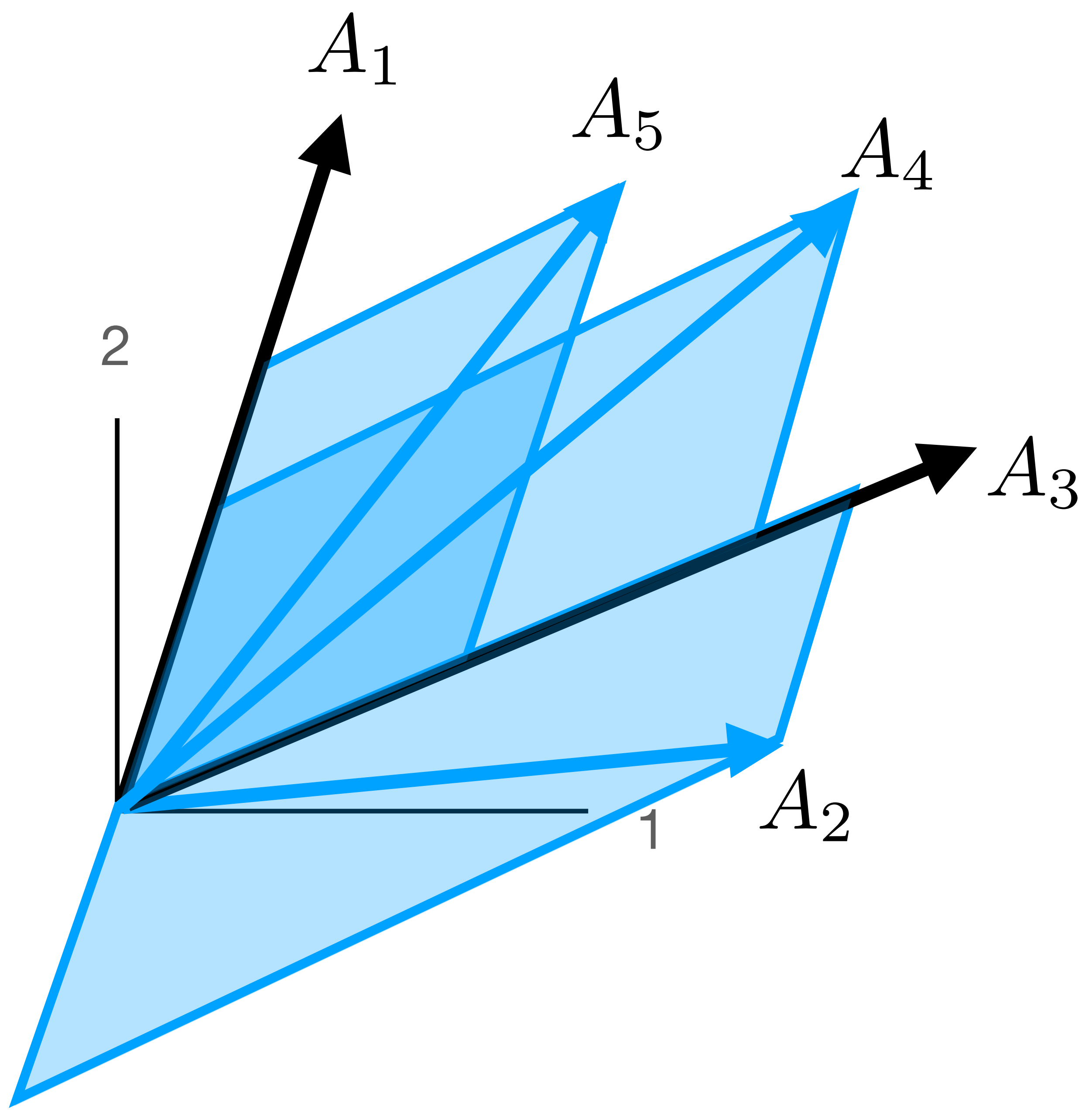
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} A_2 & A_4 & A_5 \end{bmatrix} = \begin{bmatrix} A_1B_{12} + A_3B_{32} & A_1B_{14} + A_3B_{34} & A_1B_{15} + A_3B_{35} \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ B_{33} & B_{34} & B_{35} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



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Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{15} \\ B_{42} & B_{43} & B_{45} \end{bmatrix}$$

$$A'' = A'B$$

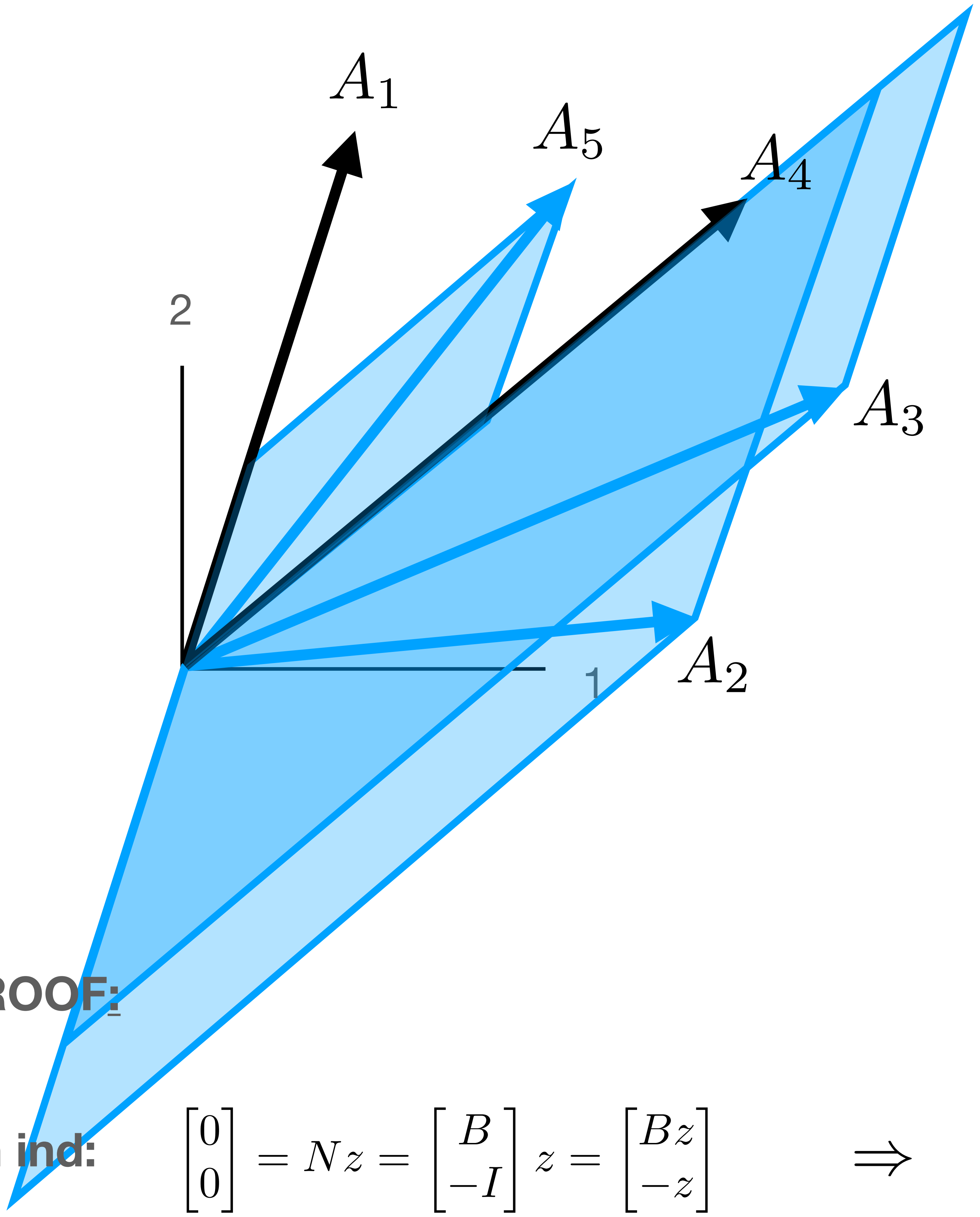
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_3 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_4 B_{42} & A_1 B_{13} + A_4 B_{43} & A_1 B_{15} + A_4 B_{45} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ B_{43} & B_{44} & B_{45} \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$

$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$

Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{14} \\ B_{52} & B_{53} & B_{54} \end{bmatrix}$$

$$A'' = A'B$$

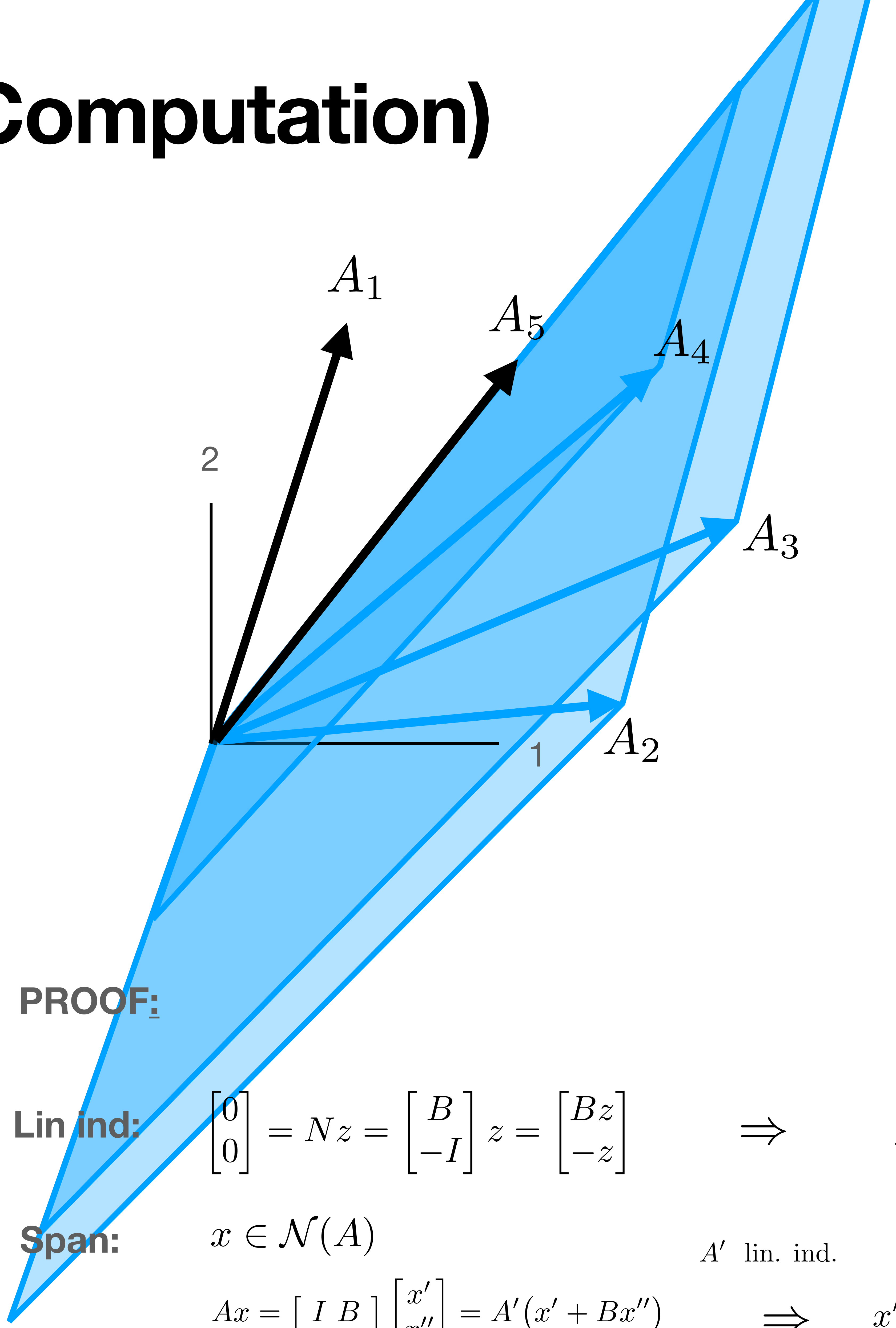
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_3 & A_4 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_5 B_{52} & A_1 B_{13} + A_5 B_{53} & A_1 B_{14} + A_5 B_{54} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ B_{53} & B_{54} & B_{55} \end{bmatrix}$$

$$AN = 0$$



PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'')$ $\quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$

$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$

Range - Row Geometry

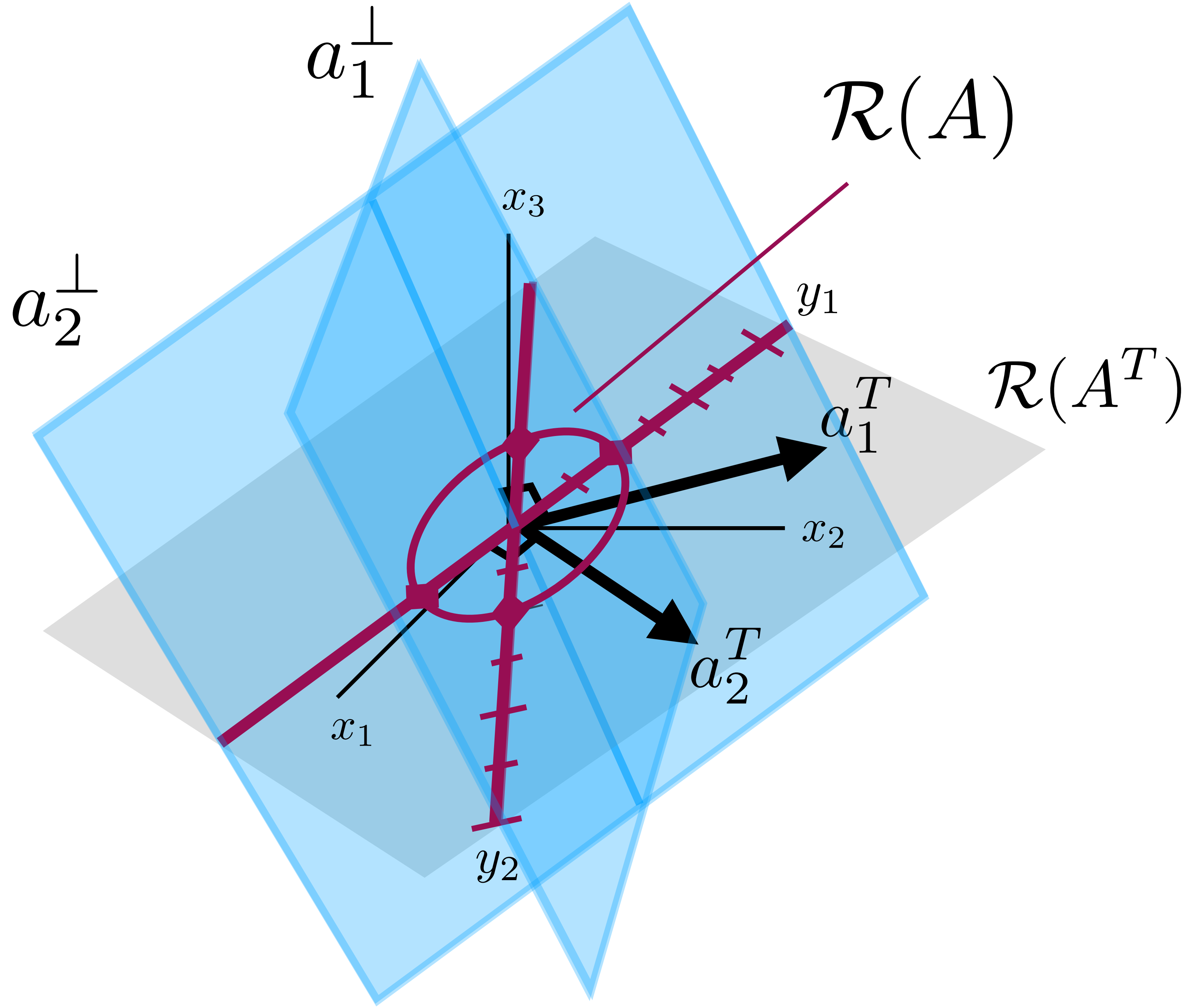
$y \in \mathcal{R}(A) \qquad y = Ax$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

$\mathcal{R}(A)$

"projection orthogonal to other rows"



Pre-image of $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is… $a_1^T x = 1$, $\left.\begin{bmatrix} a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\right\}$ …intersection of m-1 subspaces (each with dim n-1)

Pre-image of $\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ is… $a_2^T x = 1$, $\begin{bmatrix} a_1^T x \\ a_3^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ etc…

Nullspace - Row Geometry

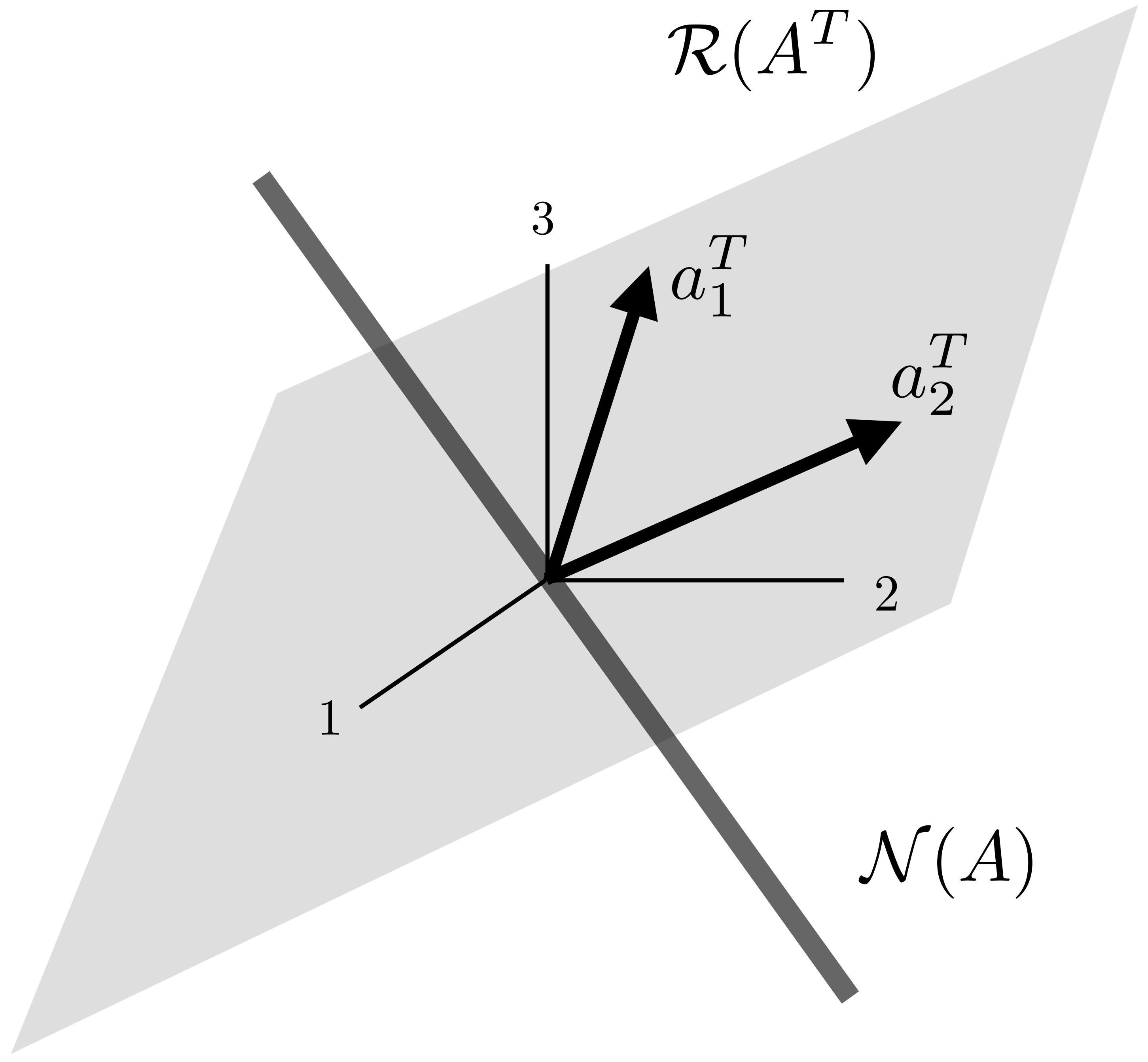
$x \in \mathcal{N}(A)$ $\quad Ax = 0$

$$\begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\mathcal{N}(A)$

"orthogonal to all rows"



x orthogonal to all rows.

(pre-image of 0)

Rank = Column Rank = Row Rank

Column Rank — num. of linear independent columns — $\dim\big(\mathcal{R}(A)\big)$

Row Rank — num. of linear independent rows — $\dim\big(\mathcal{R}(A^T)\big)$

Column Rank = Row Rank = Rank $\quad A \in \mathbb{R}^{m \times n}$

… if col rank = k write… $A = CV$

$$A = m \underbrace{\overbrace{\begin{bmatrix} | & & | \\ C_1 & \cdots & C_k \\ | & & | \end{bmatrix}}^{k}}_{\text{lin. ind. (basis) for col space}} k \underbrace{\overbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}^{n}}_{\text{coords of cols w.r.t basis}}$$

change perspective…

$$A = m \underbrace{\overbrace{\begin{bmatrix} - & c_1^T & - \\ & \vdots & \\ - & c_m^T & - \end{bmatrix}}^{k}}_{\text{coords of rows…}} k \underbrace{\overbrace{\begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_k^T & - \end{bmatrix}}^{n}}_{\text{rows in span..}}$$

row rank $\leq k$

… if row rank = k write… $A = WR$

$$A = m \underbrace{\overbrace{\begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_m^T & - \end{bmatrix}}^{k}}_{\text{coords of rows w.r.t basis}} k \underbrace{\overbrace{\begin{bmatrix} - & r_1^T & - \\ & \vdots & \\ - & r_k^T & - \end{bmatrix}}^{n}}_{\text{lin. ind. (basis) for row space}}$$

change perspective…

$$A = m \underbrace{\overbrace{\begin{bmatrix} | & & | \\ W_1 & \cdots & W_k \\ | & & | \end{bmatrix}}^{k}}_{\text{cols in span…}} k \underbrace{\overbrace{\begin{bmatrix} | & & | \\ R_1 & \cdots & R_n \\ | & & | \end{bmatrix}}^{n}}_{\text{coords of cols…}}$$

col rank $\leq k$