Controllability & Observability

Linear System Theory

Major sources:

Winter 2022 - Dan Calderone

DLTI System - Reachability

LTI Discrete Update Eqn

$A_\Delta \in \mathbb{R}^{n \times n}$ $\quad x \in \mathbb{R}^n$

$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$

Discrete Time Matrices

$$A_\Delta = e^{A \Delta t} \qquad B_\Delta = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \; d\tau$$

assuming $u[k]$ constant over Δt

Solutions

$$x[k] = A_\Delta^k x_0 + \sum_{k'=0}^{k-1} A_\Delta^{k-1-k'} B_\Delta u[k']$$

$$= A_\Delta^k x_0 + \underbrace{\begin{bmatrix} A_\Delta^{k-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}}_{U}$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of G

Reaching a particular state: x_{des}

…solve $\quad x_{\text{des}} - A_\Delta^k x_0 = GU \quad$ for U

Minimum norm solution:

$$U^* = G^T (GG^T)^{-1} \left(x_{\text{des}} - A_\Delta^k x_0 \right)$$

$$= G^T W^{-1} \left(x_{\text{des}} - A_\Delta^k x_0 \right)$$

DT Controllability Grammian (FH): $W = GG^T$

$$W = \sum_{k'=0}^{k-1} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'\,T}$$

$$= \begin{bmatrix} A_\Delta^{k-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} B_\Delta^T A_\Delta^{k-1\,T} \\ \vdots \\ B_\Delta^T A_\Delta^T \\ B_\Delta^T \end{bmatrix}$$

$$= GG^T$$

if G is fat, then W is invertible, if and only if G has full row rank

DLTI System - Reachability

LTI Discrete Update Eqn

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$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$

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assuming $u[k]$ constant over Δt

Solutions

$$x[k] = A_\Delta^k x_0 + \sum_{k'=0}^{k-1} A_\Delta^{k-1-k'} B_\Delta u[k']$$

$$= A_\Delta^k x_0 + \underbrace{\begin{bmatrix} A_\Delta^{k-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}}_{U}$$

$$A_\Delta^k = \left(e^{A\Delta t}\right)^k = e^{Ak\Delta t} = e^{At} \qquad t = k\Delta t$$

Reachability/Controllability

...where can you drive the system to?

reachable space = range of G

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$$= G^T W^{-1} \left(x_{\text{des}} - A_\Delta^k x_0\right)$$

by Cayley-Hamilton

$$\mathcal{R}(G) = \mathcal{R}\left(\begin{bmatrix} A_\Delta^{n-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}\right)$$

...since $\quad A_\Delta^{k'} = \beta_{n-1} A_\Delta^{n-1} + \cdots + \beta_1 A_\Delta^1 + \beta_0 I$

for $\quad k' > n-1$

CLTI System - Reachability

LTI Continuous ODE $A = \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau$$

$$= e^{A(t-t_0)}x_0 + \tilde{G}(u[t_0, t])$$

Operator

- infinite-dimensional input $u[t_0, t]$
- n dimensional output

$$\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)}B(\cdot)\, d\tau$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of $\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)}B(\cdot)\, d\tau$

Reaching a particular state: x_{des} at time t

…solve $x_{\text{des}} - e^{A(t-t_0)}x_0 = \tilde{G}(u)$ for u

Minimum norm solution:

…works with infinite-dimensional operators too!

…recall **in DT** $U^* = G^T W^{-1}\big(x_{\text{des}} - A_{\Delta}^{k}x_0\big)$ $W = \sum_{k'=0}^{k-1} A_{\Delta}^{k'}B_{\Delta}B_{\Delta}^{T}A_{\Delta}^{k'T}$

in CT **CT Controllability Grammian (FH):**

$$\tilde{W} = \int_{t_0}^{t} e^{A(t-\tau)}BB^Te^{A^T(t-\tau)}\, d\tau \in \mathbb{R}^{n\times n}$$

Solution:

$$u^*(\tau) = B^Te^{A^T(t-\tau)}\tilde{W}^{-1}\big(x_{\text{des}} - e^{A(t-t_0)}x_0\big)$$

DLTI System - Observability

LTI Discrete Update Eqn

$A_\Delta \in \mathbb{R}^{n\times n}$ $\quad x \in \mathbb{R}^n$

$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$
$$y[k] = Cx[k] + Du[k]$$

Discrete Time Matrices

$$A_\Delta = e^{A\Delta t} \qquad B_\Delta = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \; d\tau$$

assuming $u[k]$ constant over Δt

Solutions

$$y[k] = Cx[k] + Du[k] = CA_\Delta^k x_0 + \sum_{k'=0}^{k-1} CA_\Delta^{k-1-k'} B_\Delta u[k'] + Du[k]$$

Observations over time:

no controls, with noise $v_{k'} \sim \mathcal{N}(0, \sigma I)$

normal distribution

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}}_{H} x_0 + \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

Observability

…can you estimate the initial state from measurements

unobservable subspace = null space of H

Least Squares Solution

$$x_0 = (H^T H)^{-1} H^T Y$$
$$= X^{-1} H^T Y$$

DT Observability Grammian (FH):

$$X = \sum_{k'=0}^{k} {A_\Delta^{k'}}^T C^T C A_\Delta^{k'}$$
$$= \begin{bmatrix} C^T & A_\Delta^T C^T & {A_\Delta^2}^T C^T & \cdots & {A_\Delta^k}^T C^T \end{bmatrix} \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}$$
$$= H^T H$$

if H is tall, then X is invertible
if and only if H has full col rank

DLTI System - Observability

LTI Discrete Update Eqn $A_\Delta \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$
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$$y[k] = Cx[k] + Du[k] = CA_\Delta^k x_0 + \sum_{k'=0}^{k-1} CA_\Delta^{k-1-k'} B_\Delta u[k'] + Du[k]$$

Observations over time: with controls, with noise $v_{k'} \sim \mathcal{N}(0, \sigma I)$

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}}_{H} x_0 + \underbrace{\begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB_\Delta & D & 0 & \cdots & 0 \\ CA_\Delta B_\Delta & CB_\Delta & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA_\Delta^{k-1}B_\Delta & CA_\Delta^{k-2}B_\Delta & CA_\Delta^{k-3}B_\Delta & \cdots & D \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ \vdots \\ u[k] \end{bmatrix}}_{U} + \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

Observability

...can you estimate the initial state from measurements

unobservable subspace = null space of H

Least Squares Solution

$$x_0 = (H^T H)^{-1} H^T \left(Y - \mathbf{G}U\right)$$
$$= X^{-1} H^T \left(Y - \mathbf{G}U\right)$$

DT Observability Grammian (FH):

$$X = \sum_{k'=0}^{k} A_\Delta^{k'\,T} C^T C A_\Delta^{k'}$$
$$= \begin{bmatrix} C^T & A_\Delta^T C^T & A_\Delta^{2\,T} C^T & \cdots & A_\Delta^{k\,T} C^T \end{bmatrix} \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}$$
$$= H^T H$$

if H is tall, then X is invertible
if and only if H has full col rank

DT Controllability - Cayley Hamilton

Discrete Time $A_\Delta \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

Range of G $=$ Range of M

where $G = \left[\ A_\Delta^k B_\Delta \ \cdots \ A_\Delta^n B_\Delta \ A_\Delta^{n-1} B_\Delta \ \cdots \ A_\Delta B_\Delta \ B_\Delta\ \right]$ $M = \left[\ A_\Delta^{n-1} B_\Delta \ \cdots \ A_\Delta B_\Delta \ B_\Delta \right]$

Proof:

…by Cayley Hamilton $A_\Delta^k = \beta_{(n-1)k} A_\Delta^{n-1} + \cdots + \beta_{1k} A_\Delta + \beta_{0k} I$

$$\left[\ A_\Delta^k B_\Delta \ \cdots \ A_\Delta^n B_\Delta \ A_\Delta^{n-1} B_\Delta \ \cdots \ A_\Delta B_\Delta \ B_\Delta\ \right] = \left[\ A_\Delta^{n-1} B_\Delta \ \cdots \ A_\Delta B_\Delta \ B_\Delta \right] \begin{bmatrix} \beta_{(n-1)k} I & \cdots & \beta_{(n-1)n} I & I & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \beta_{1k} I & \cdots & \beta_{1n} I & 0 & \cdots & I & 0 \\ \beta_{0k} I & \cdots & \beta_{0n} I & 0 & \cdots & 0 & I \end{bmatrix}$$

Every column of G is a linear combination of columns of M

The columns of G include the columns of M.

… range of G $=$ range of M

CT Controllability - Cayley Hamilton

Continuous Time $A \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^n$

Range of $\tilde{G}(\cdot) =$ Range of M

where $\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)} B(\cdot)\; d\tau$ $\quad M = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix}$

Proof:

…by Cayley Hamilton $e^{A(t-\tau)} = \beta_{n-1}(\tau)A^{n-1} + \cdots + \beta_1(\tau)A + \beta_0(\tau)I$

if $x \in$ $\text{range}\Big(\tilde{G}(\,\cdot\,)\Big)$ $\exists\, u(\tau)$ s.t. $x = \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\; d\tau$

$$= \int_{t_0}^{t} \Big(\beta_{n-1}(\tau)A^{n-1} + \cdots + \beta_1(\tau)A + \beta_0(\tau)I\Big) Bu(\tau)\; d\tau$$

$$= \left(\int_{t_0}^{t} \beta_{n-1}(\tau)u(\tau)\; d\tau\right) A^{n-1}B + \cdots + \left(\int_{t_0}^{t} \beta_1(\tau)u(\tau)\; d\tau\right) AB + \left(\int_{t_0}^{t} \beta_0(\tau)u(\tau)\; d\tau\right) B$$

x is a linear combination of the columns of M … range of $\tilde{G}$ is a subset of range of M

DT Observability - Cayley Hamilton

Discrete Time $A_\Delta \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

Nullspace H = Nullspace of M

where

$$H = \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^{n-1} \\ CA_\Delta^n \\ \vdots \\ CA_\Delta^k \end{bmatrix} \qquad M = \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^{n-1} \end{bmatrix}$$

Proof:

…by Cayley Hamilton $A_\Delta^k = \beta_{k(n-1)} A_\Delta^{n-1} + \cdots + \beta_{k1} A_\Delta + \beta_{k0} I$

$$\begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^{n-1} \\ CA_\Delta^n \\ \vdots \\ CA_\Delta^k \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \\ \beta_{n0} I & \beta_{n1} I & \cdots & \beta_{n(n-1)} I \\ \vdots & \vdots & & \vdots \\ \beta_{k0} I & \beta_{k1} I & \cdots & \beta_{k(n-1)} I \end{bmatrix} \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^{n-1} \end{bmatrix}$$

If x in nullspace of M, then it is in the nullspace of H.

The identity block in the top of the coefficient matrix shows that the columns are linearly independent. Thus it has a trivial nullspace. Thus if x is in the null space of H then it has to be in the null space of M.

DT Controllability Tests

The following statements are equivalent

1. (A_Δ, B_Δ) **is controllable**

2. **There is no left eigenvector** W_i^T **s.t.** $W_i^T B_\Delta = 0$

3. **Controllability Matrix Test**

$$M = \begin{bmatrix} A_\Delta^{n-1} B_\Delta & \cdots & A_\Delta^2 B_\Delta & A_\Delta B_\Delta & B_\Delta \end{bmatrix}$$

full row rank (rank n)

4. **PBH Test**

$$\begin{bmatrix} A_\Delta - \lambda I & B \end{bmatrix}$$

has full row rank for every $\lambda \in \mathrm{eig}(A_\Delta)$

5. **The Grammian matrix W is invertible**

$$W = \sum_{k'=0}^{k} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'\,T}$$

for $k \geq n-1$

Intuition:

A system is not controllable if there is a left eigenvector orthogonal to all columns of B

Left eigenvector W_i^T $W_i^T B_\Delta = 0$

Not controllable

The corresponding right eigenvector V_i cannot be reached and is thus called an **uncontrollable mode.**

because eigenvectors are A-invariant (they don't change directions under the action of A)… if an eigenvector is not affected by B, then that eigen-mode will just evolve on it's own forever.

DT Controllability Tests

The following statements are equivalent

1. (A_Δ, B_Δ) **is controllable**

2. There is no left eigenvector W_i^T **s.t.** $W_i^T B_\Delta = 0$

3. Controllability Matrix Test

$$M = \left[\begin{array}{ccccc} A_\Delta^{n-1} B_\Delta & \cdots & A_\Delta^2 B_\Delta & A_\Delta B_\Delta & B_\Delta \end{array} \right]$$

full row rank (rank n)

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$$\left[\begin{array}{cc} A_\Delta - \lambda I & B \end{array} \right]$$

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5. The Grammian matrix W is invertible

$$W = \sum_{k'=0}^{k} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'^T}$$

for $k \geq n-1$

2 & 3

for diagonalizable $A_\Delta = V D_\Delta V^{-1}$

not 2 implies not 3

$$W_i^T M = \left[\begin{array}{cccc} W_i^T A_\Delta^{n-1} B_\Delta & \cdots & W_i^T A_\Delta B_\Delta & W_i^T B_\Delta \end{array} \right]$$

$$= \left[\begin{array}{cccc} \lambda_{i\Delta}^{n-1} W_i^T B_\Delta & \cdots & \lambda_{i\Delta} W_i^T B_\Delta & W_i^T B_\Delta \end{array} \right]$$

$$= \left[\begin{array}{cccc} \lambda_{i\Delta}^{n-1} 0 & \cdots & \lambda_{i\Delta} 0 & 0 \end{array} \right]$$

CT Controllability Tests

The following statements are equivalent

1. (A,B) is controllable

2. There is no left eigenvector W_i^T **s.t.** $W_i^T B = 0$

3. Controllability Matrix Test

$$M = \left[\begin{array}{cccc} A^{n-1}B & \cdots A^2B & AB & B \end{array} \right]$$

full row rank (rank n)

4. PBH Test

$$\left[\begin{array}{cc} A - \lambda I & B \end{array} \right]$$

has full row rank for every $\lambda \in \mathrm{eig}(A)$

5. The Grammian matrix W is invertible

$$W = \int_0^t e^{A\tau} B B^T e^{A^T \tau} \, d\tau \qquad \text{for} \quad t > 0$$

Intuition:

A system is not controllable if there is a left eigenvector orthogonal to all columns of B

Left eigenvector	W_i^T	$W_i^T B = 0$
		Not controllable

The corresponding right eigenvector V_i cannot be reached and is thus called an **uncontrollable mode.**

because eigenvectors are A-invariant (they don't change directions under the action of A)… if an eigenvector is not affected by B, then that eigen-mode will just evolve on it's own forever.

DT Observability Tests

The following statements are equivalent

1. (A_Δ, C) is observable

2. There is no right eigenvector V_i s.t. $CV_i = 0$

3. Observability Matrix Test

$$M = \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^{n-1} \end{bmatrix}$$

full col rank (rank n)

4. PBH Test

$$\begin{bmatrix} A_\Delta - \lambda I \\ C \end{bmatrix}$$

has full col rank for every $\lambda \in \mathrm{eig}(A_\Delta)$

5. The Grammian matrix W is invertible

$$W = \sum_{k'=0}^{k} {A_\Delta^{k'}}^T C^T C A_\Delta^{k'} \qquad \text{for} \quad k \geq n-1$$

Intuition:

A system is not observable if there is a right eigenvector orthogonal to all rows of C

Right eigenvector	V_i	$CV_i = 0$
		Not observable

Since V_i is in the null space of C it never shows up in the output y

because eigenvectors are A-invariant (they don't change directions under the action of A)… if an eigenvector is not "seen" by C, then that eigen-mode will never show in the output.

CT Observability Tests

The following statements are equivalent

1. (A,C) is controllable

2. There is no right eigenvector V_i **s.t.** $CV_i = 0$

3. Observability Matrix Test

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

full col rank (rank n)

4. PBH Test

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has full col rank for every $\lambda \in \mathrm{eig}(A)$

5. The Grammian matrix W is invertible

$$W = \int_0^t e^{A^T \tau} C^T C e^{A\tau} \, d\tau \qquad \text{for} \quad t > 0$$

Intuition:

A system is not observable if there is a right eigenvector orthogonal to all rows of C

Right eigenvector V_i $CV_i = 0$

Not observable

Since V_i is in the null space of C it never shows up in the output y

because eigenvectors are A-invariant (they don't change directions under the action of A)… if an eigenvector is not "seen" by C, then that eigen-mode will never show in the output.