Column Geometry - Fundamental Theorem

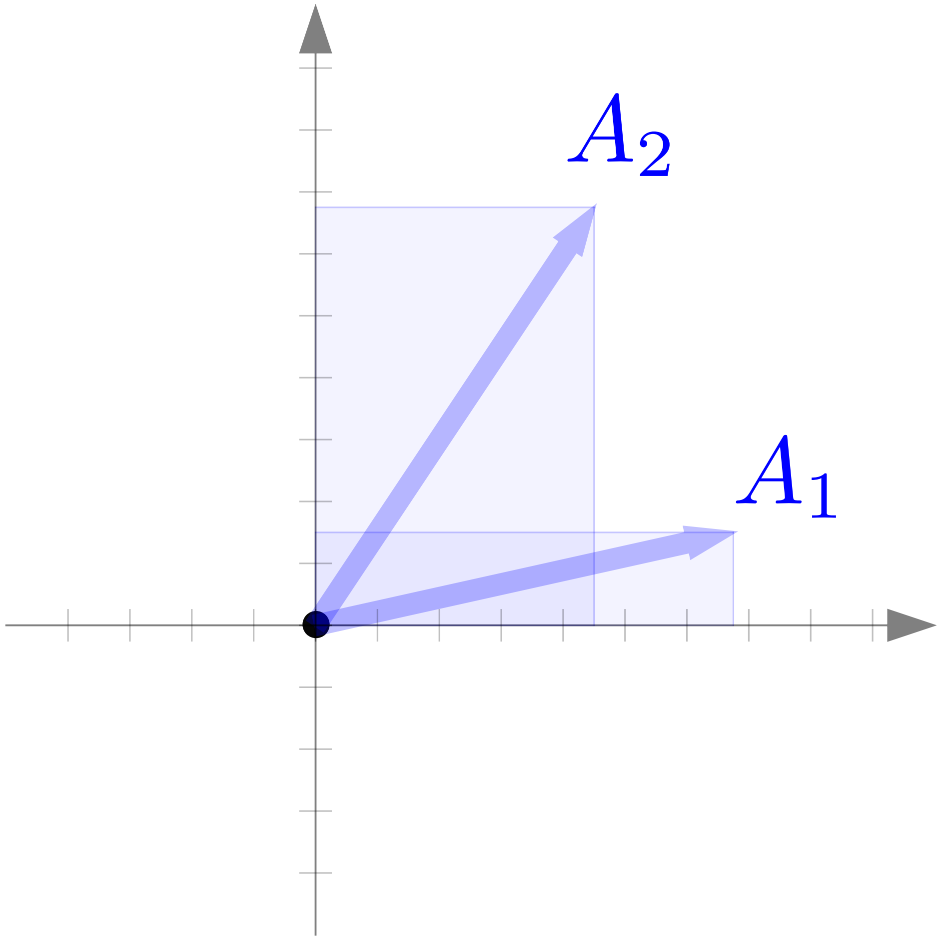
Linear Algebra

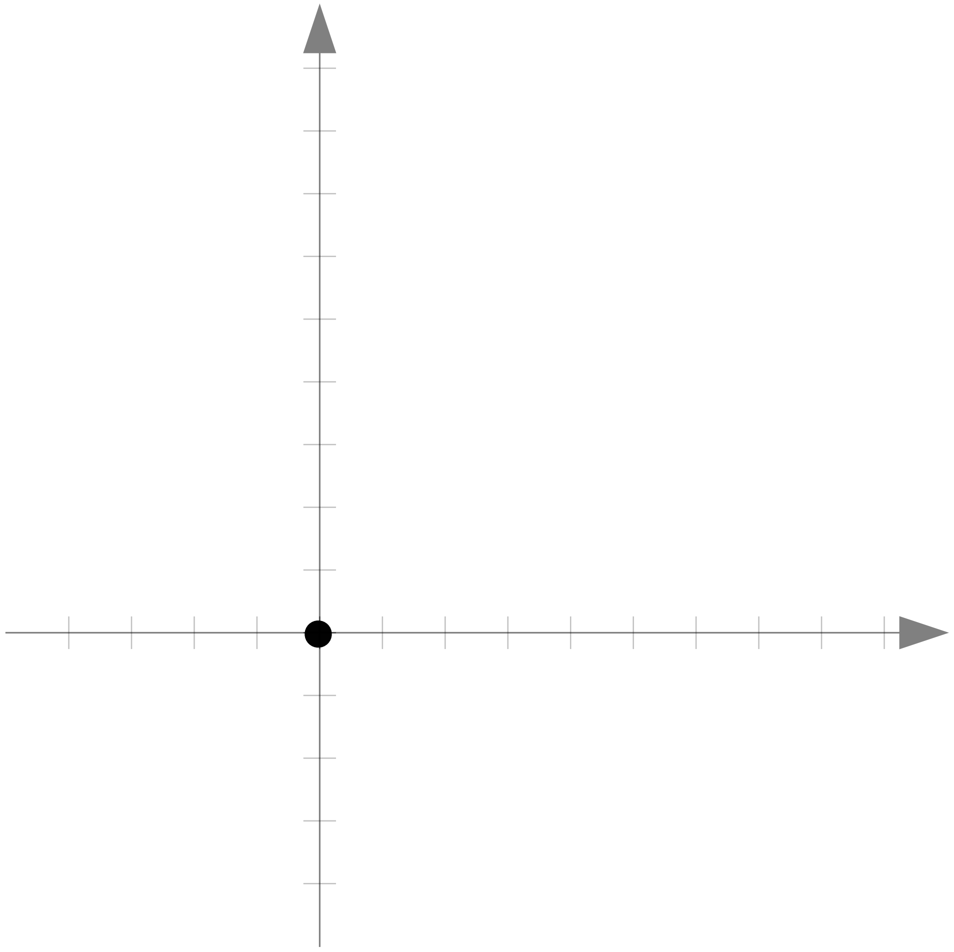
Summer 2023 - Dan Calderone

Range of A

Range of A

"span of columns"

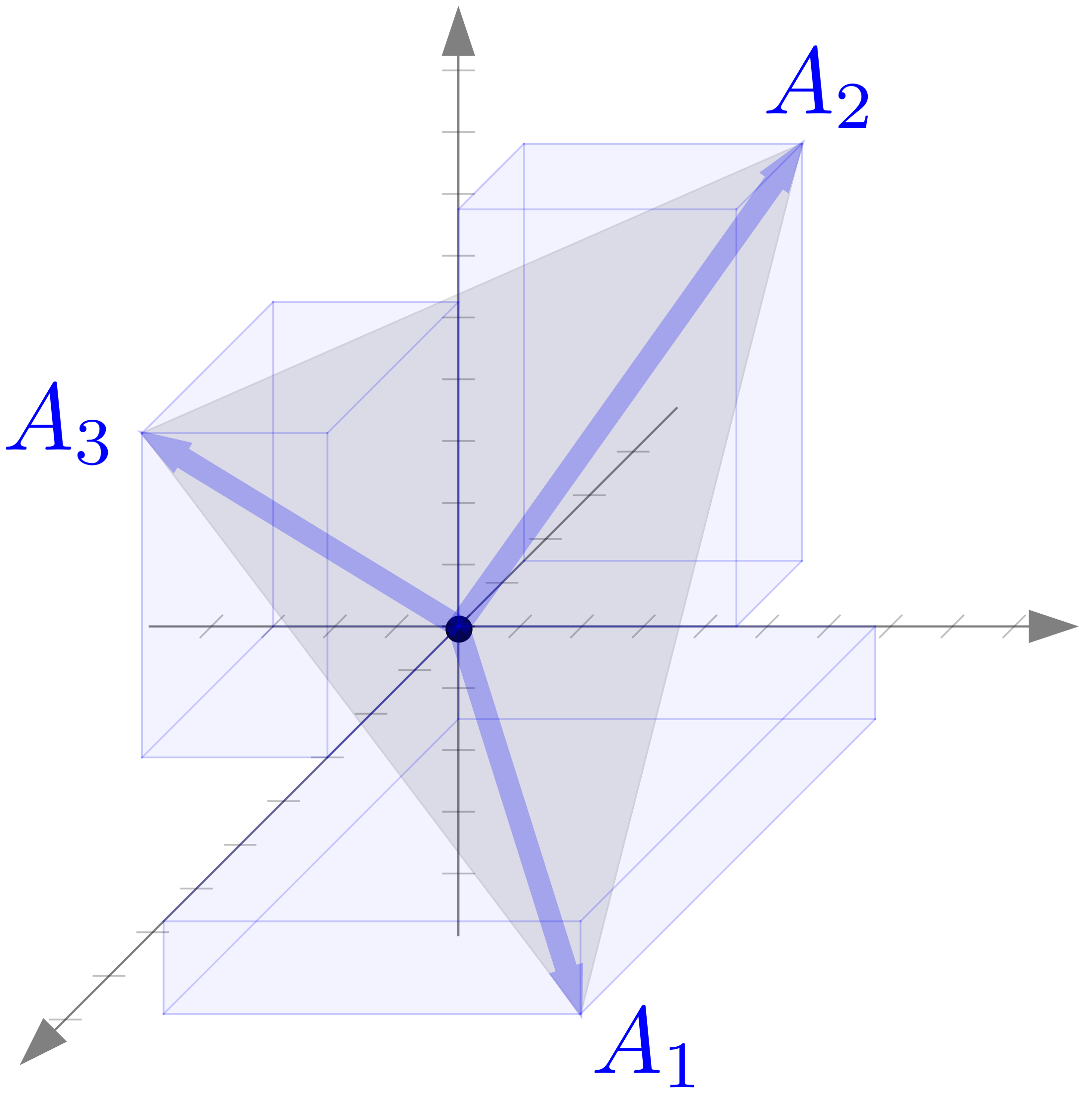


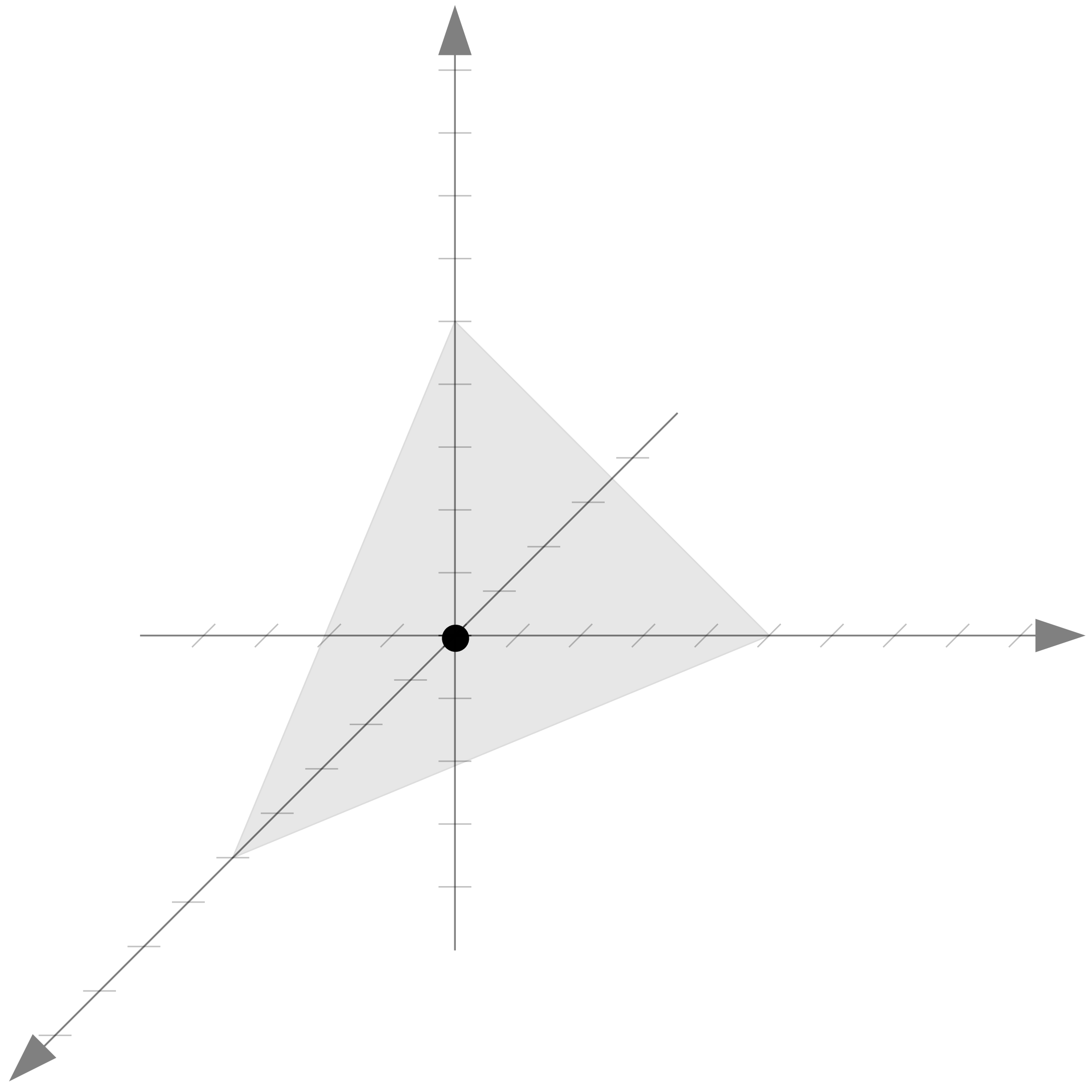


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Range of A

"span of columns"

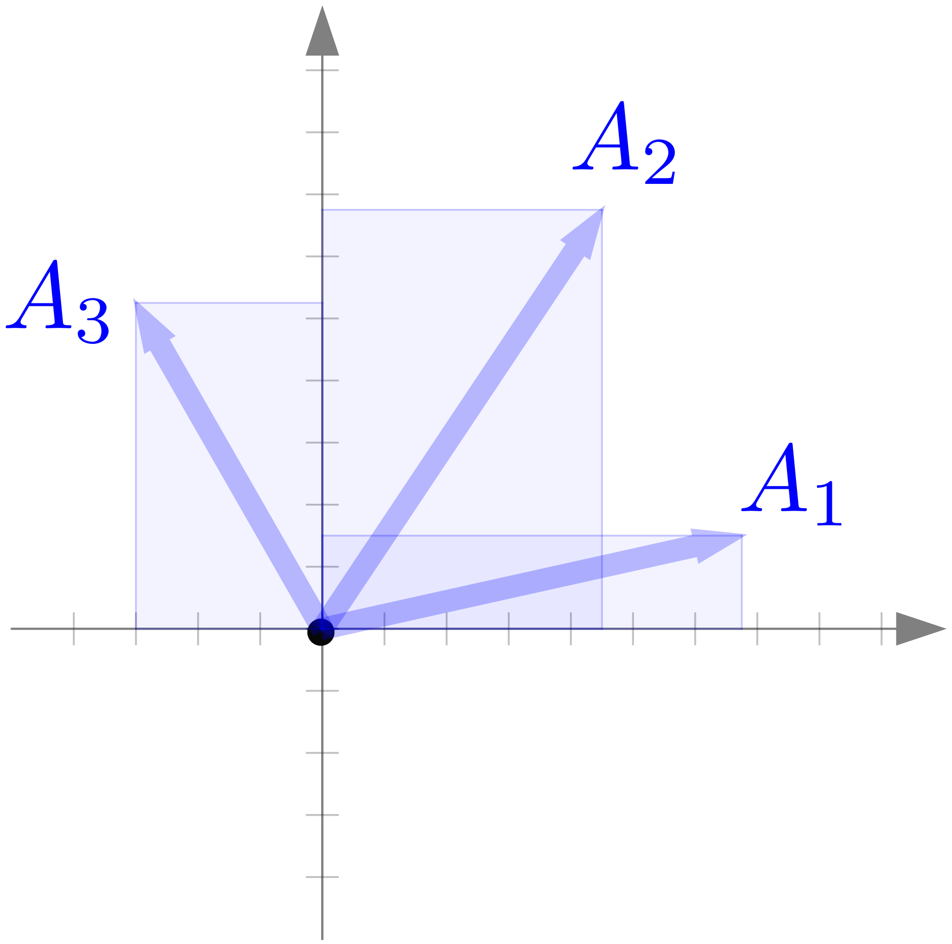


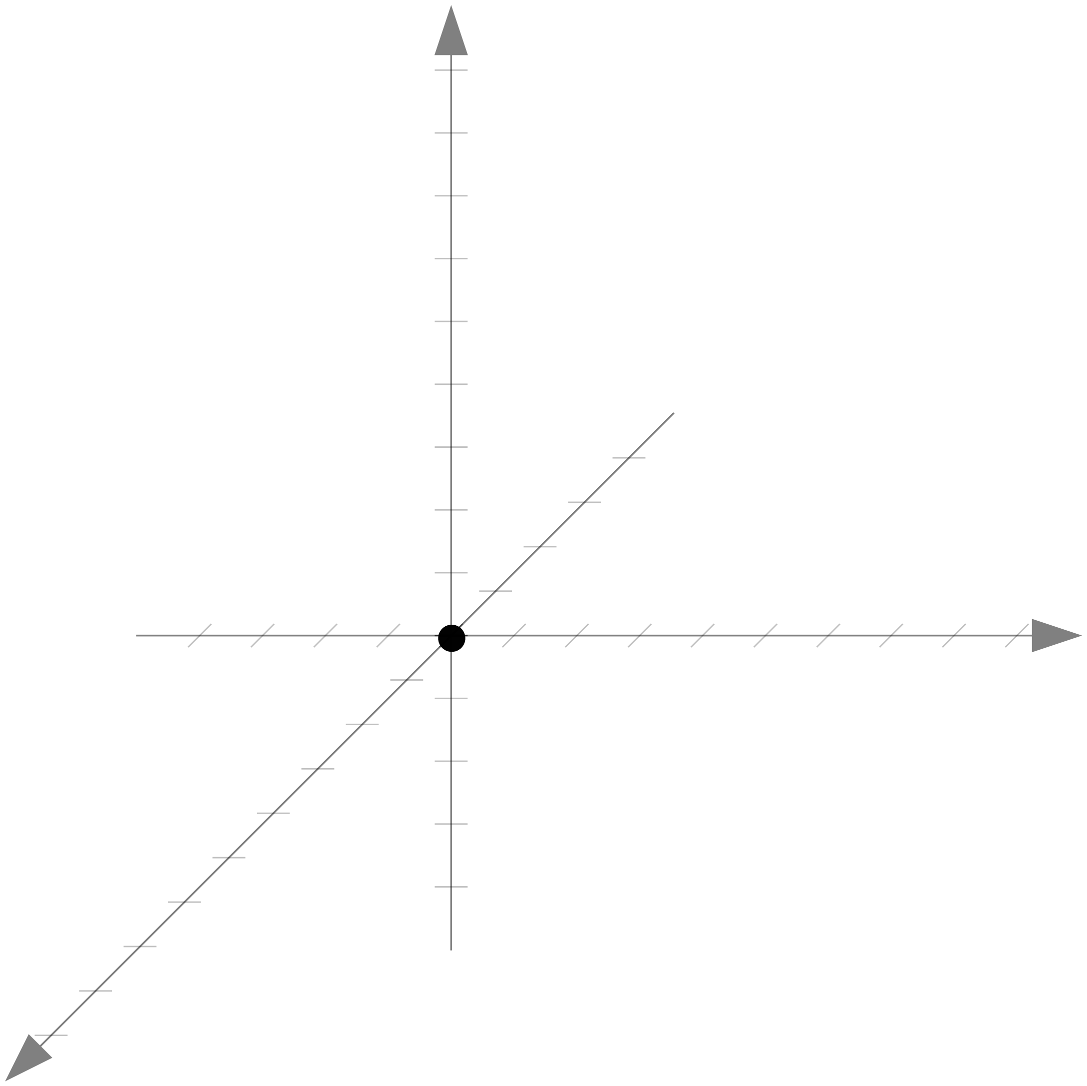


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Range of A

"span of columns"

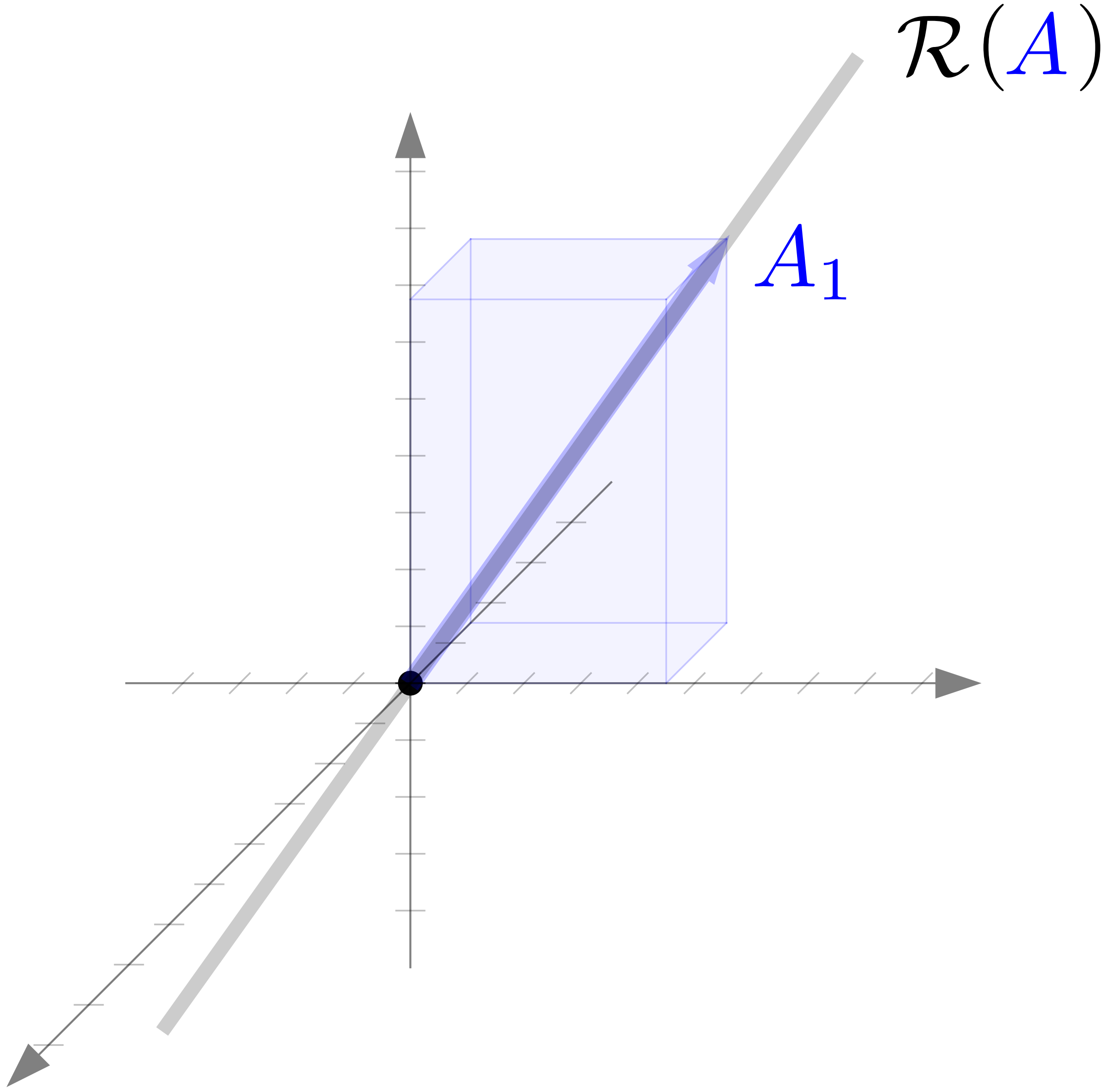




$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Range of A

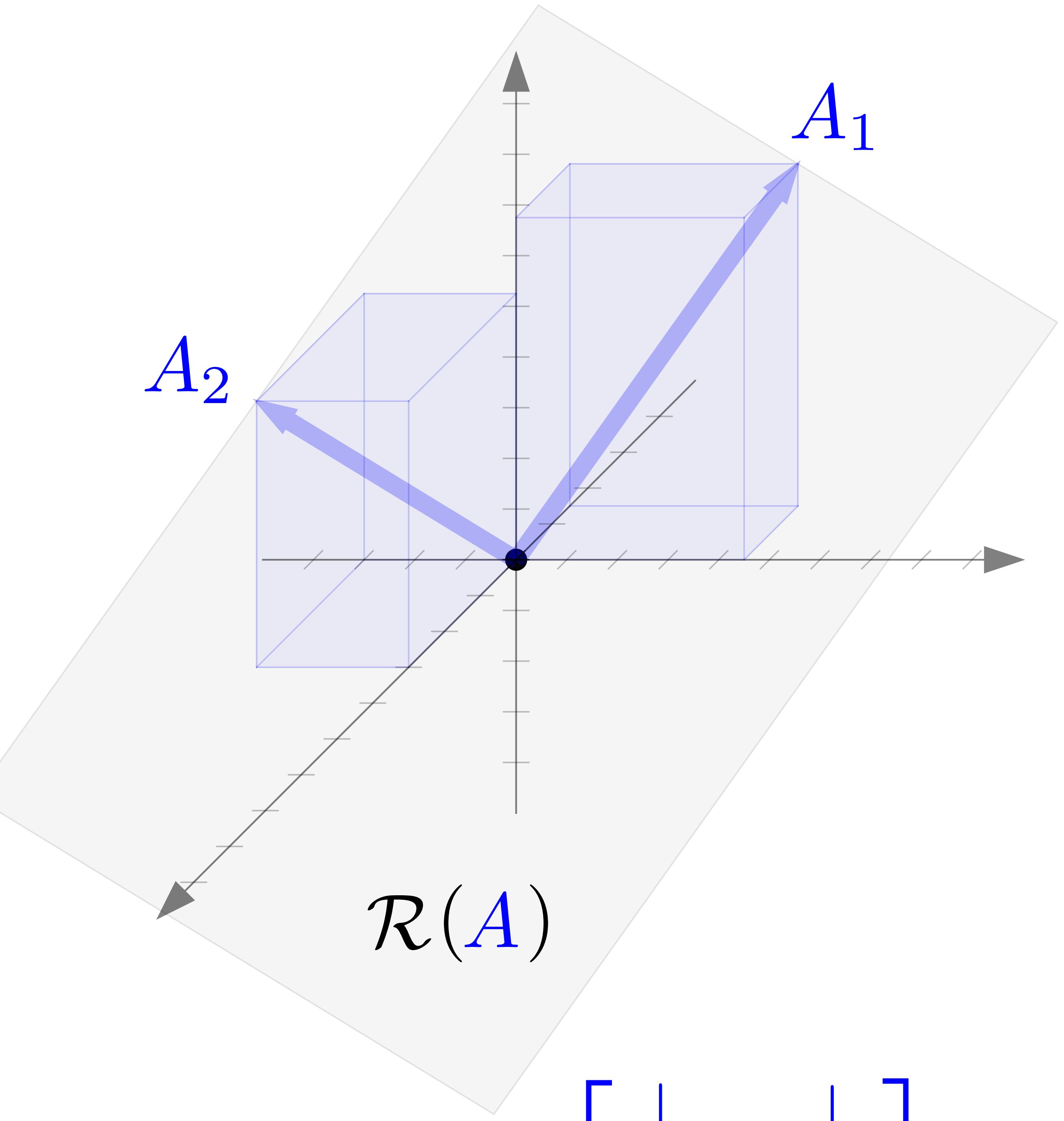
"span of columns"

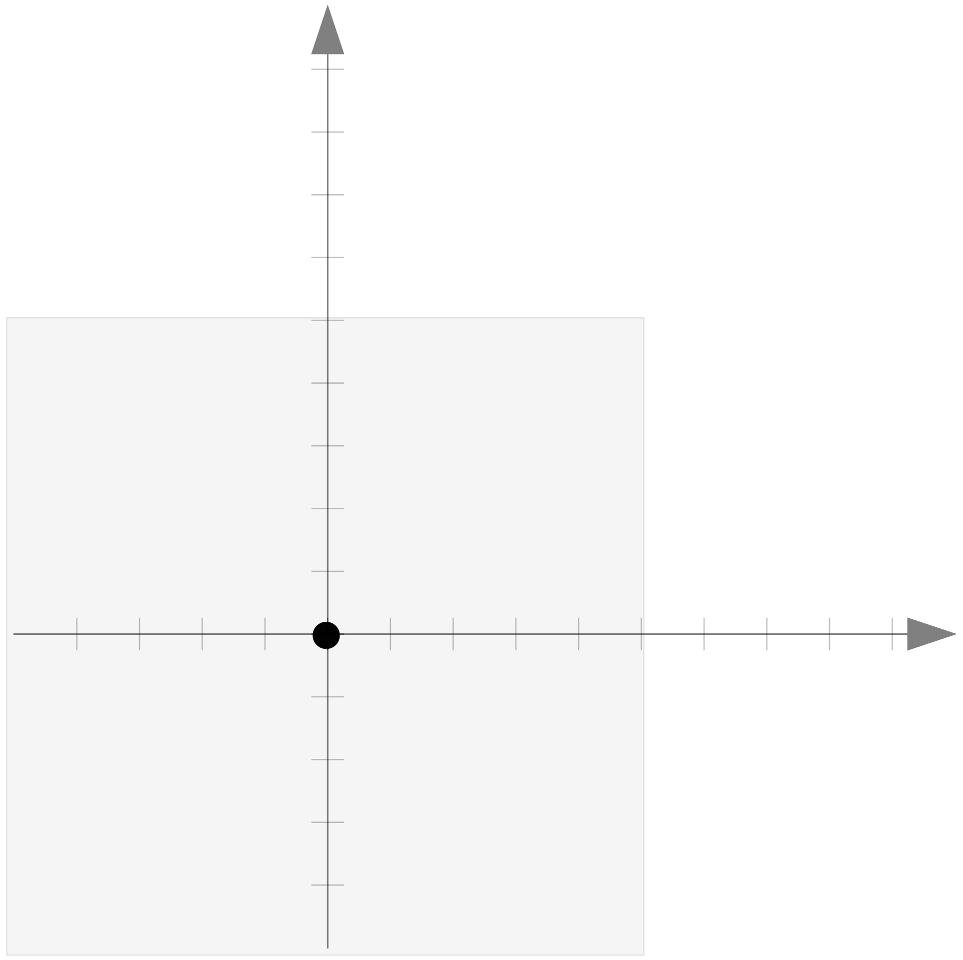


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1$$

Range of A

"span of columns"



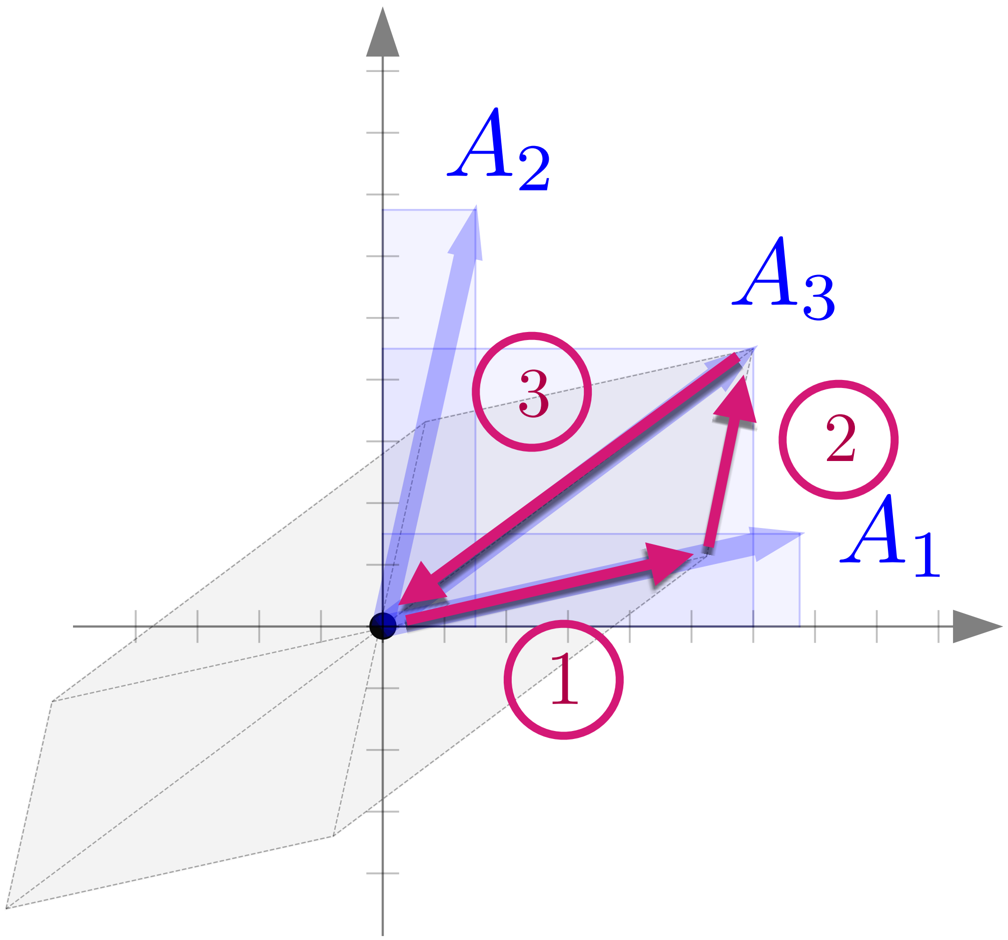


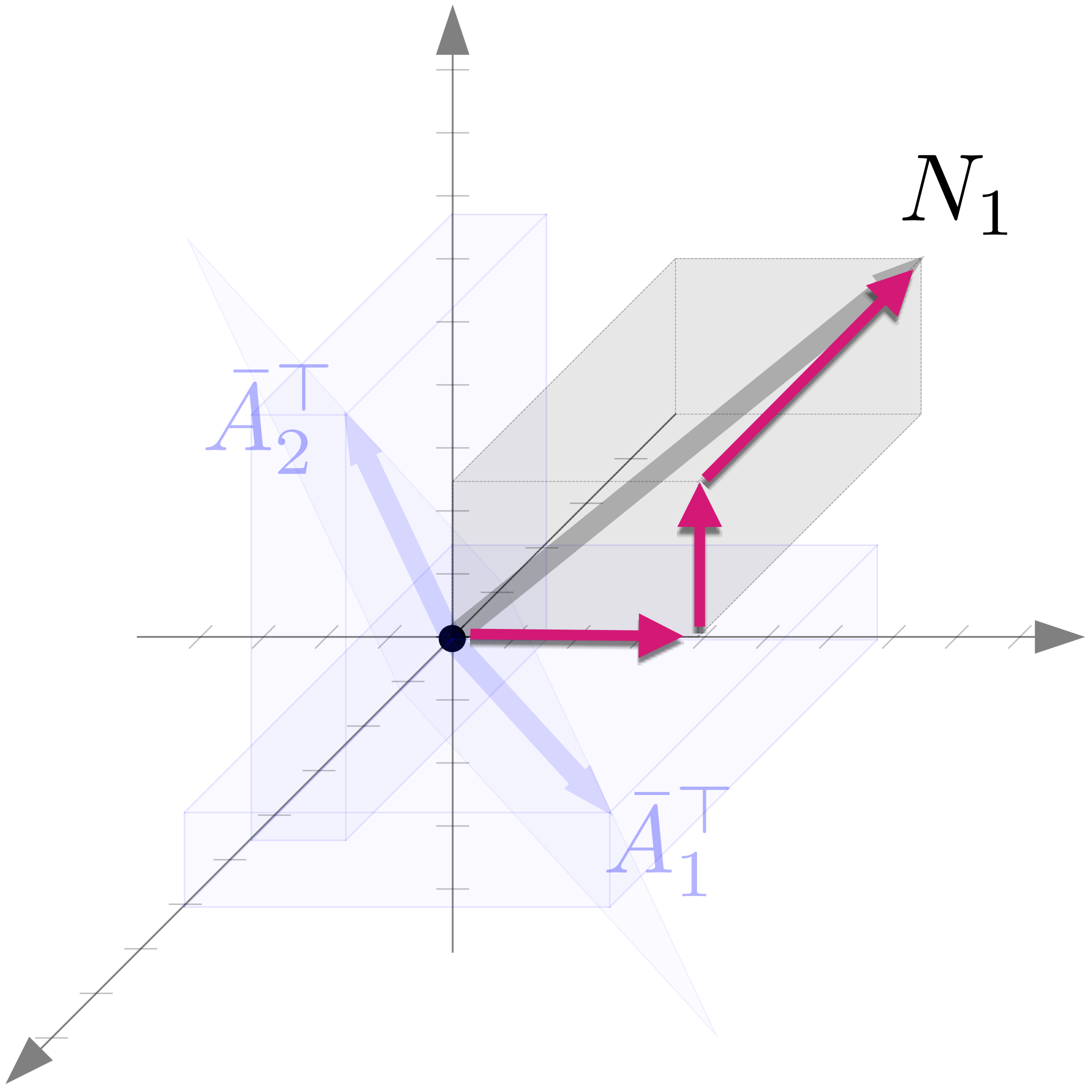
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Nullspace of A

Nullspace of A

“coordinates of 0”

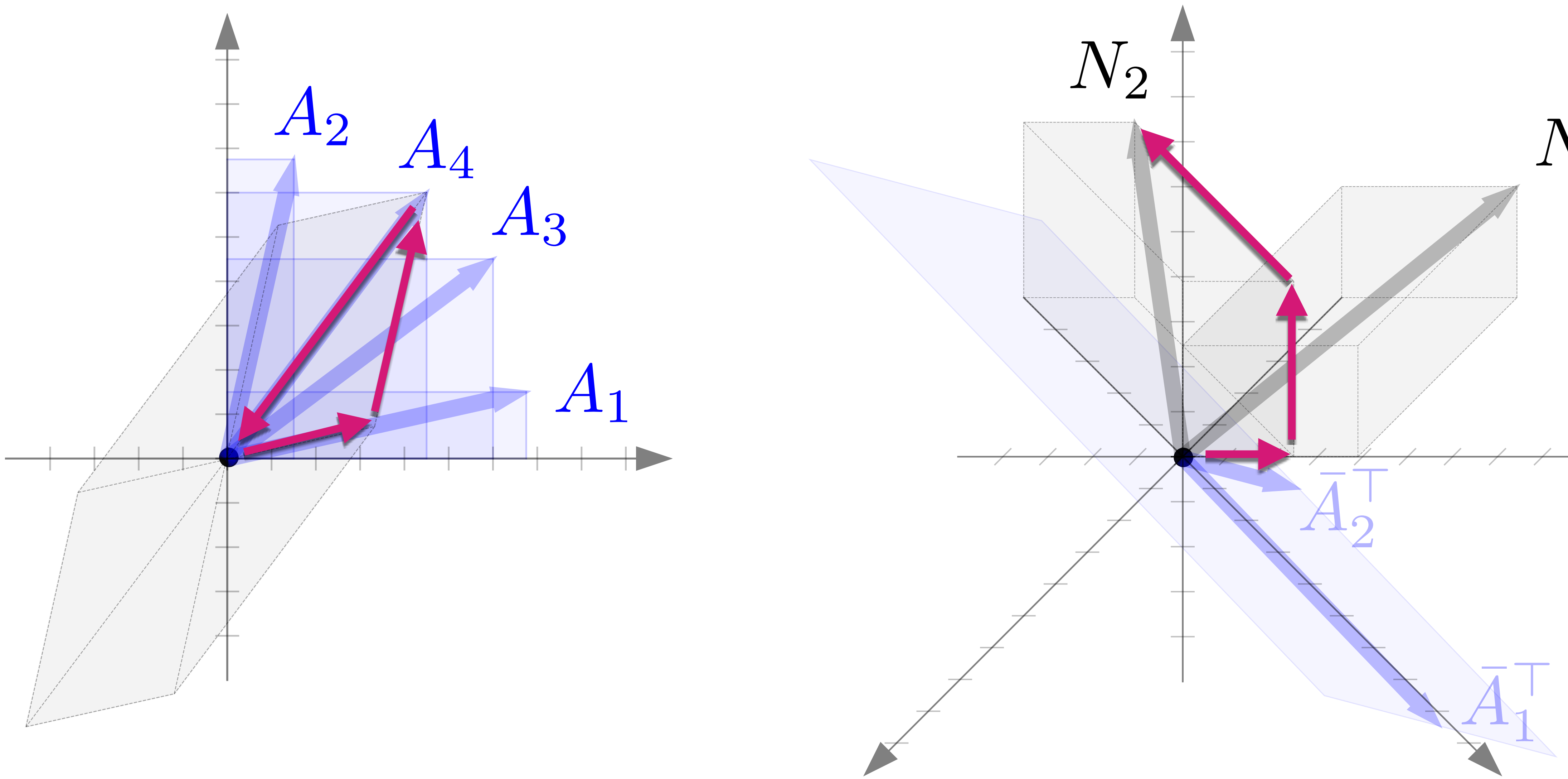




$$\begin{bmatrix} | \\ 0 \\ | \end{bmatrix} = \left[\underbrace{\begin{matrix} | & | \\ A_1 & A_2 \\ | & | \end{matrix}}_{\text{lin. ind.}} \;\; \underbrace{\begin{matrix} | \\ A_3 \\ | \end{matrix}}_{\text{lin. dep.}} \right] \underset{N}{\begin{bmatrix} B_{11} \\ B_{21} \\ -1 \end{bmatrix}} = \underset{\text{①}}{\begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} B_{11}} + \underset{\text{②}}{\begin{bmatrix} | \\ A_2 \\ | \end{bmatrix} B_{21}} - \underset{\text{③}}{\begin{bmatrix} | \\ A_3 \\ | \end{bmatrix}}$$

Nullspace of A

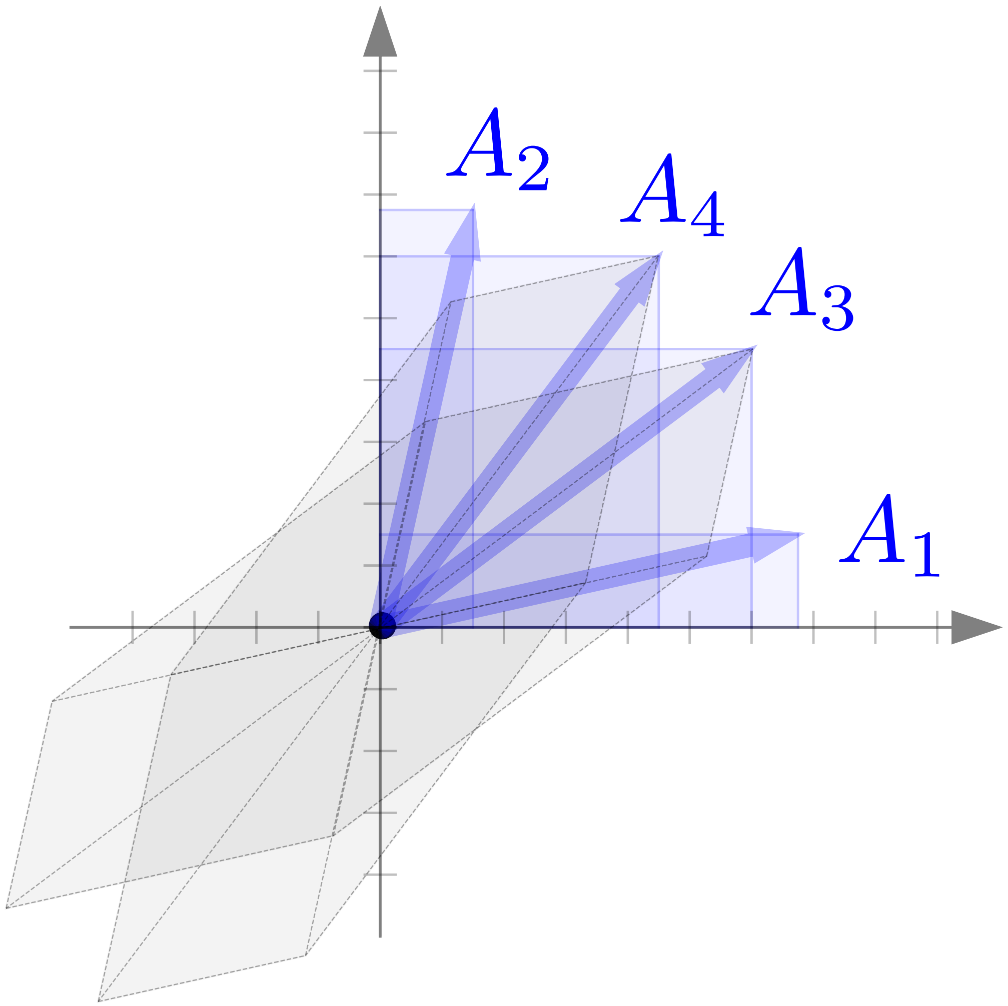
“coordinates of 0”

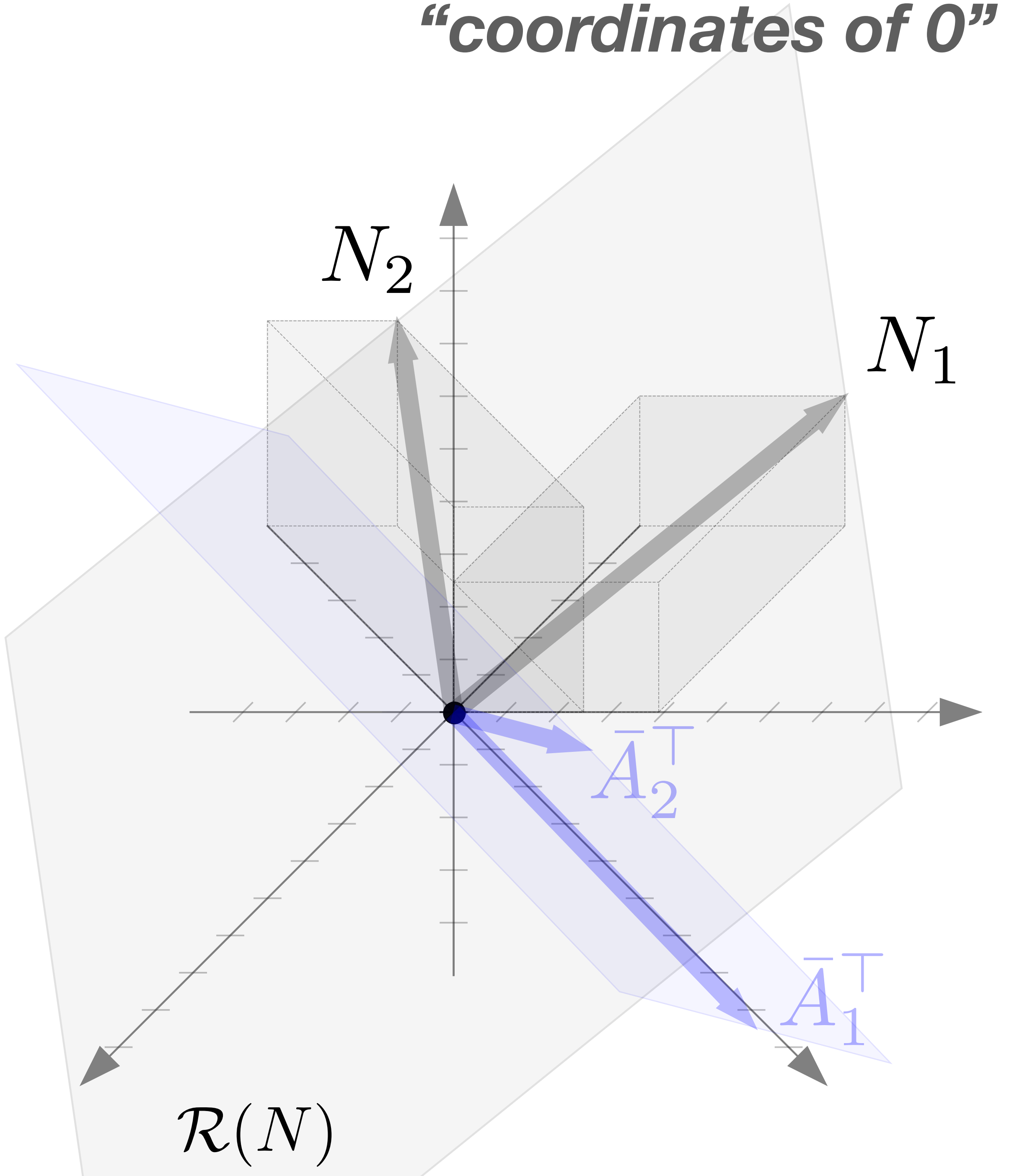


$$\begin{bmatrix} | \\ 0 \\ | \end{bmatrix} = \left[\begin{matrix} \underbrace{\begin{matrix} | & | \\ A_1 & A_2 \\ | & | \end{matrix}}_{\text{lin. ind.}} & \underbrace{\begin{matrix} | & | \\ A_3 & A_4 \\ | & | \end{matrix}}_{\text{lin. dep.}} \end{matrix}\right] \underbrace{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}}_{N} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} B_{12} + \begin{bmatrix} | \\ A_2 \\ | \end{bmatrix} B_{22} - \begin{bmatrix} | \\ A_4 \\ | \end{bmatrix}$$

Nullspace of A

“coordinates of 0”



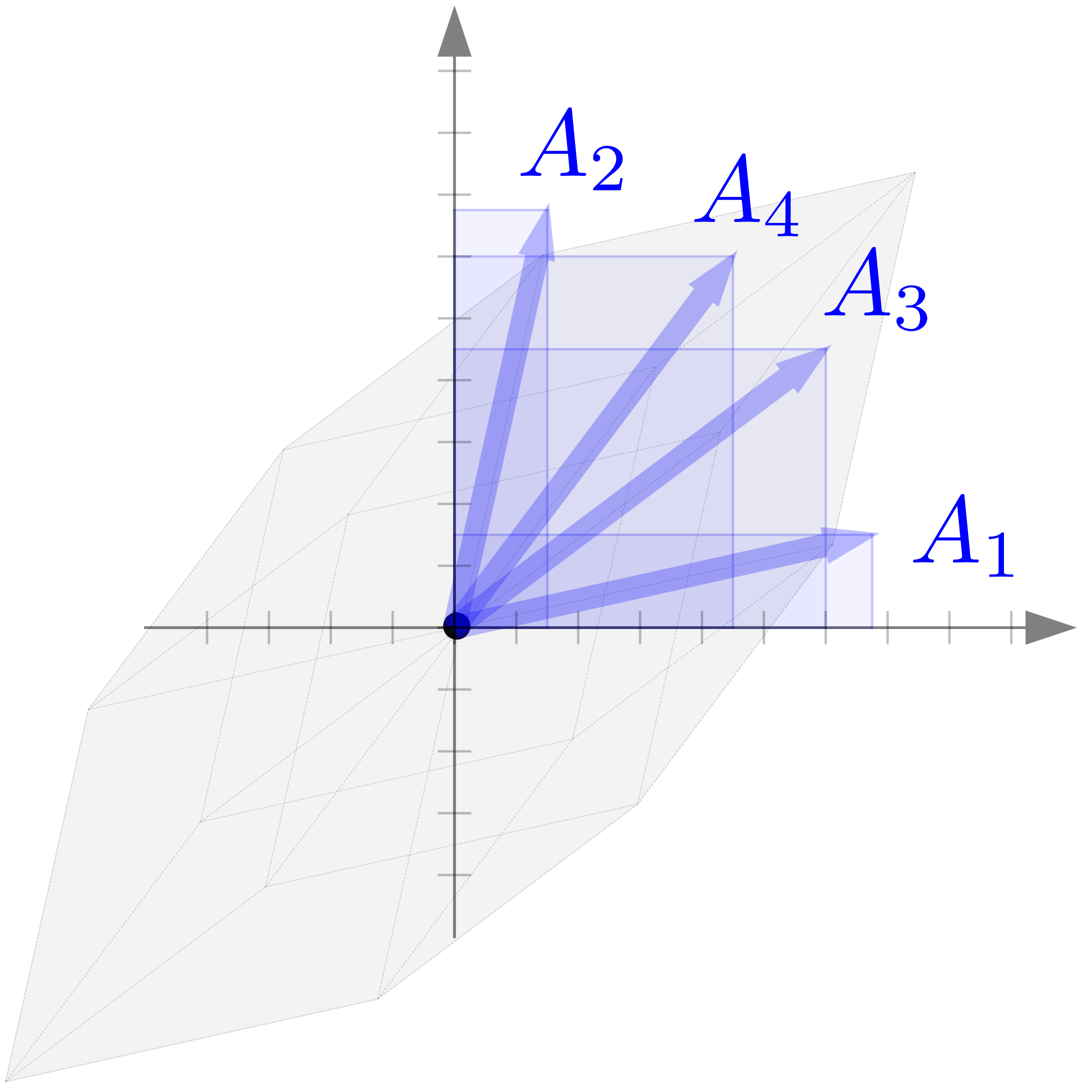


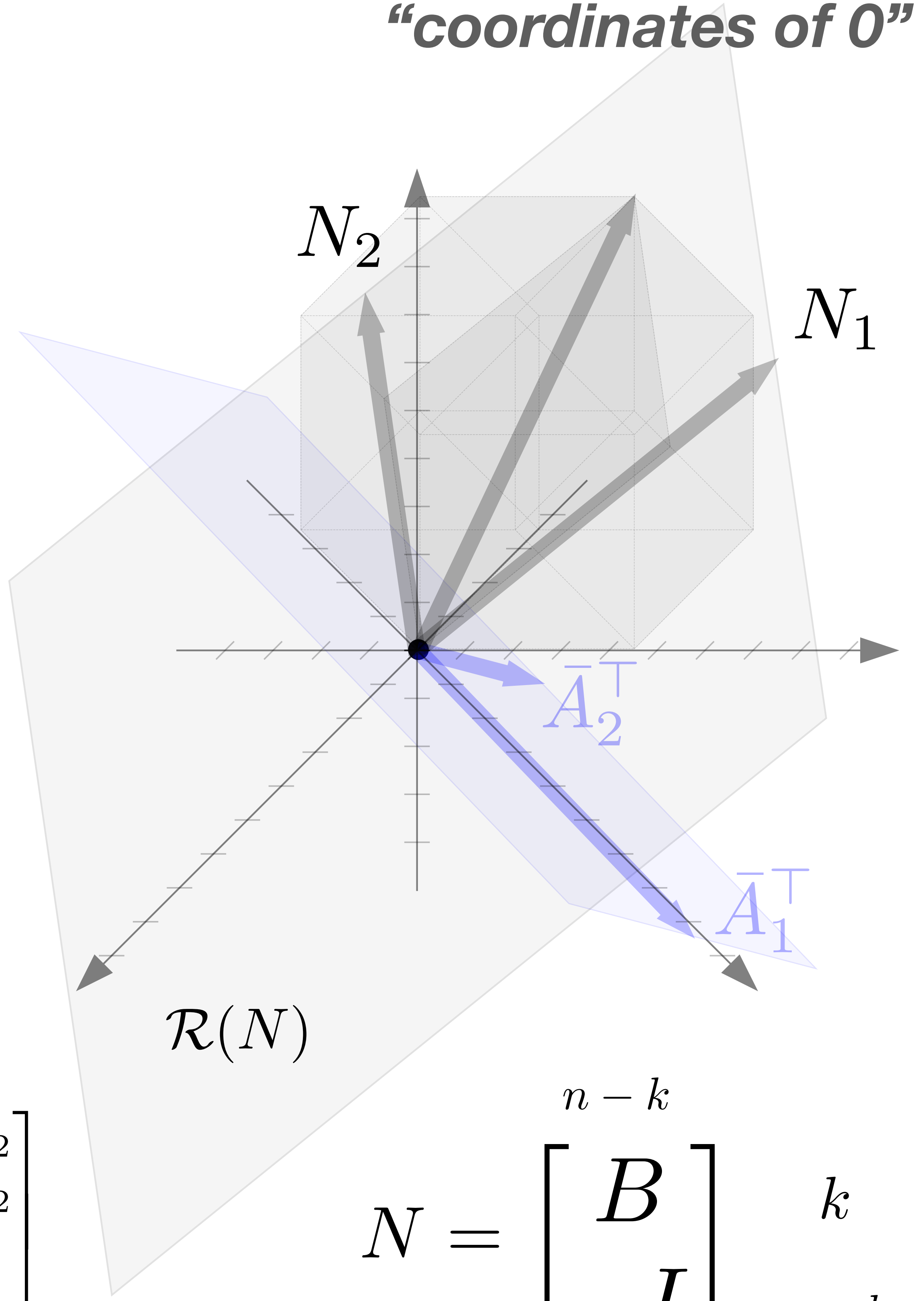
$$\begin{bmatrix}|\\0\\|\end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix}|&|\\A_1&A_2\\|&|\end{matrix}}_{\text{lin. ind.}} & \underbrace{\begin{matrix}|&|\\A_3&A_4\\|&|\end{matrix}}_{\text{lin. dep.}}\end{bmatrix} \underbrace{\begin{bmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\\-1&0\\0&-1\end{bmatrix}}_{N}$$

$$N = \begin{bmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\\-1&0\\0&-1\end{bmatrix}$$

Nullspace of A

“coordinates of 0”





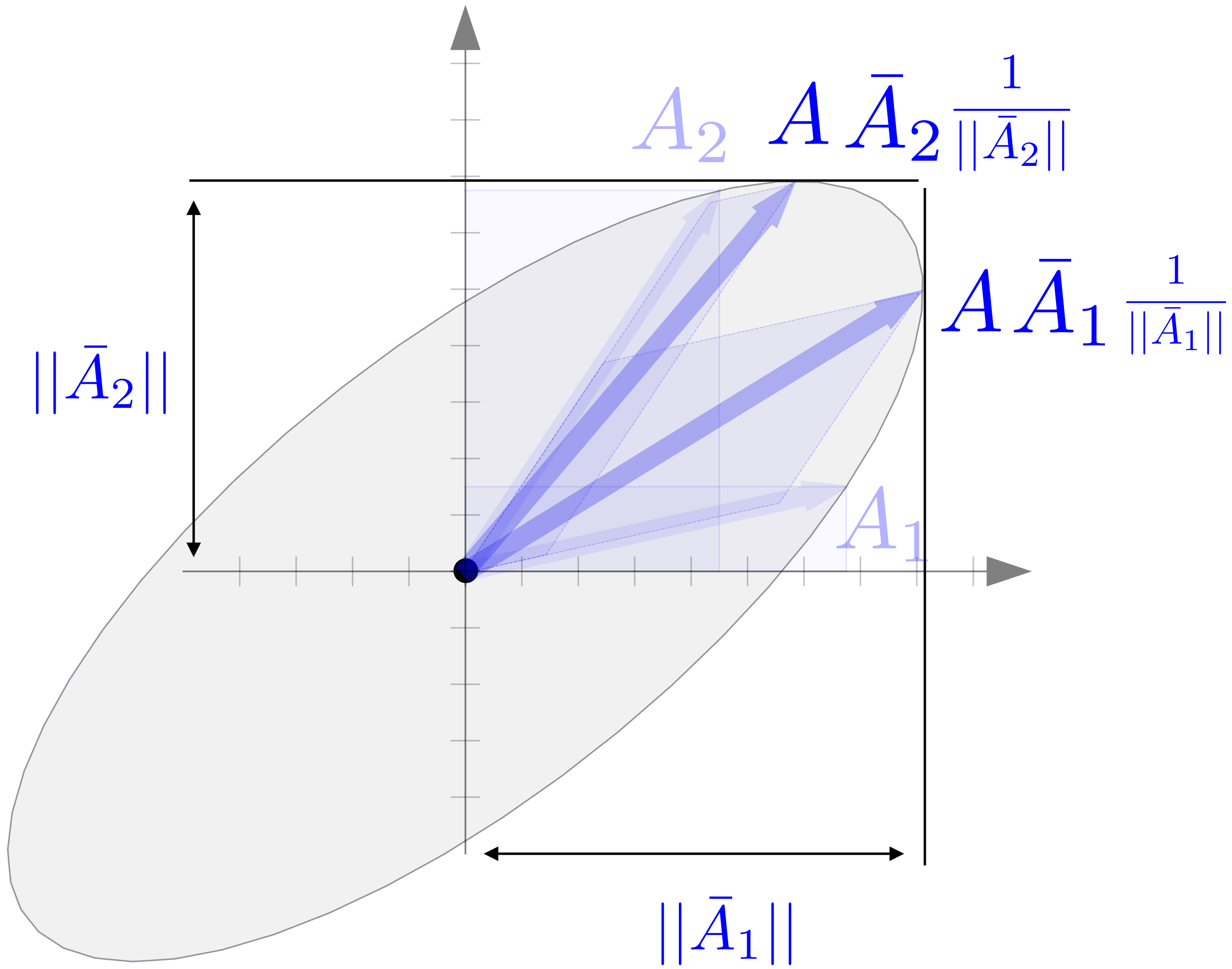
$$\begin{bmatrix} | \\ 0 \\ | \end{bmatrix} = \left[\overset{\operatorname{rank}(A)=k}{\underbrace{\begin{matrix} | & | \\ A_1 & A_2 \\ | & | \end{matrix}}_{\text{lin. ind.}}}\quad \overset{n-k}{\underbrace{\begin{matrix} | & | \\ A_3 & A_4 \\ | & | \end{matrix}}_{\text{lin. dep.}}}\right] \underset{N}{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

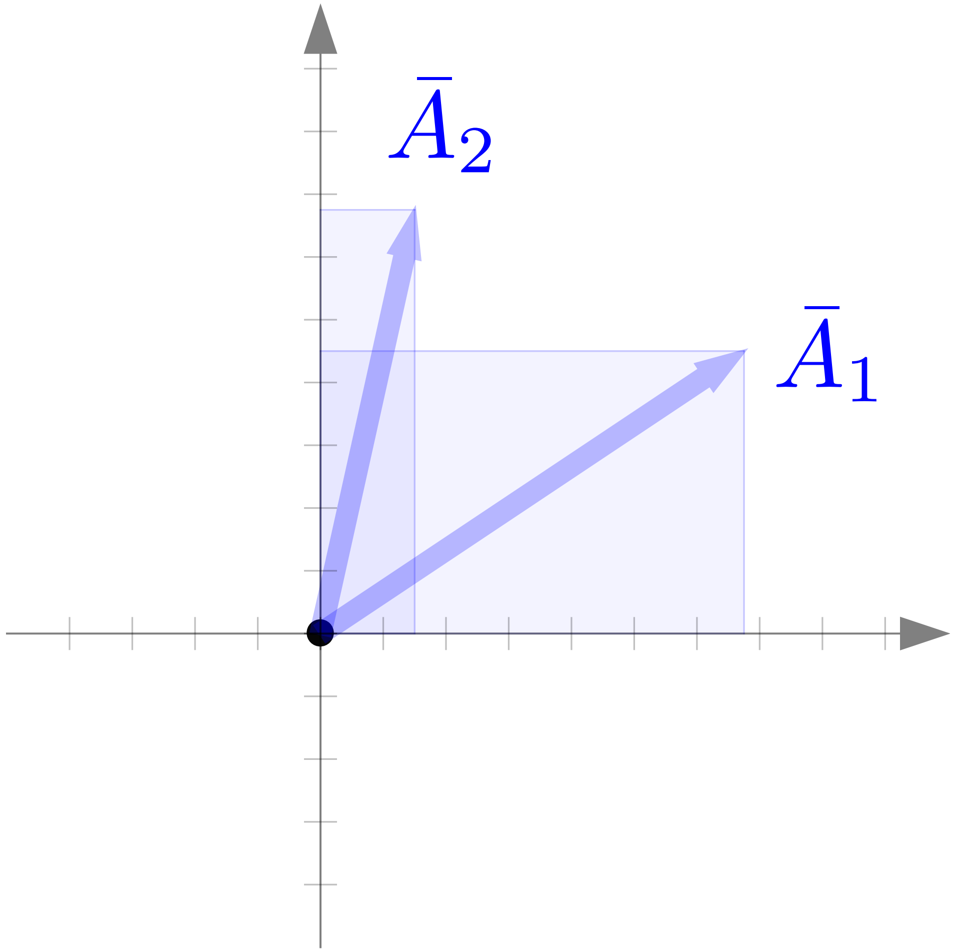
$$N = \begin{array}{cc} n-k & \\ \begin{bmatrix} B \\ -I \end{bmatrix} & \begin{matrix} k \\ n-k \end{matrix} \end{array}$$

Range of A^T

Range of A^T

“maximizing gain along ea. axis”

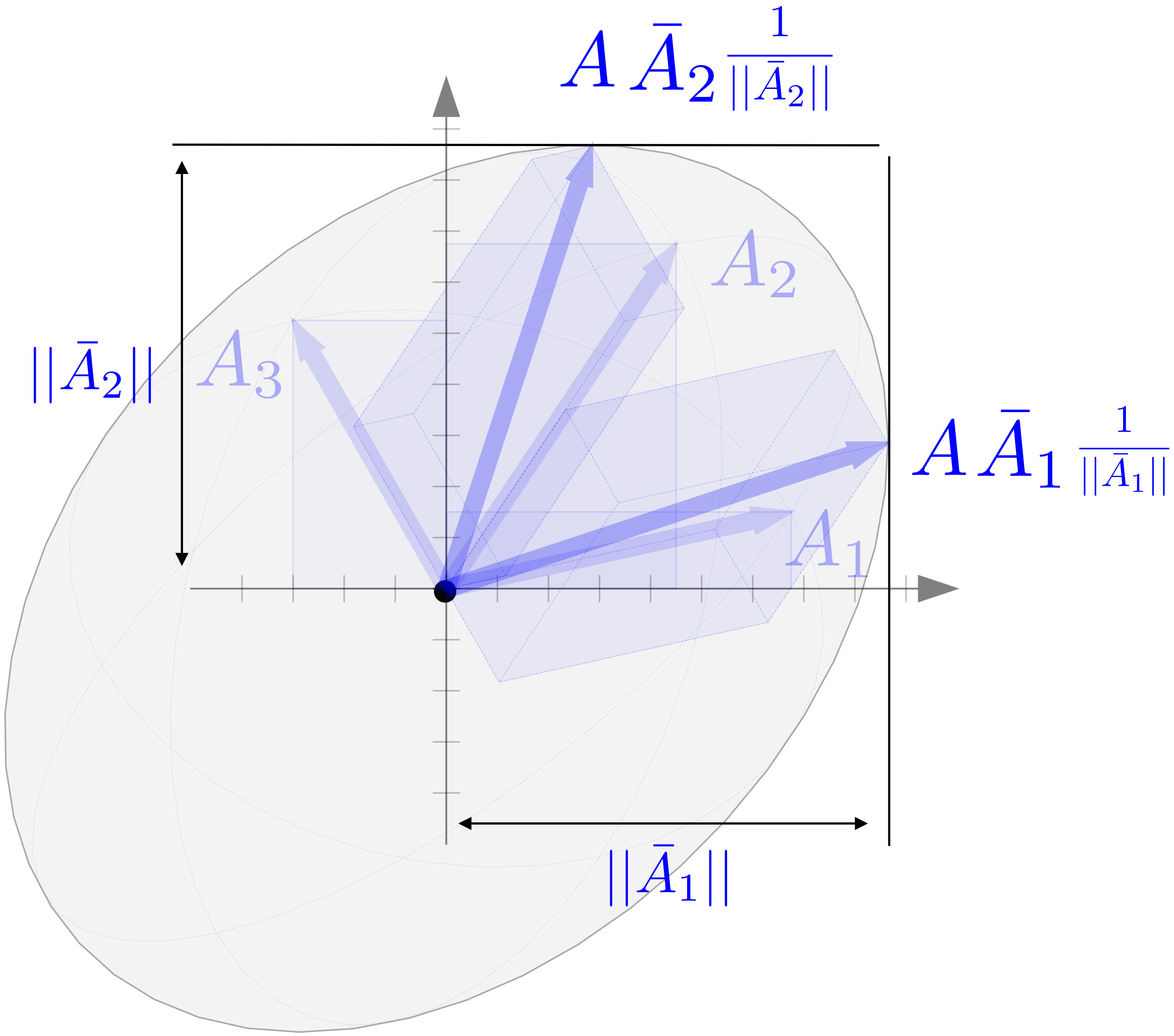


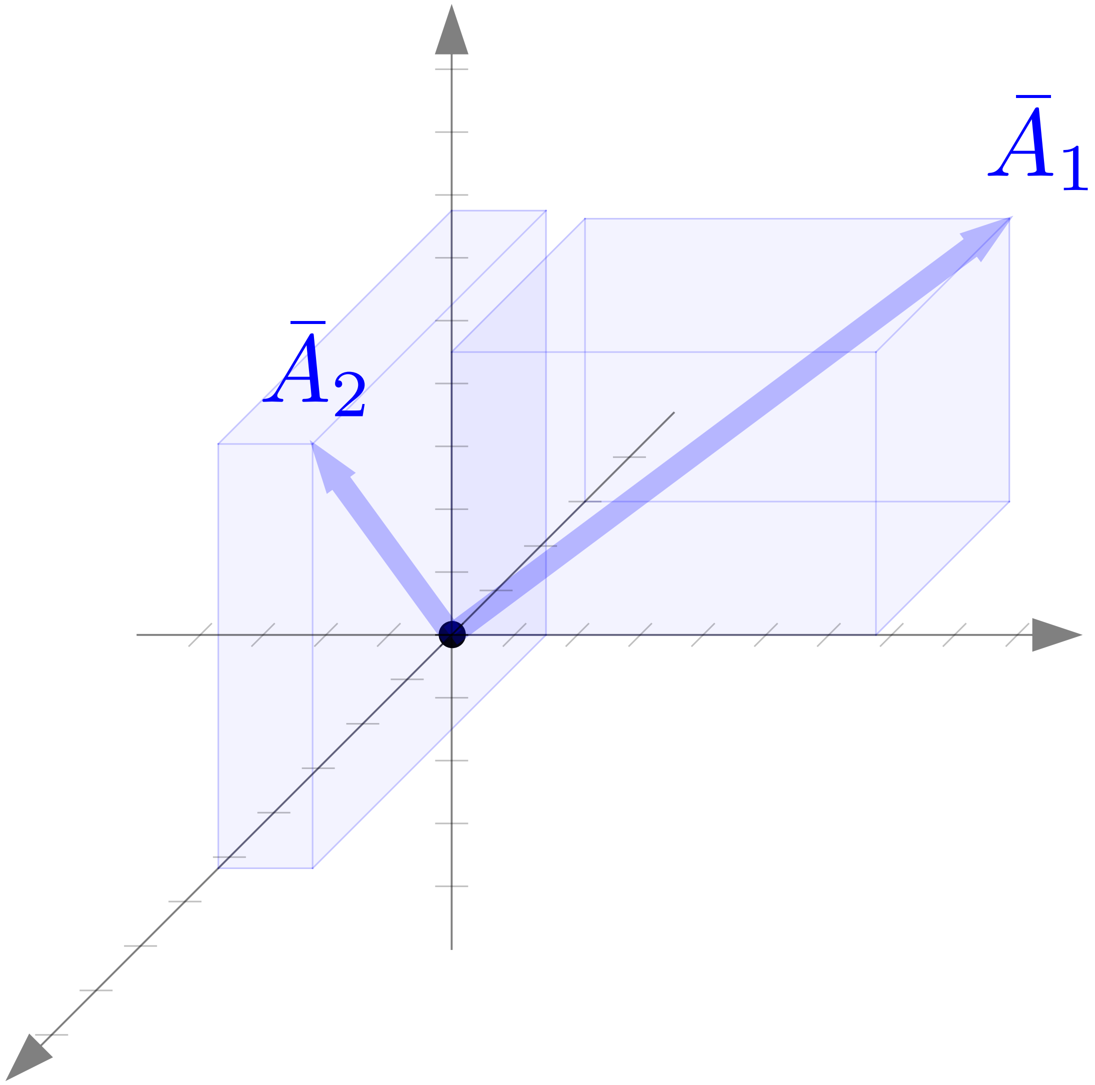


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} - & \bar{A}_1^\top & - \\ - & \bar{A}_2^\top & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Range of A^T

“maximizing gain along ea. axis”

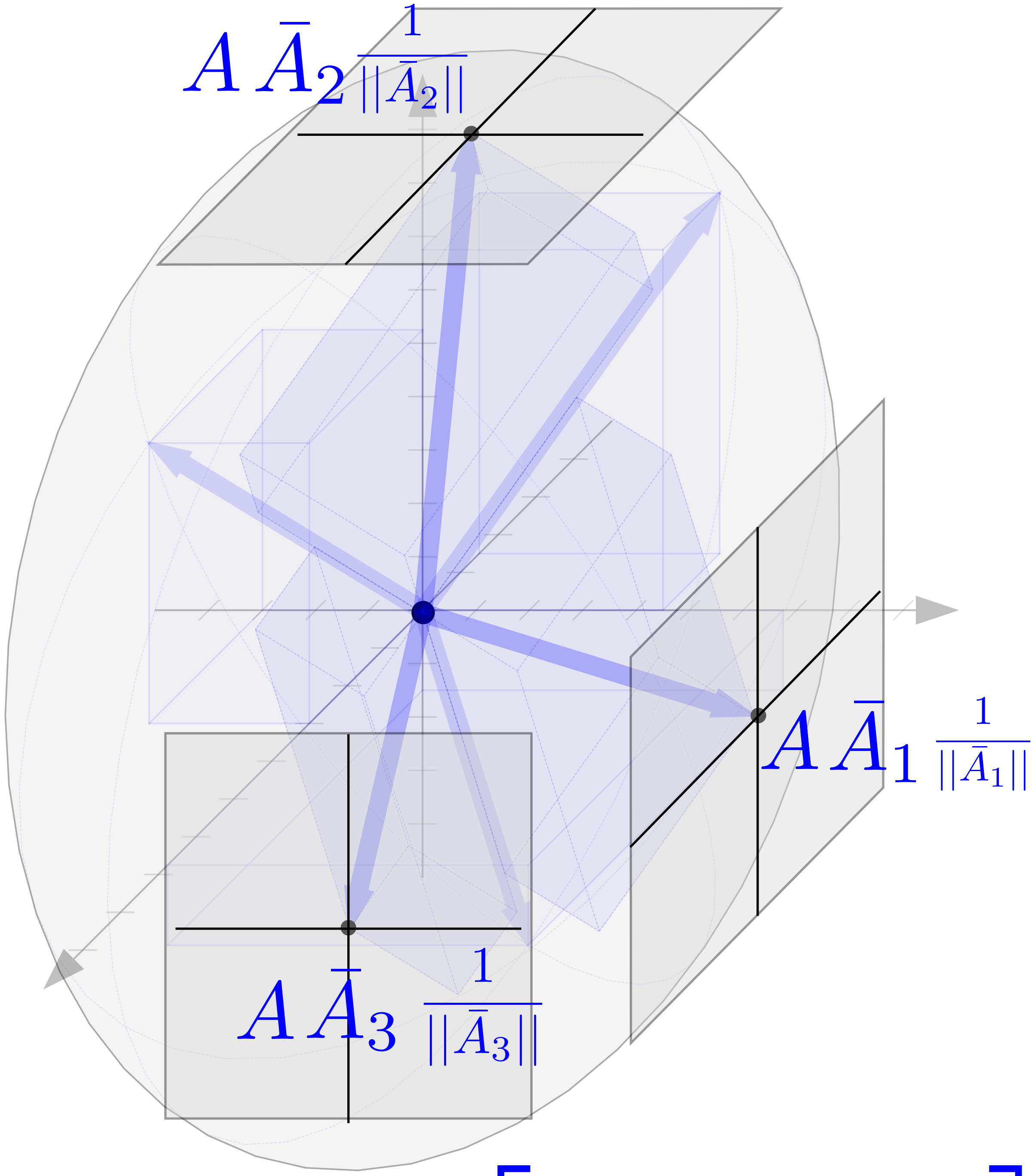


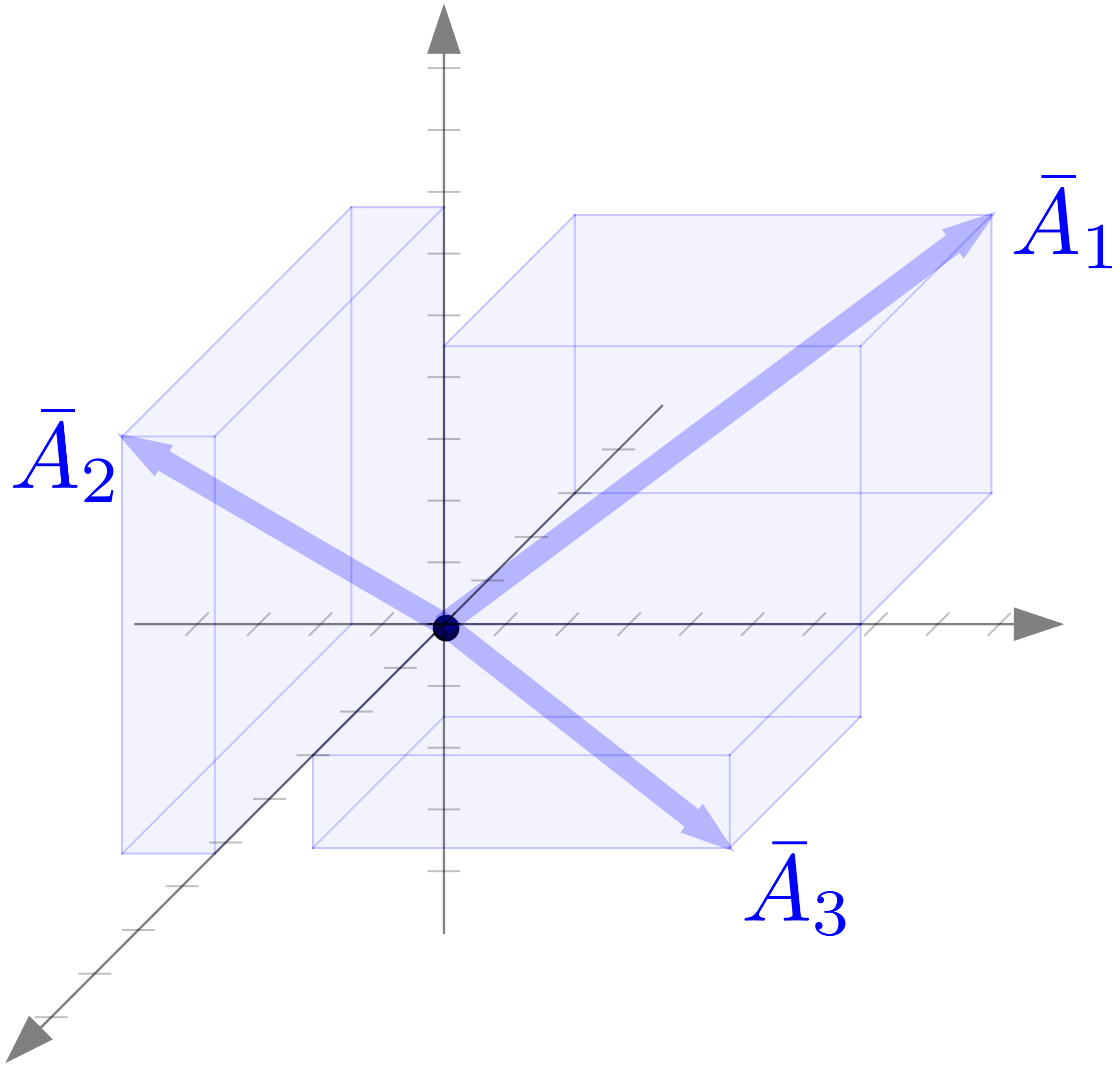


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} - & \bar{A}_1^\top & - \\ - & \bar{A}_2^\top & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Range of A^T

"maximizing gain along ea. axis"

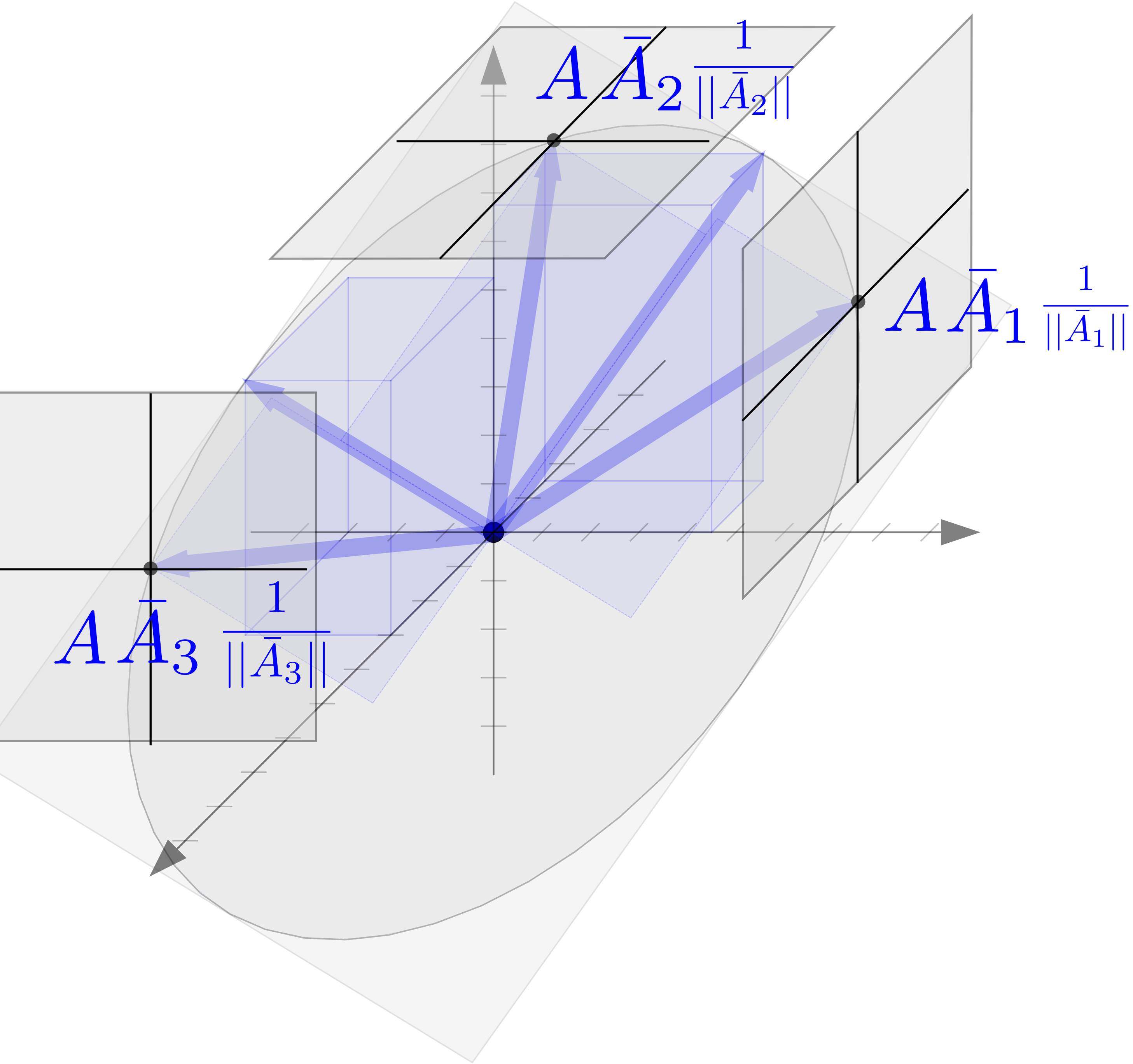


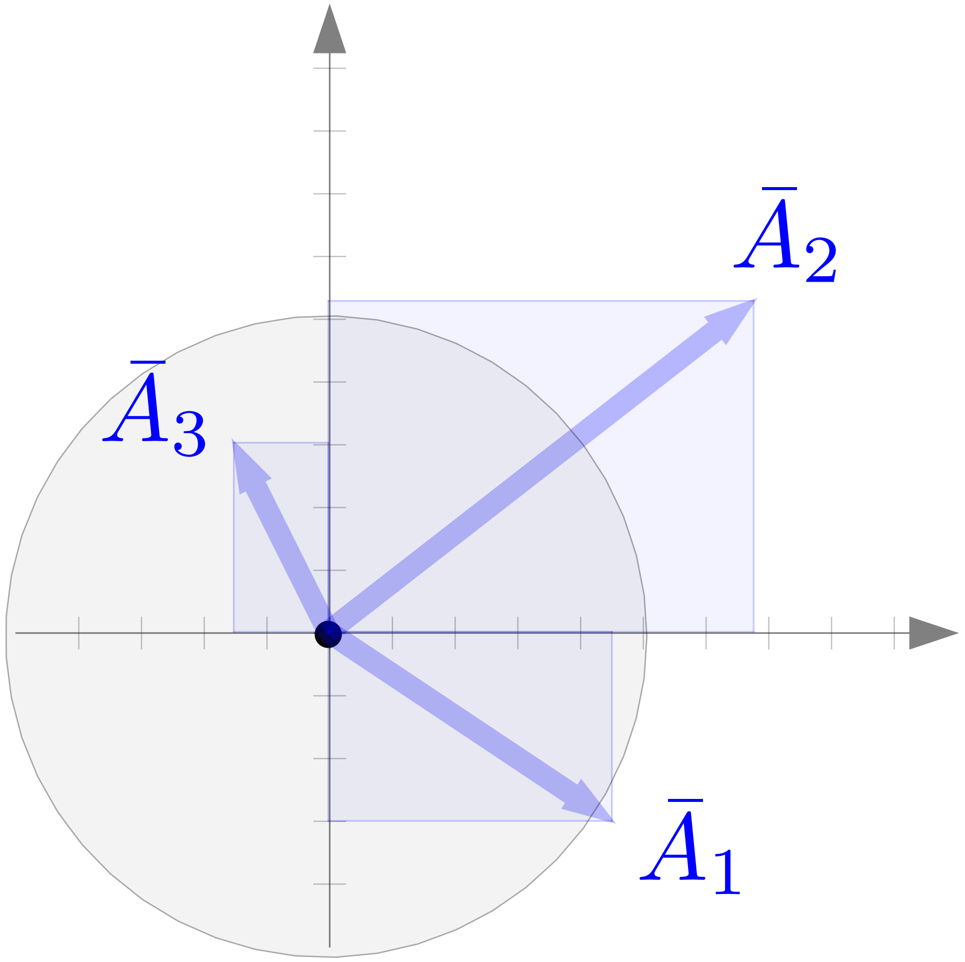


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} - & \bar{A}_1^\top & - \\ - & \bar{A}_2^\top & - \\ - & \bar{A}_3^\top & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Range of A^T

“maximizing gain along ea. axis”

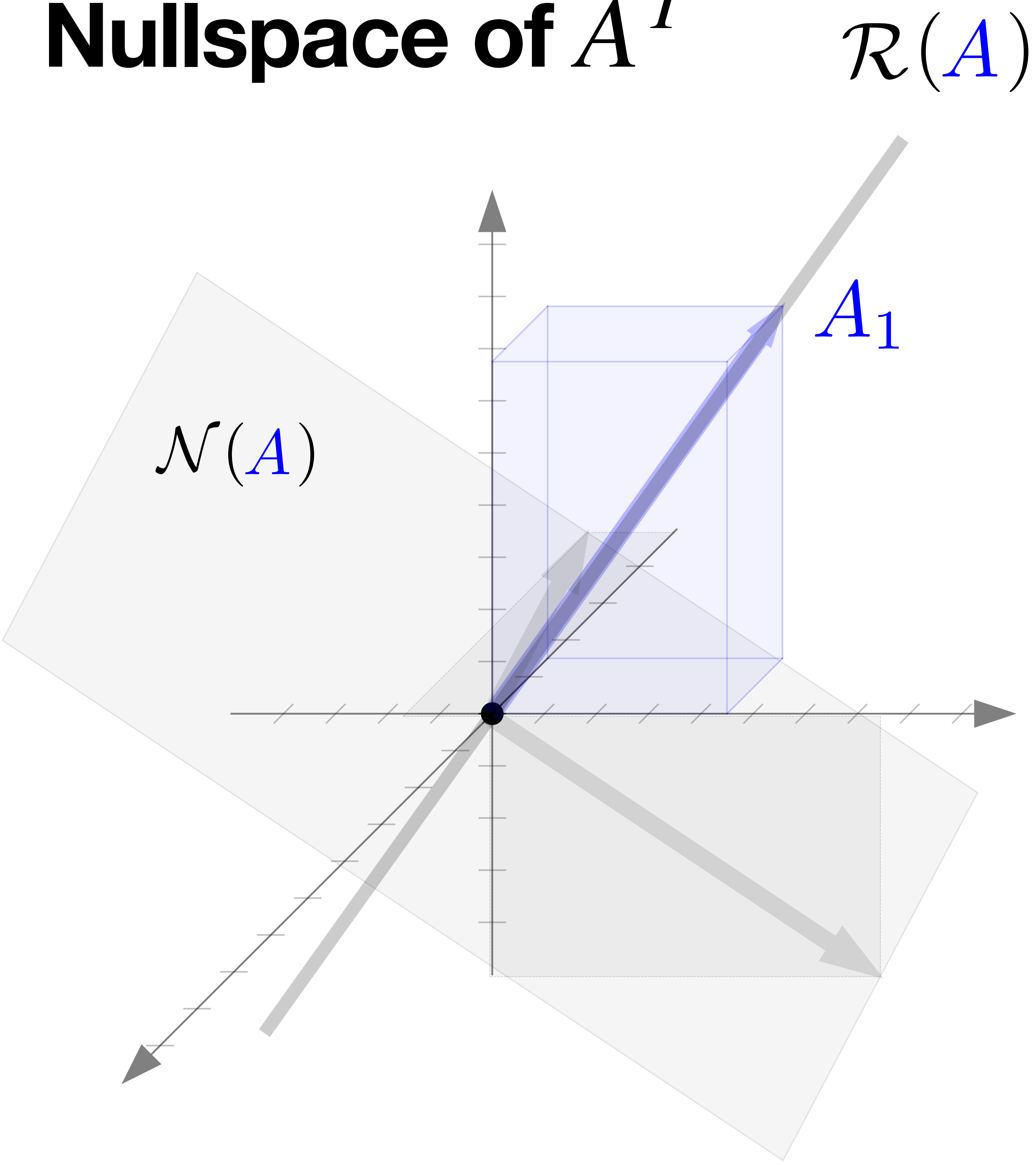




Nullspace of A^T

Nullspace of A^T

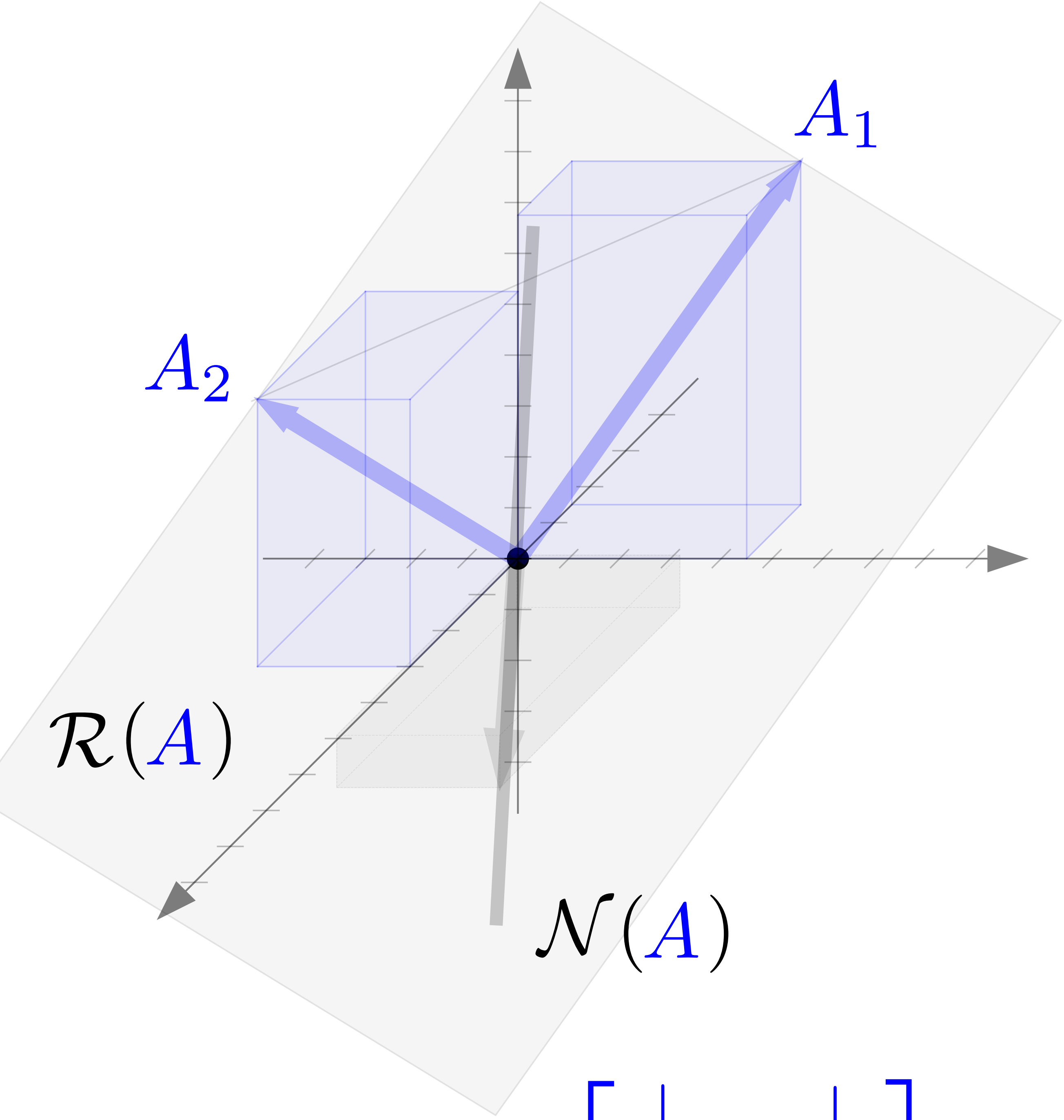
“orthogonal to columns”

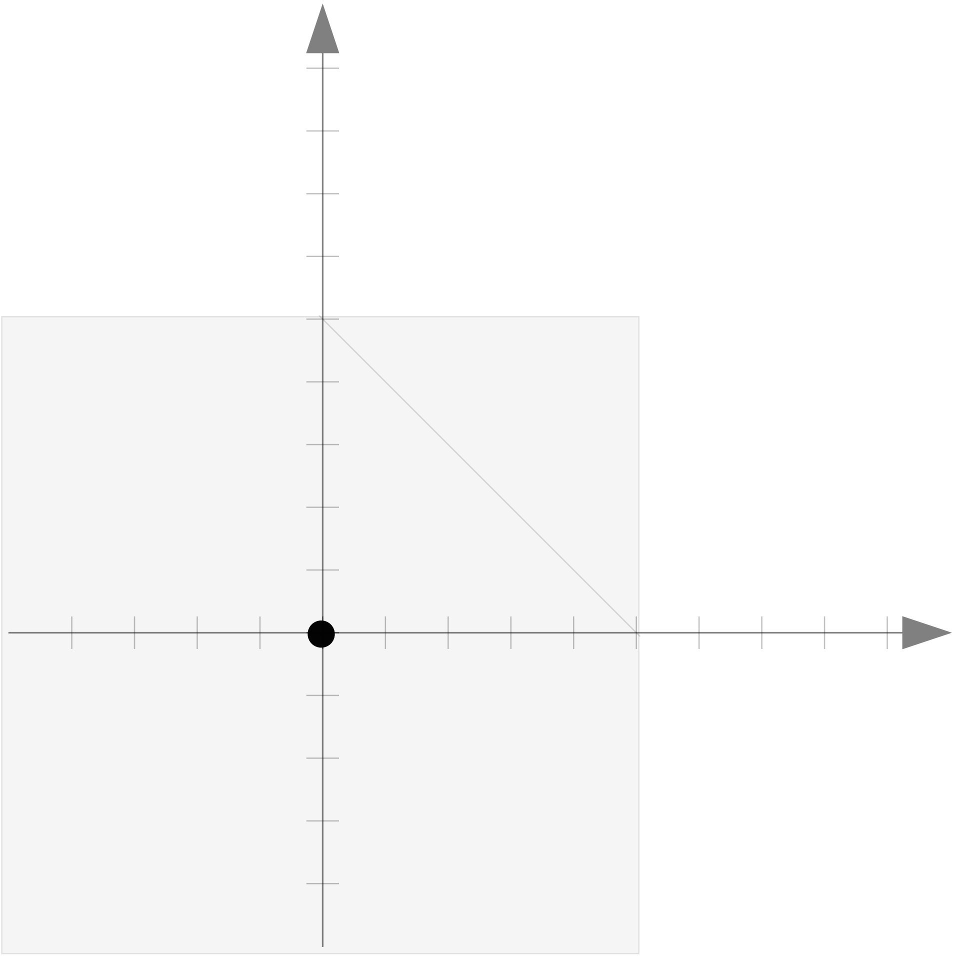


$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1$$

Nullspace of A^T

“orthogonal to columns”



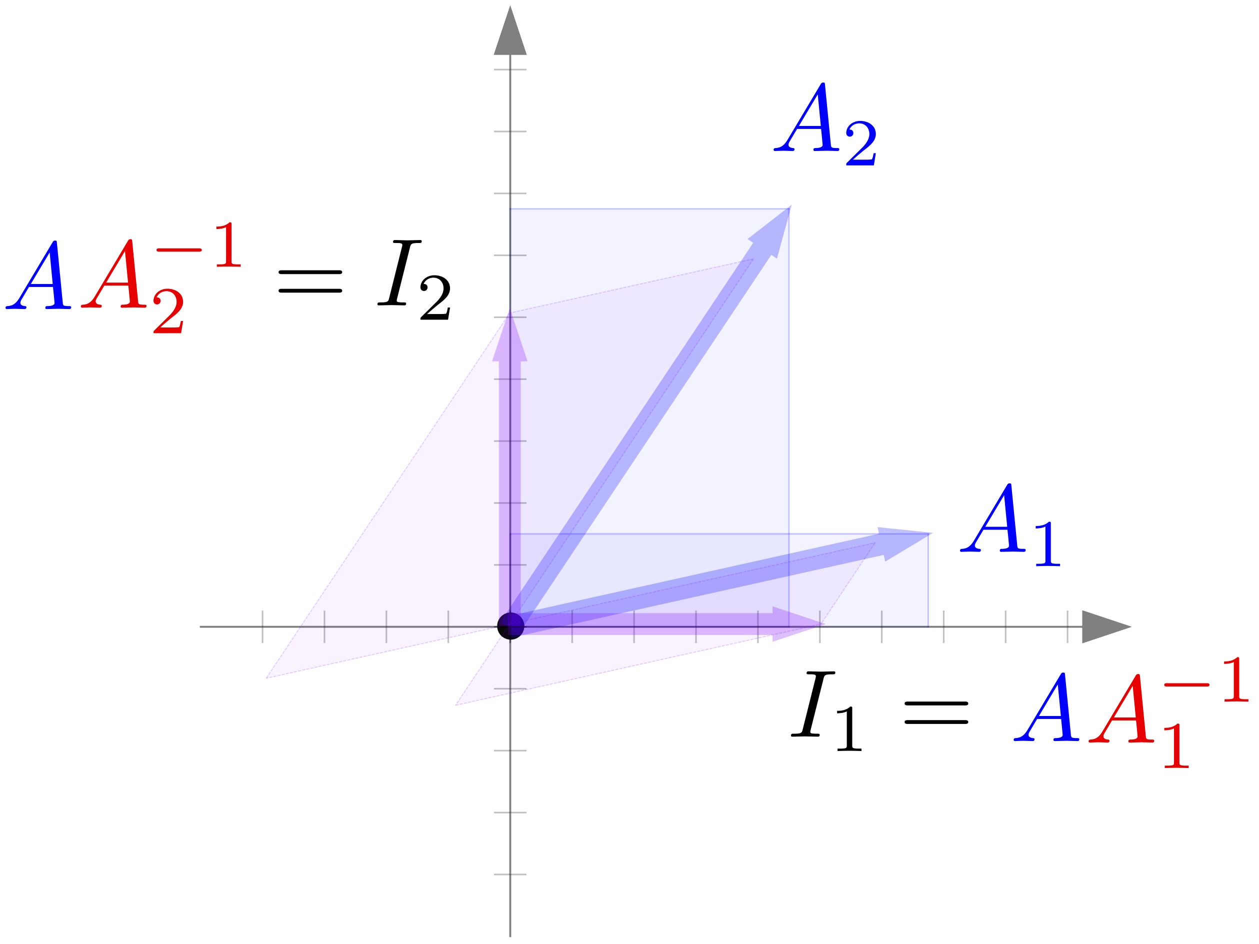


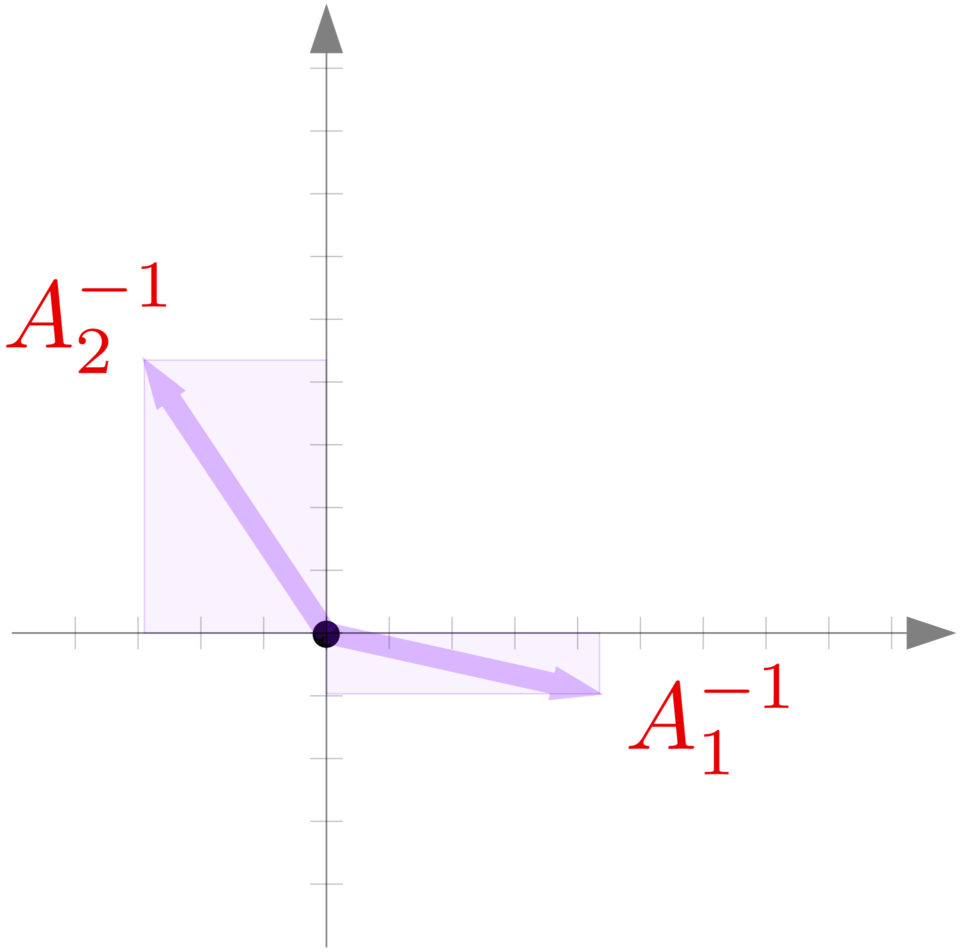
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Inverses

Inverse of A

"output only along ea. axis"

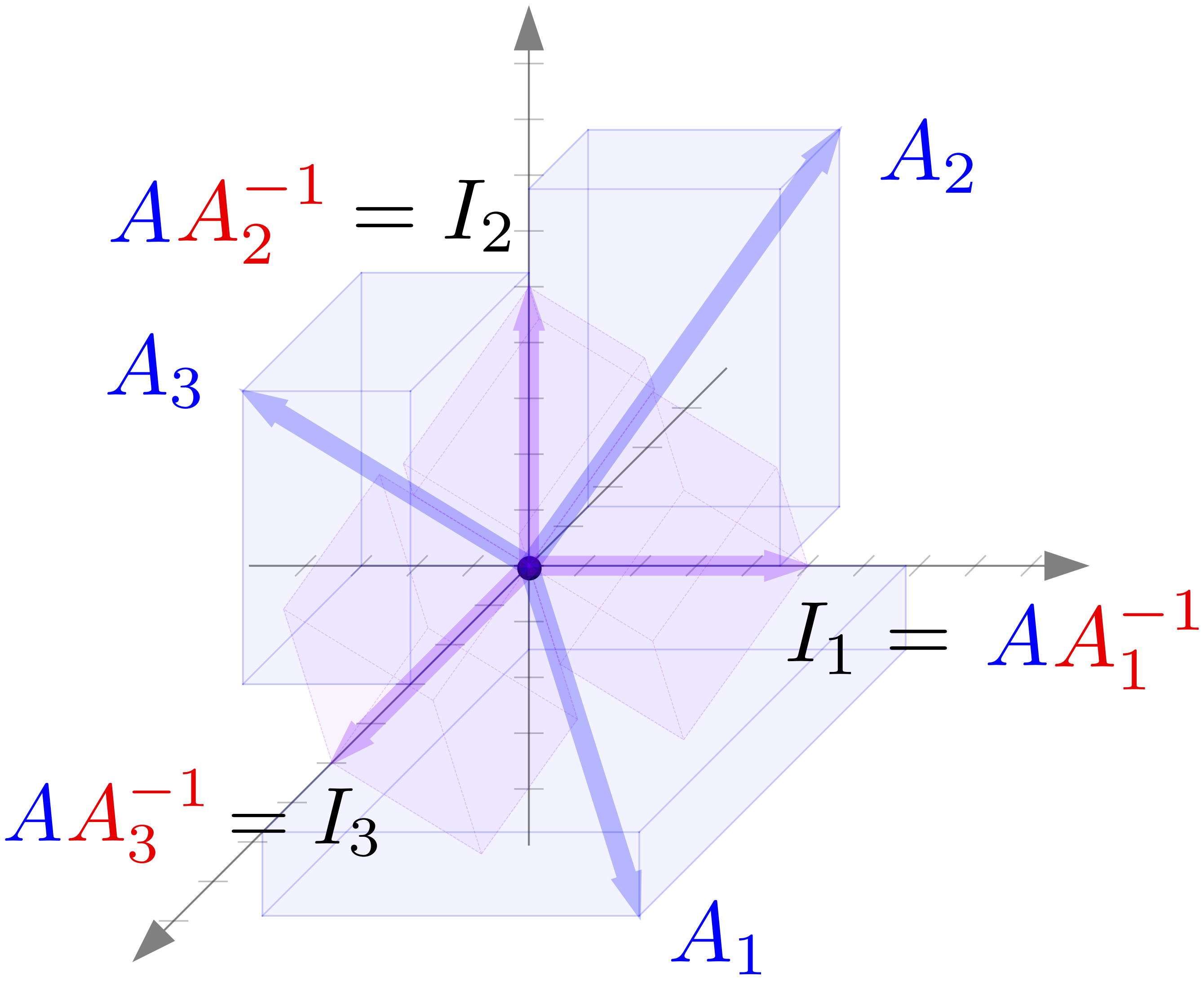


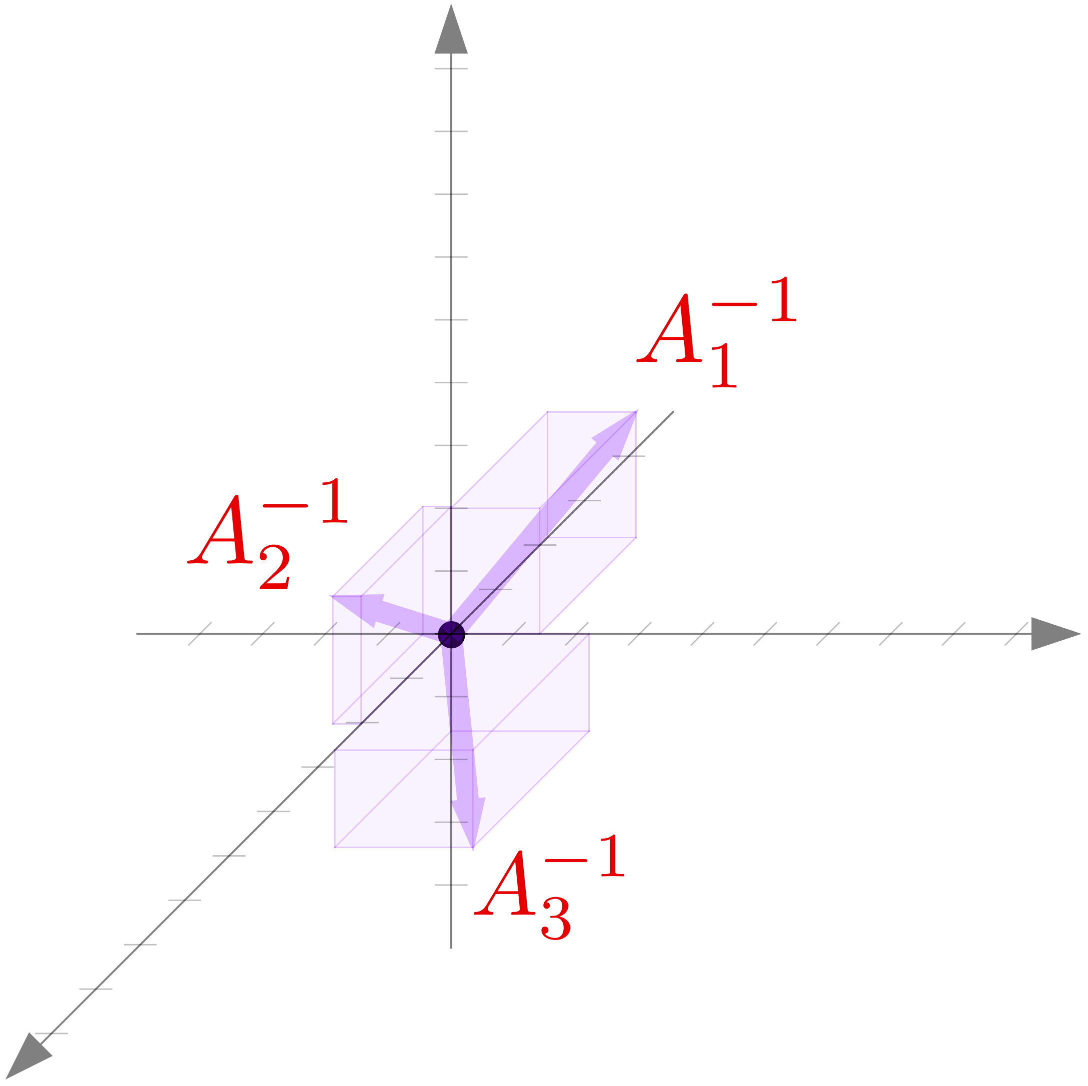


$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \begin{bmatrix} | & | \\ A_1^{-1} & A_2^{-1} \\ | & | \end{bmatrix}$$

Inverse of A

"output only along ea. axis"

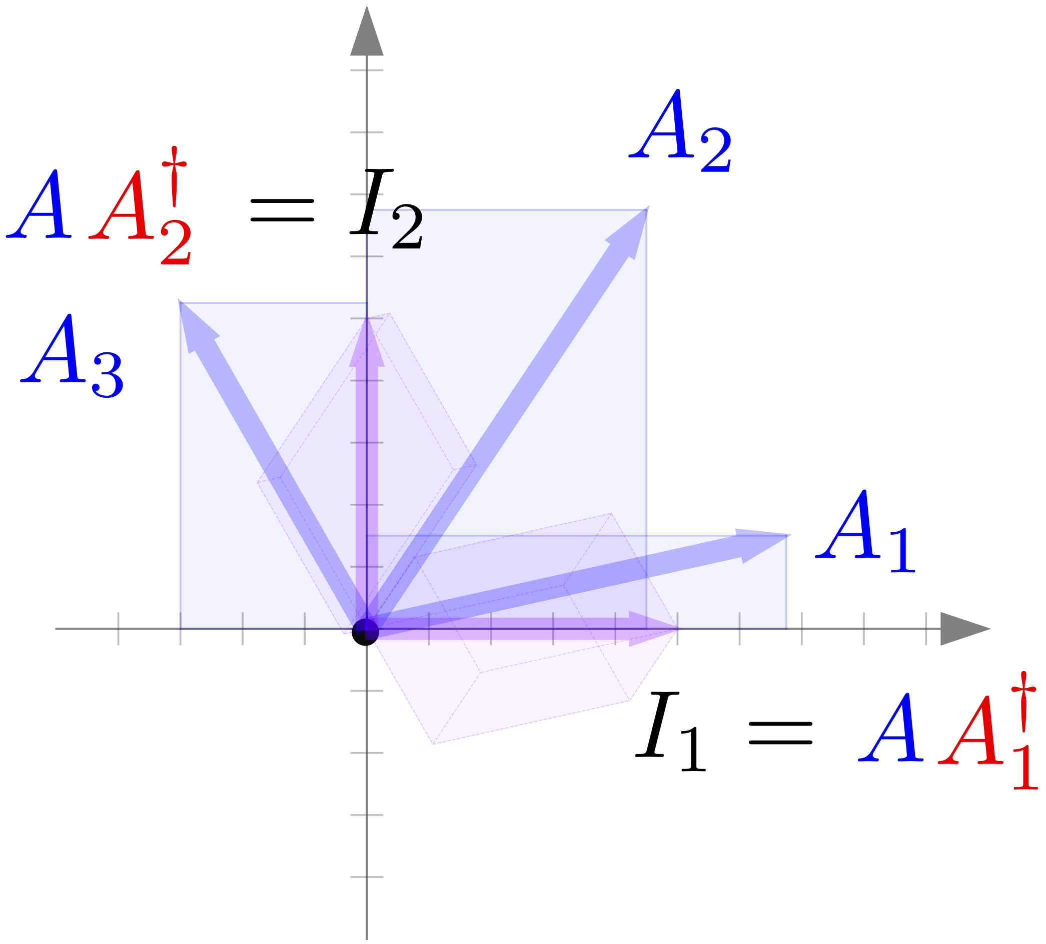


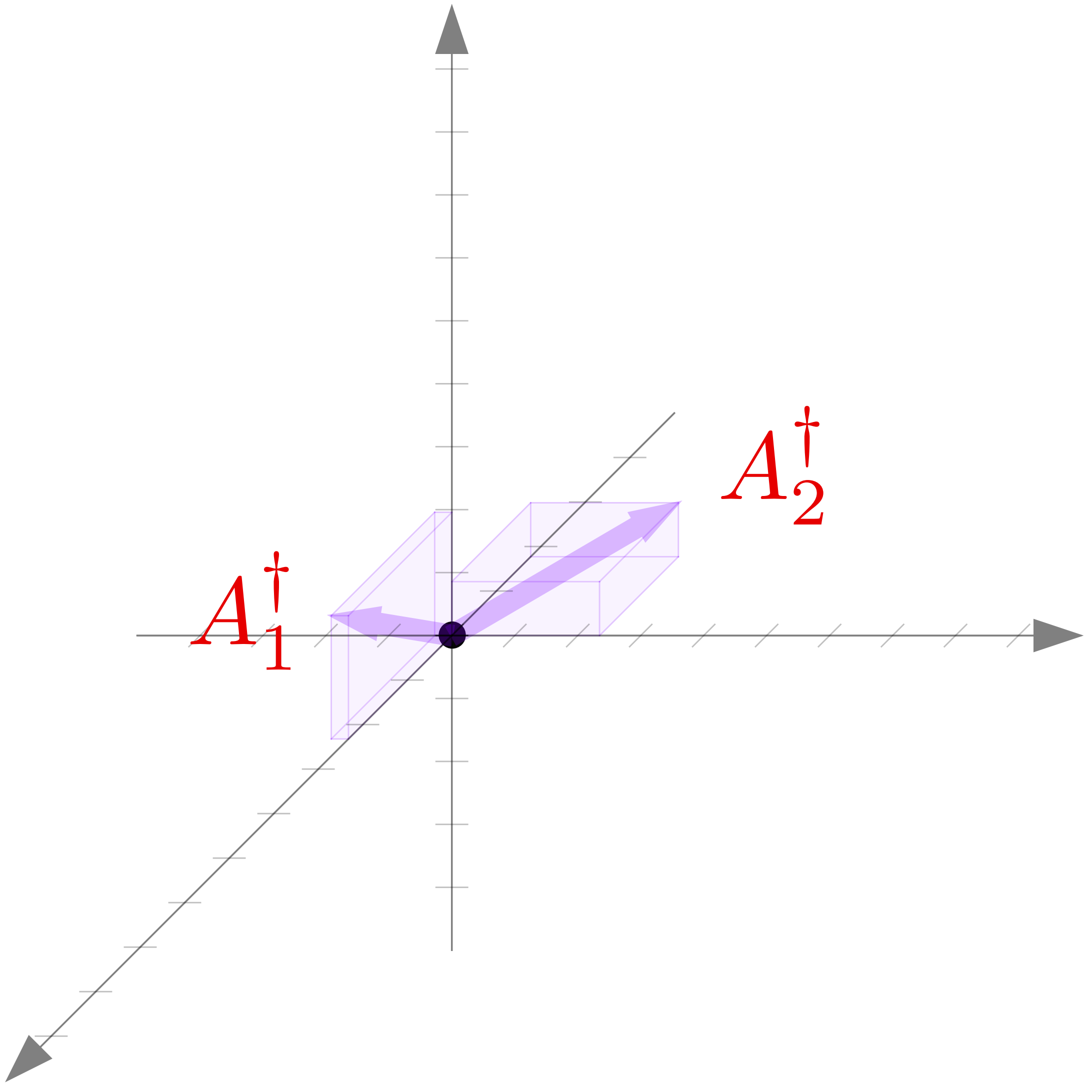


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vert & \vert & \vert \\ A_1 & A_2 & A_3 \\ \vert & \vert & \vert \end{bmatrix} \begin{bmatrix} \vert & \vert & \vert \\ A_1^{-1} & A_2^{-1} & A_3^{-1} \\ \vert & \vert & \vert \end{bmatrix}$$

Right-Inverse of A

"output only along ea. axis"





$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix}\begin{bmatrix} | & | \\ A_1^\dagger & A_2^\dagger \\ | & | \end{bmatrix}$$