# Eigenvectors & Eigenvalues:

## Linear Algebra

Major sources:

Winter 2022 - Dan Calderone

# Eigenvectors & Eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$

**Eigenvalue/Eigenvector Problem**

$A$ transforms $\mathbb{R}^n$ ***…which directions stay unchanged?*** → **Eigenvectors**

…within those directions…

***…how much do vectors get stretched*** → **Eigenvalues**

**Eigenvector Equation**

$$Ax = x\lambda$$

**Eigenvector** $x \in \mathbb{C}^n$ **Eigenvalue** $\lambda \in \mathbb{C}$

Spans of eigenvectors (& generalized eigenvectors) are called **A-invariant subspaces**

## Eigenvalues:

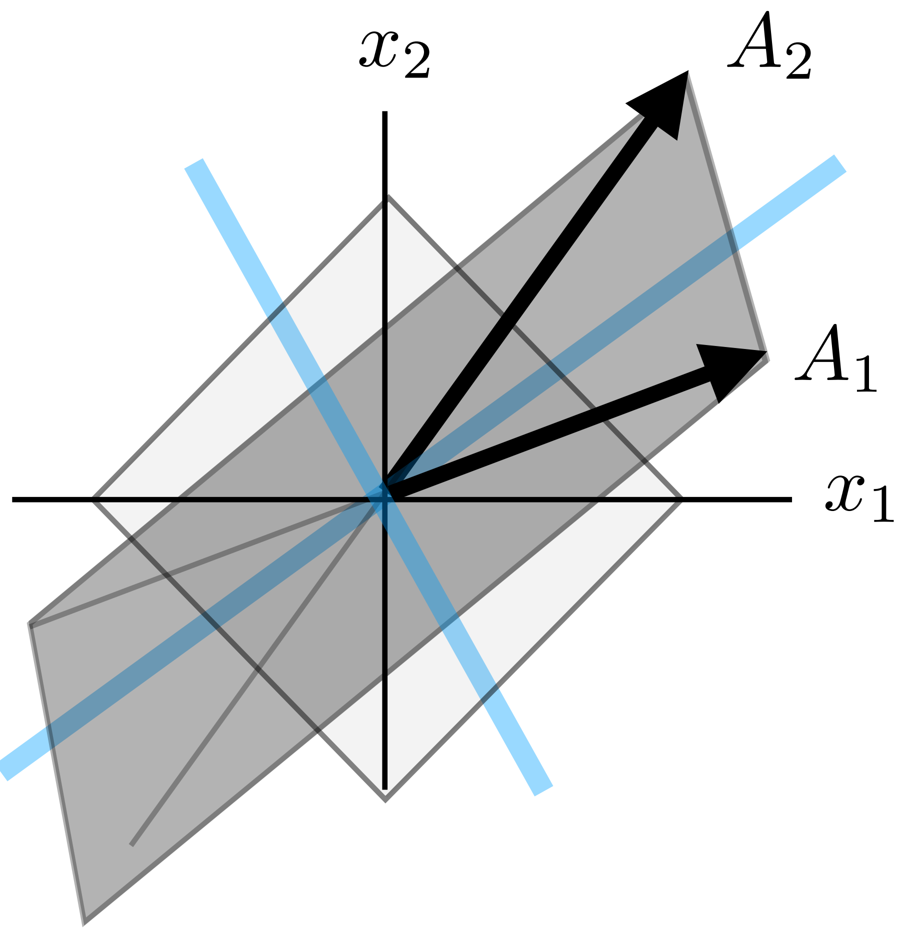
Fundamental property of matrices
Do **not** change with coordinate/similarity transformations

## Eigenvectors:

…coordinate dependent (do change with coordinate/similarity transformations)

**Picture Examples:**

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \in \mathbb{R}^{2\times 2}$$



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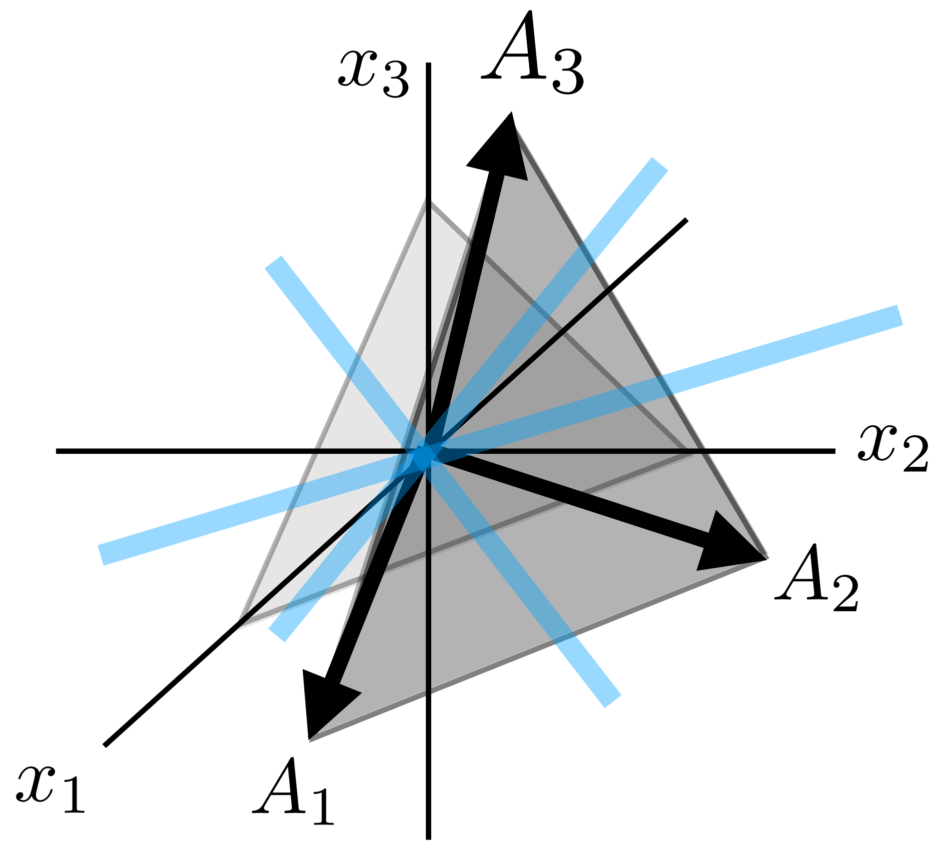

# Eigenvector/Eigenvalue equation

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For any eigenvalue $\lambda \in \mathbb{C}$

**Right Eigenvector:** $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0 \qquad v \in \mathcal{N}(A - \lambda I)$$

**Left Eigenvectors:** $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0 \qquad w^* \in \mathcal{N}^L(A - \lambda I) = 0$$

For any eigenvalue, right and left eigenvectors come in pairs since $A - \lambda I$ drops row and column rank at the same time

Eigenvectors exist only for values of s where $A - sI$ drops rank…

…how to characterize…. ➡ $sI - A$ drops rank only when $\det(sI - A) = 0$

**Characteristic Polynomial**

$\mathrm{char}_A(s) = \det(sI - A)$ **n-th order polynomial** ➡ **n roots**

**Roots are eigenvalues:** $\lambda$ solution to $\mathrm{char}_A(s) = 0$

Fundamental
Theorem of Algebra

**Picture Examples:**

$\mathbb{R}^{2 \times 2}$

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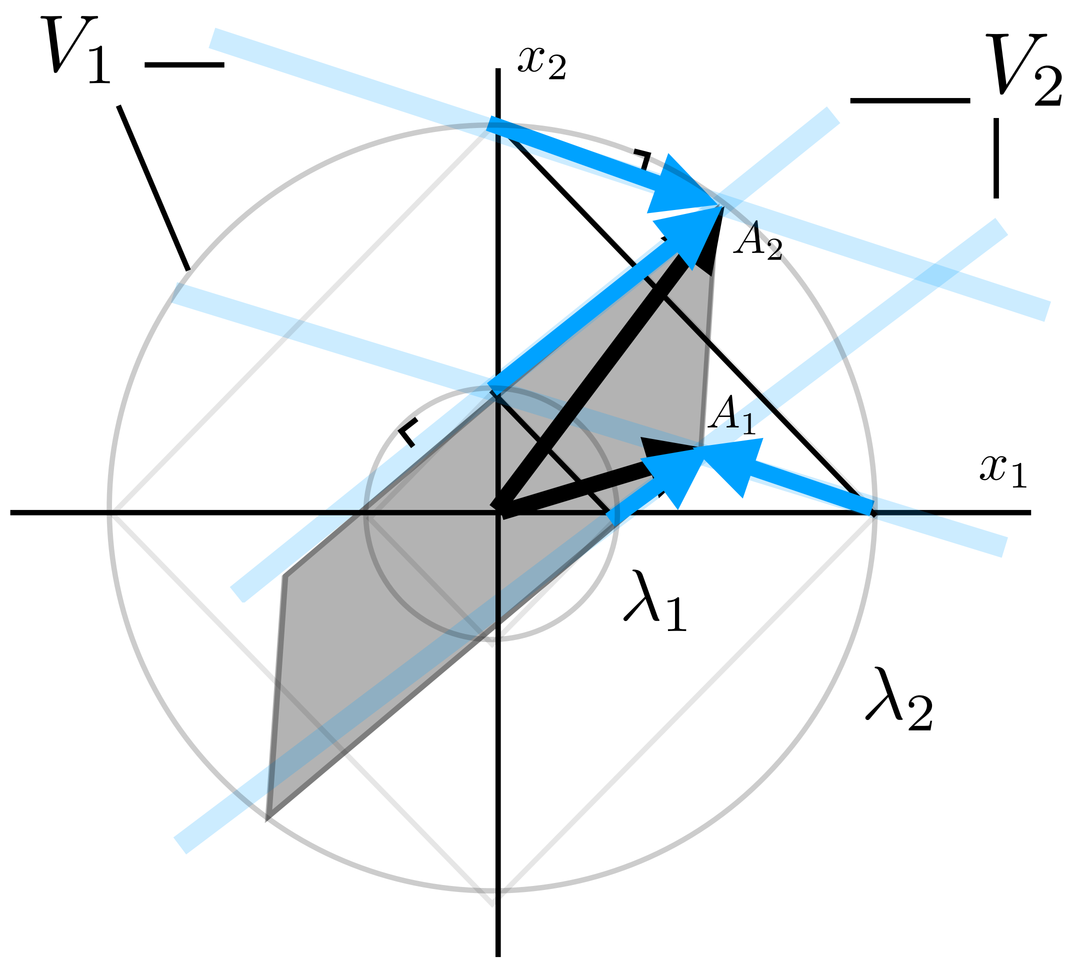

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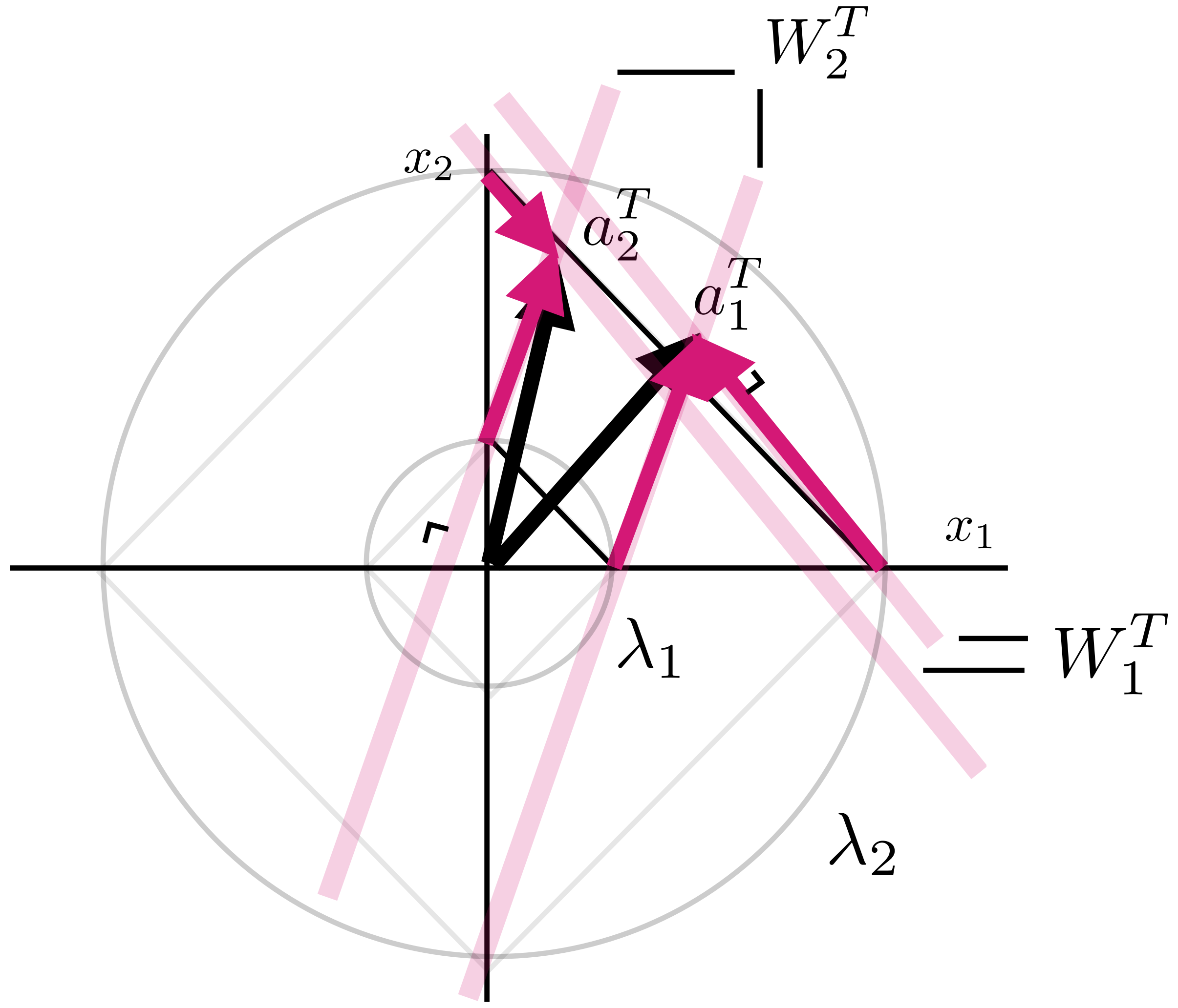

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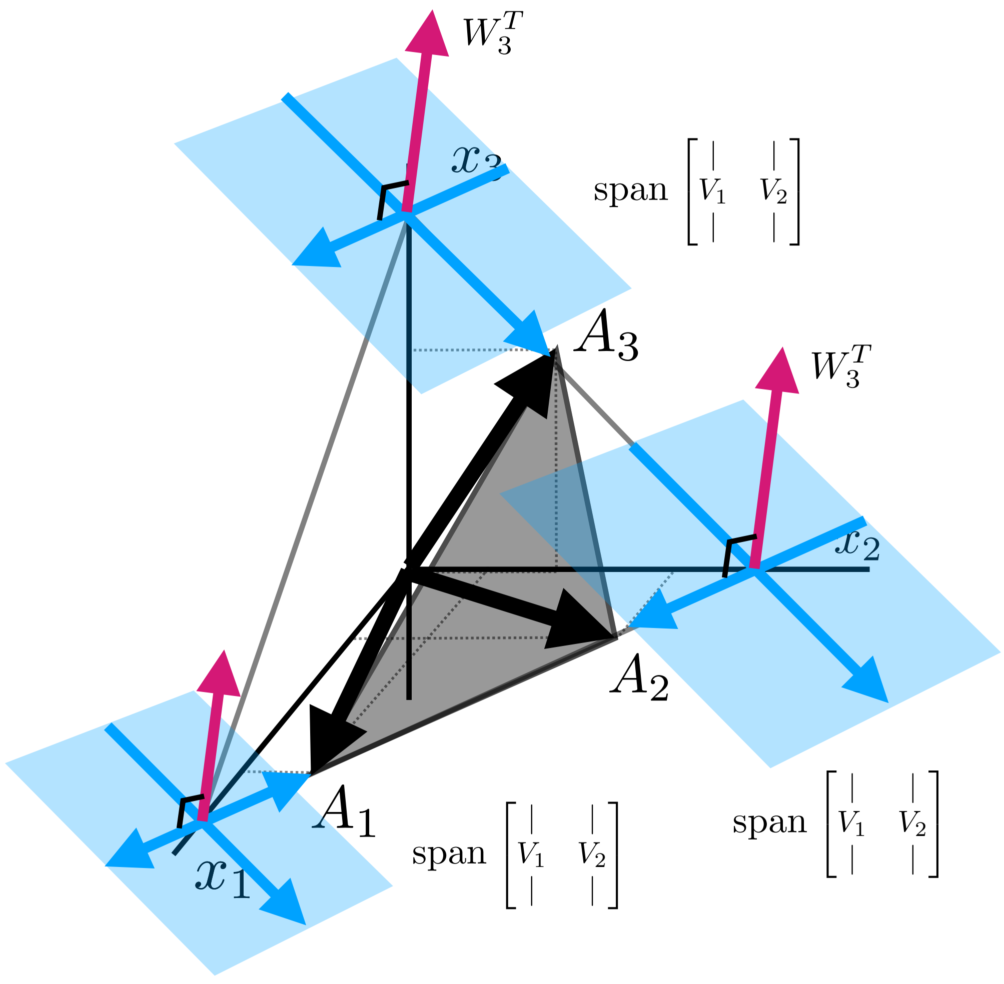

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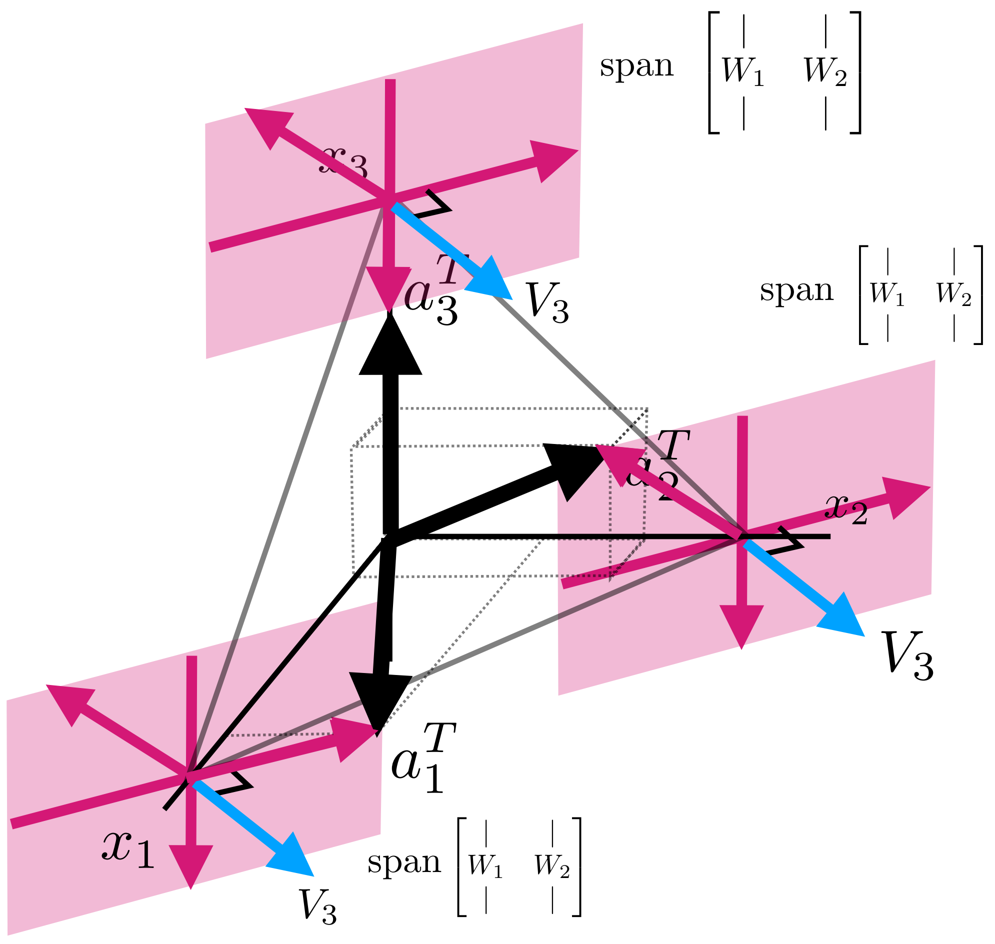

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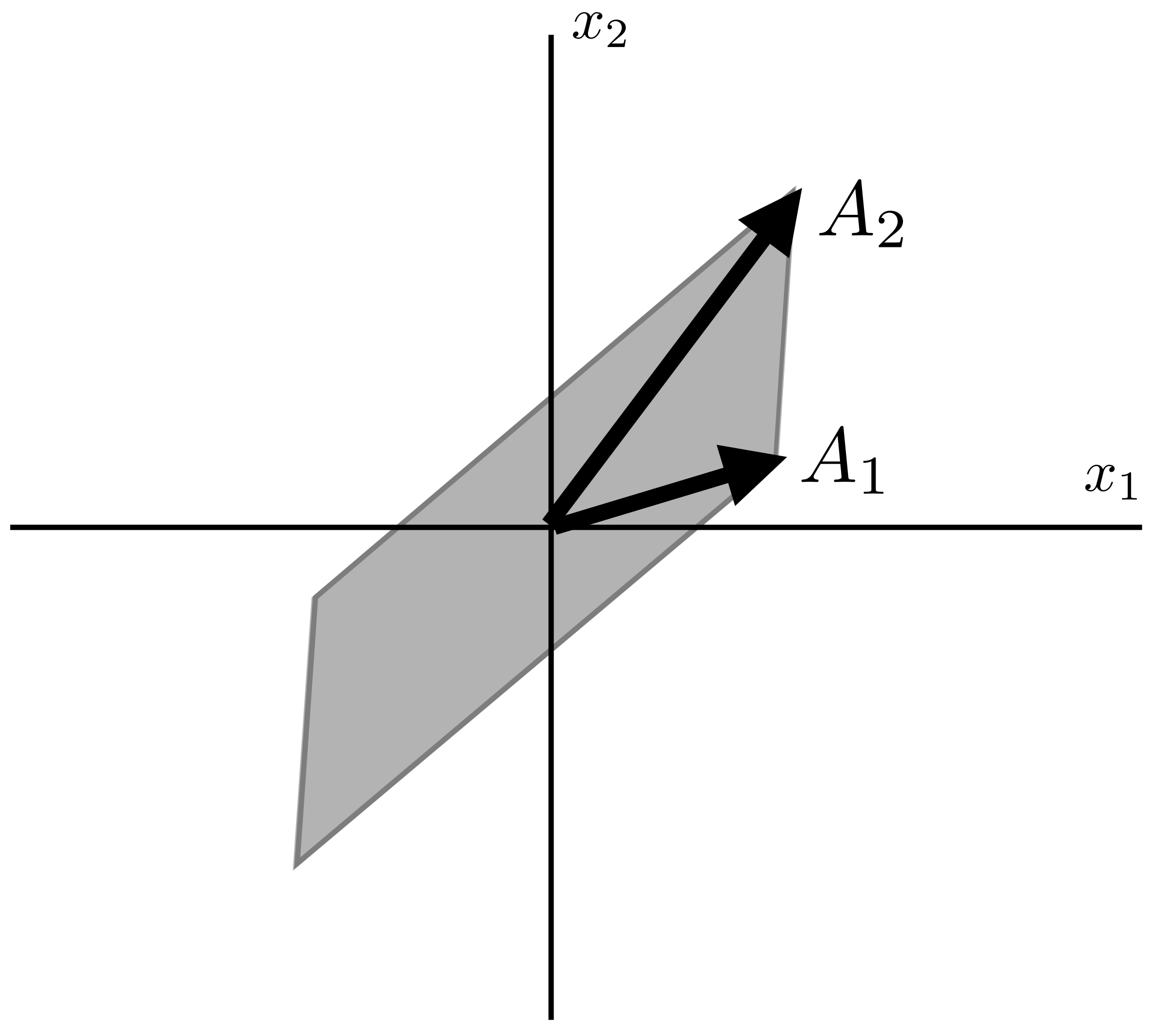

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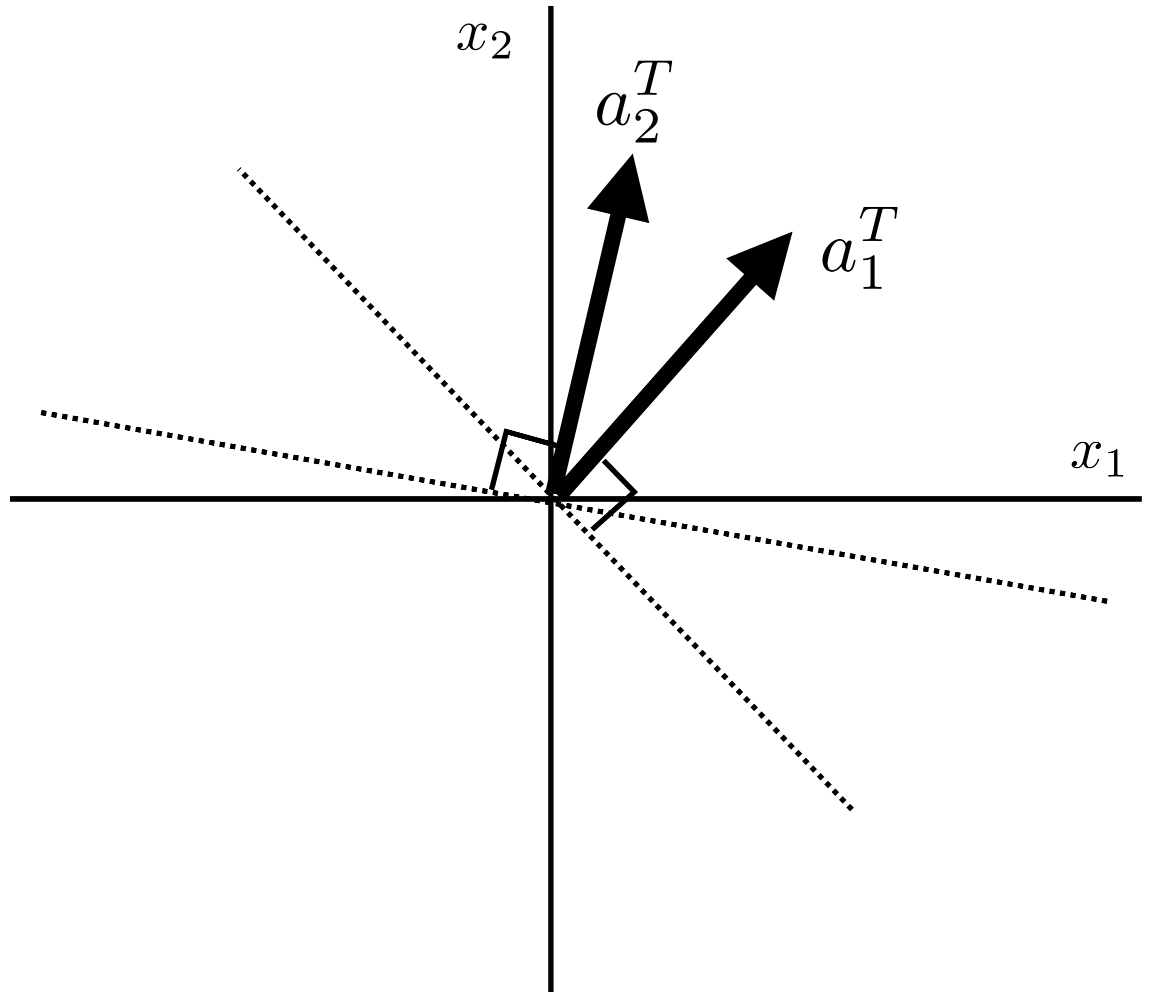

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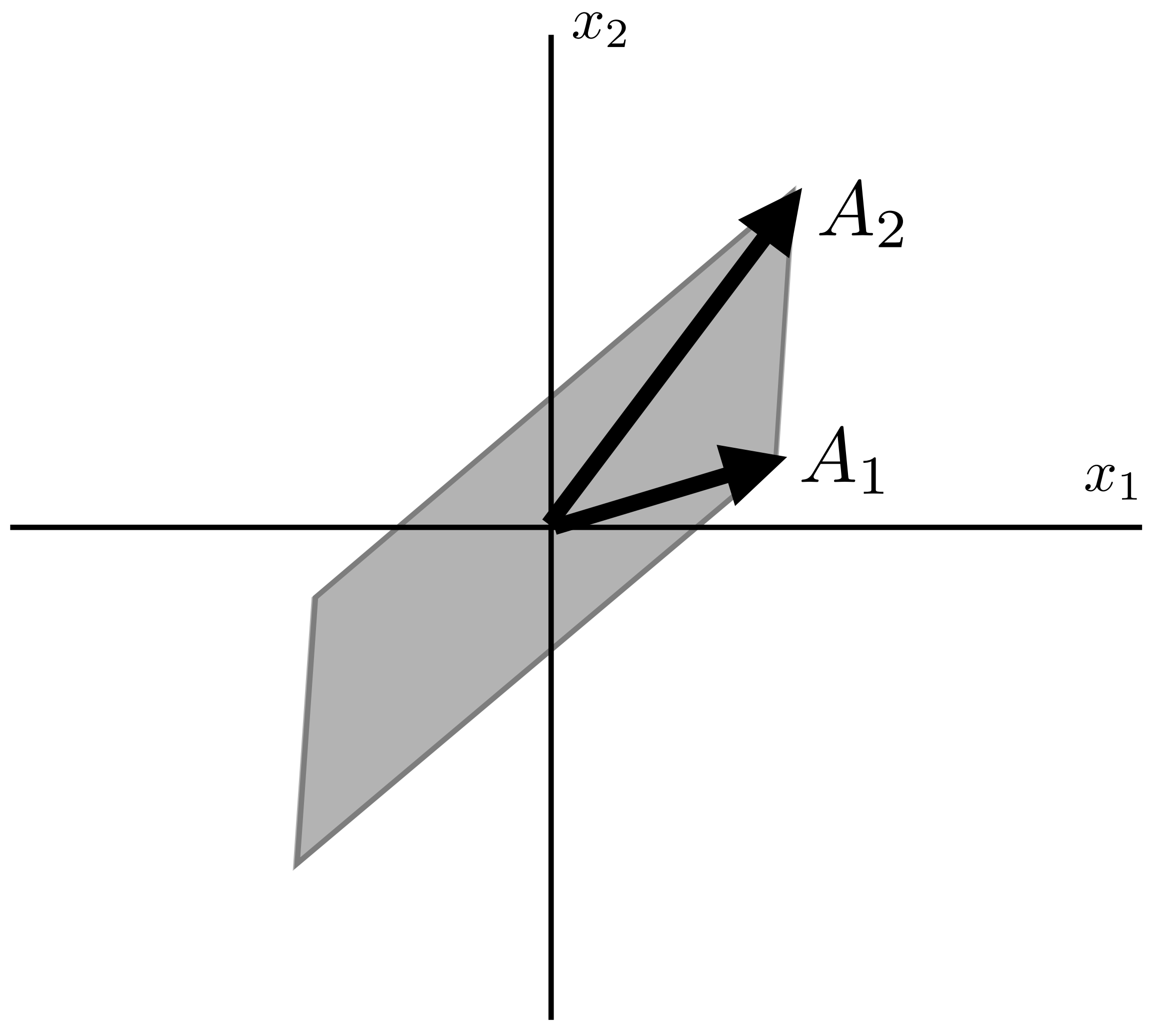

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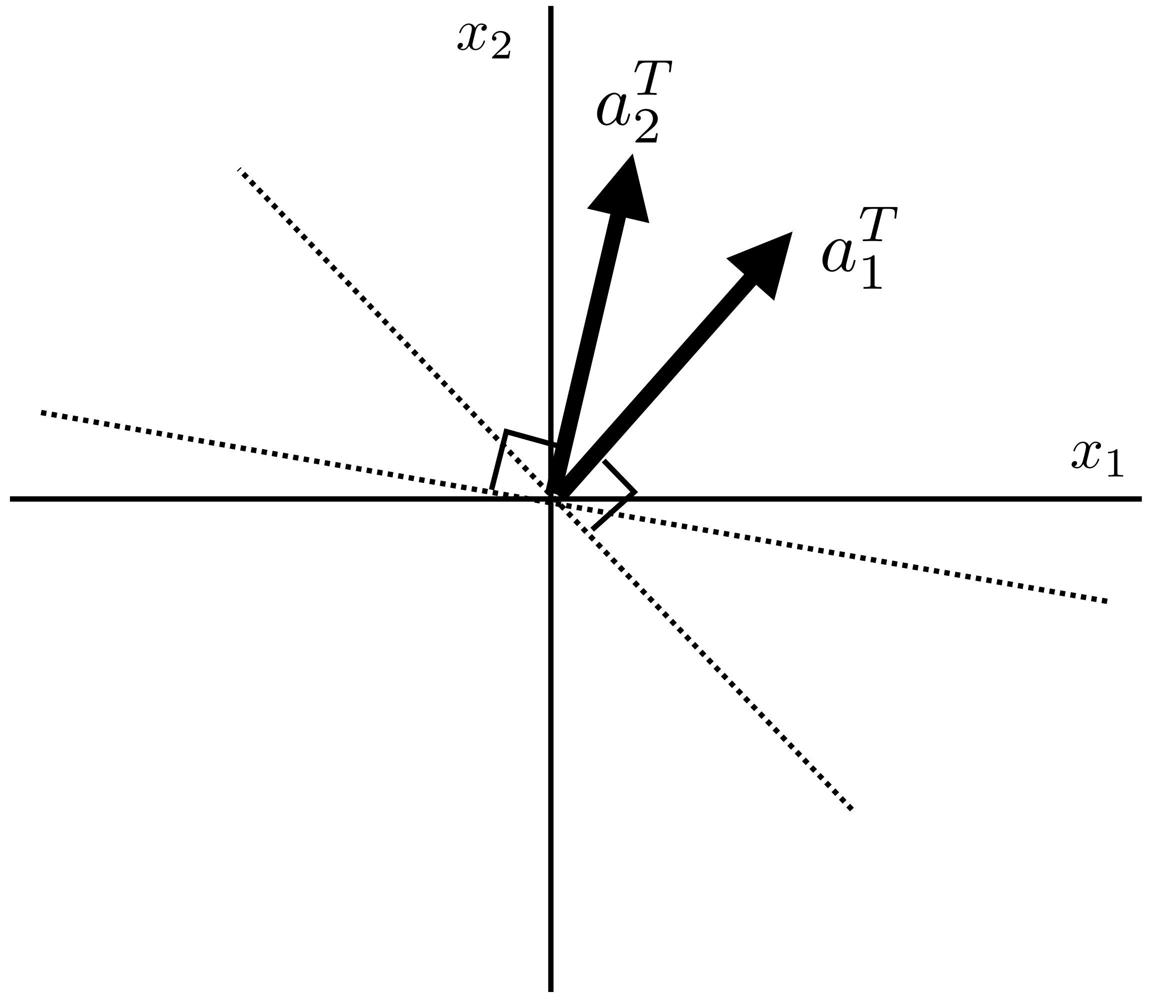

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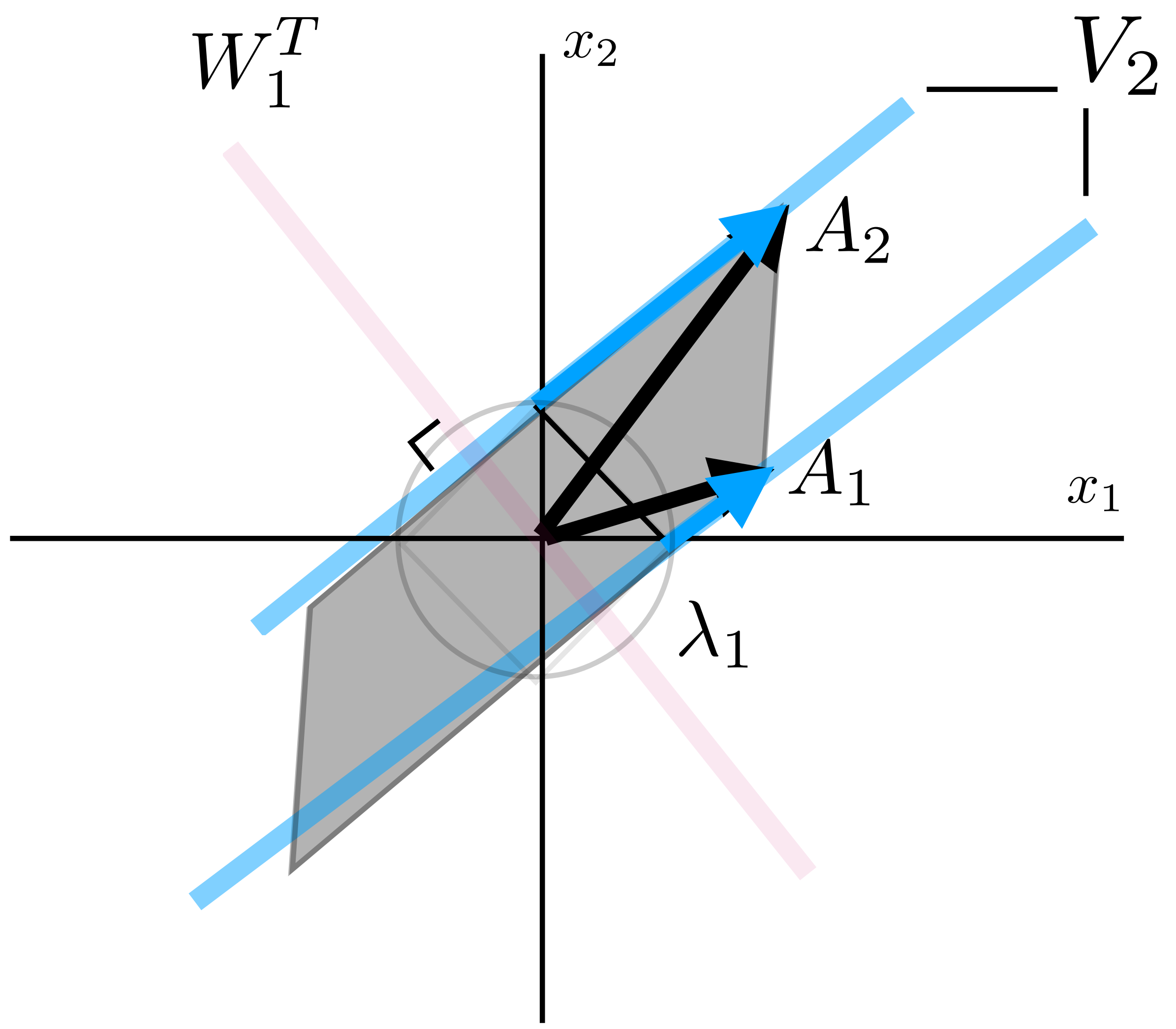

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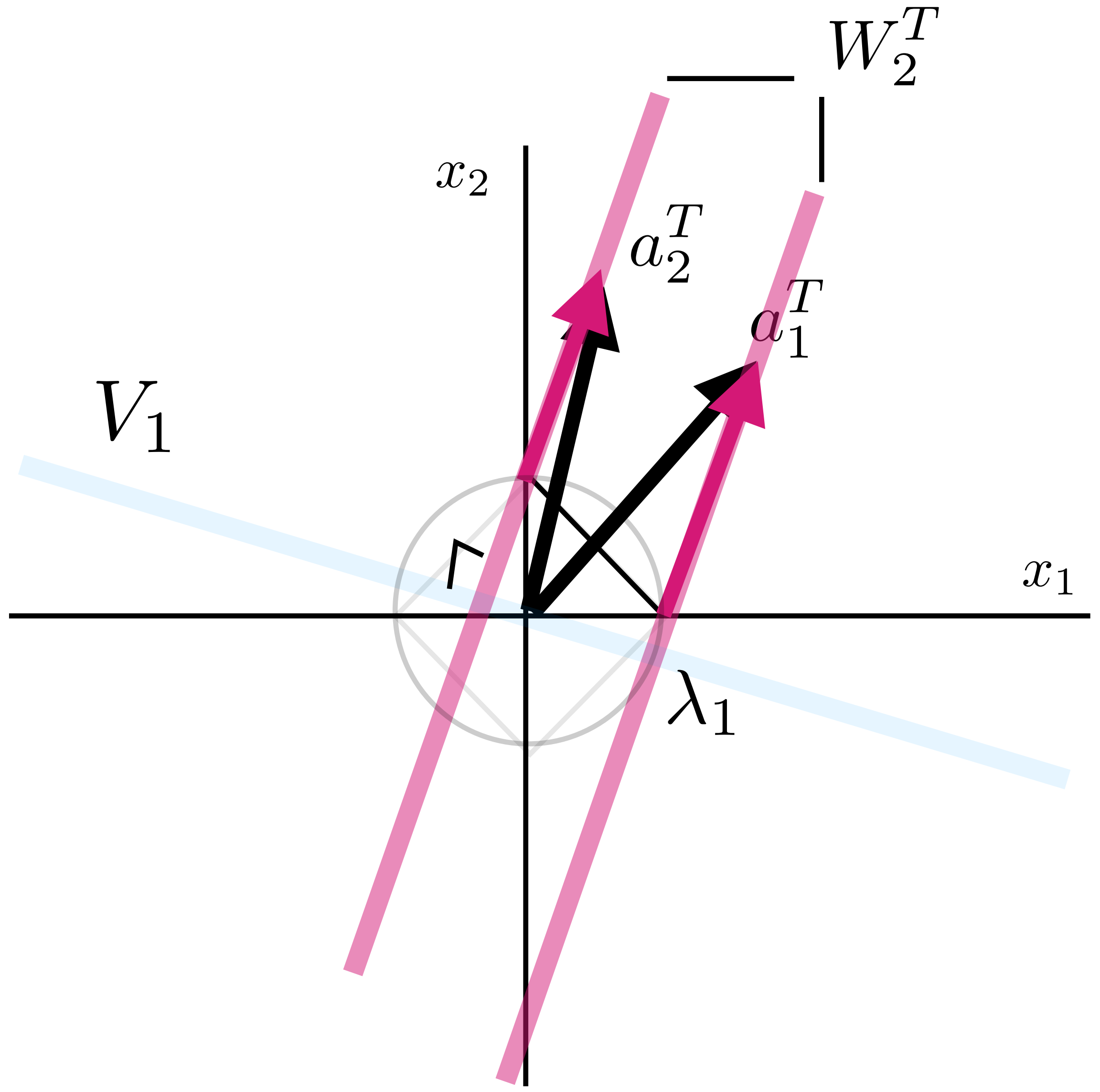

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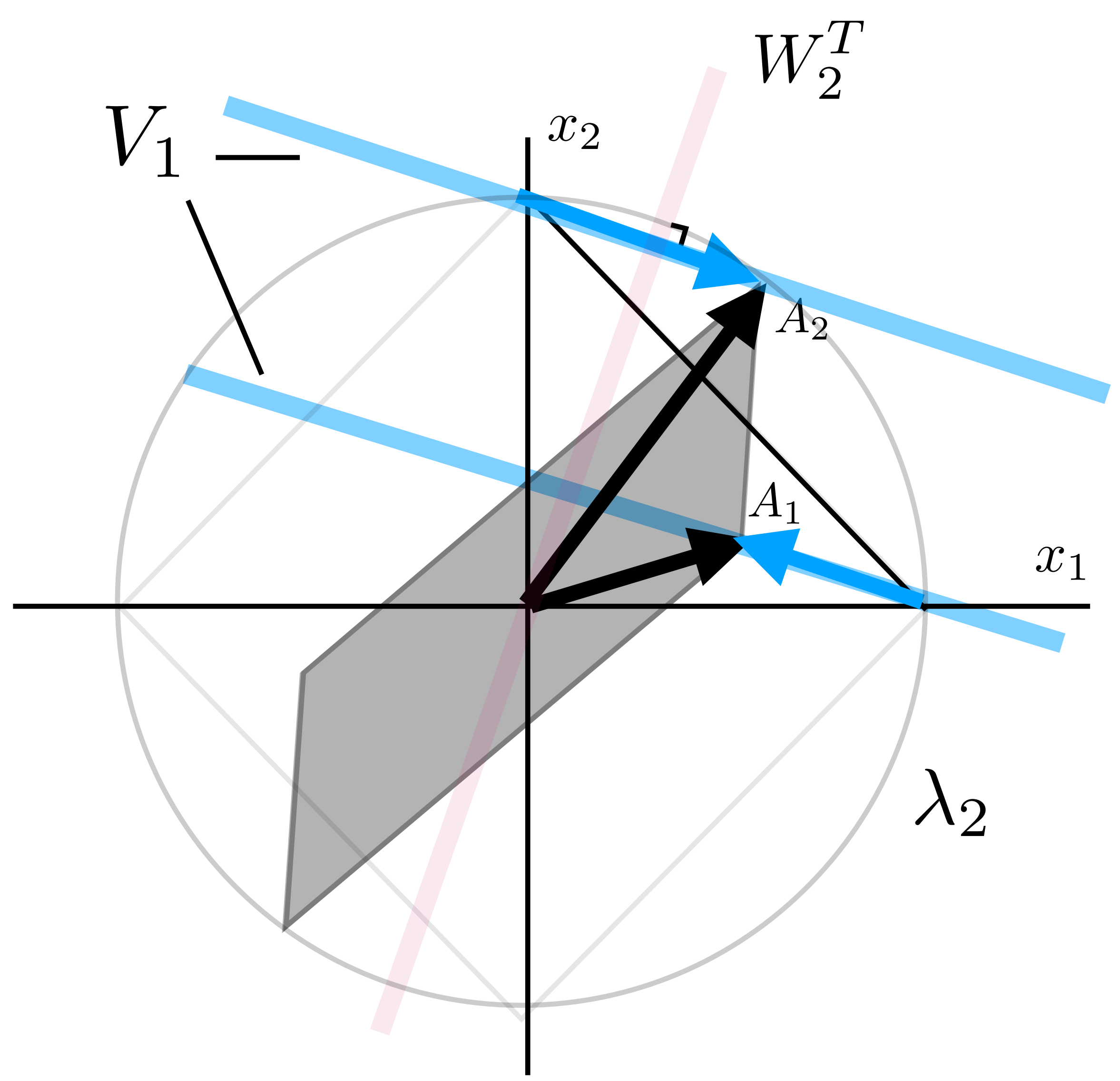

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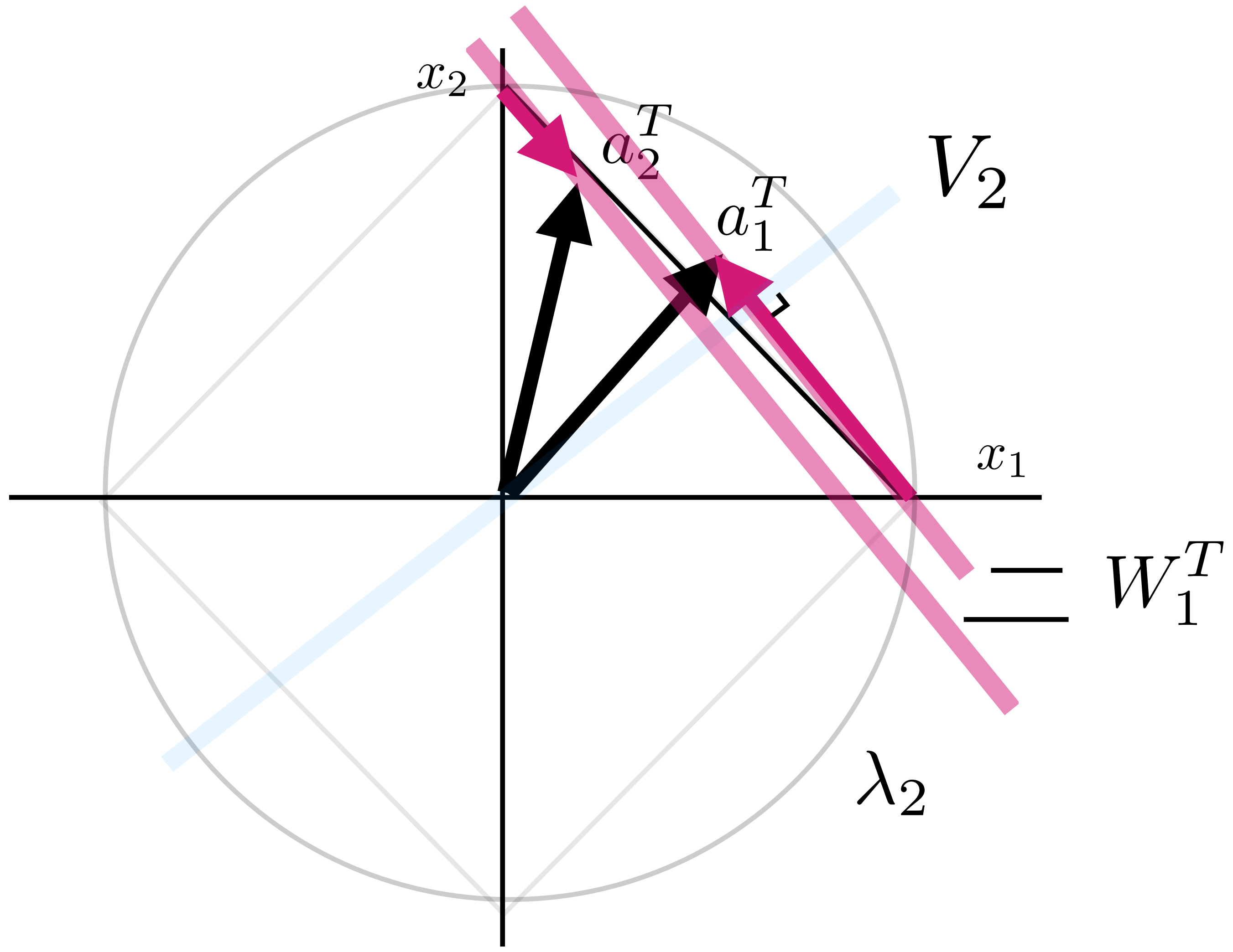

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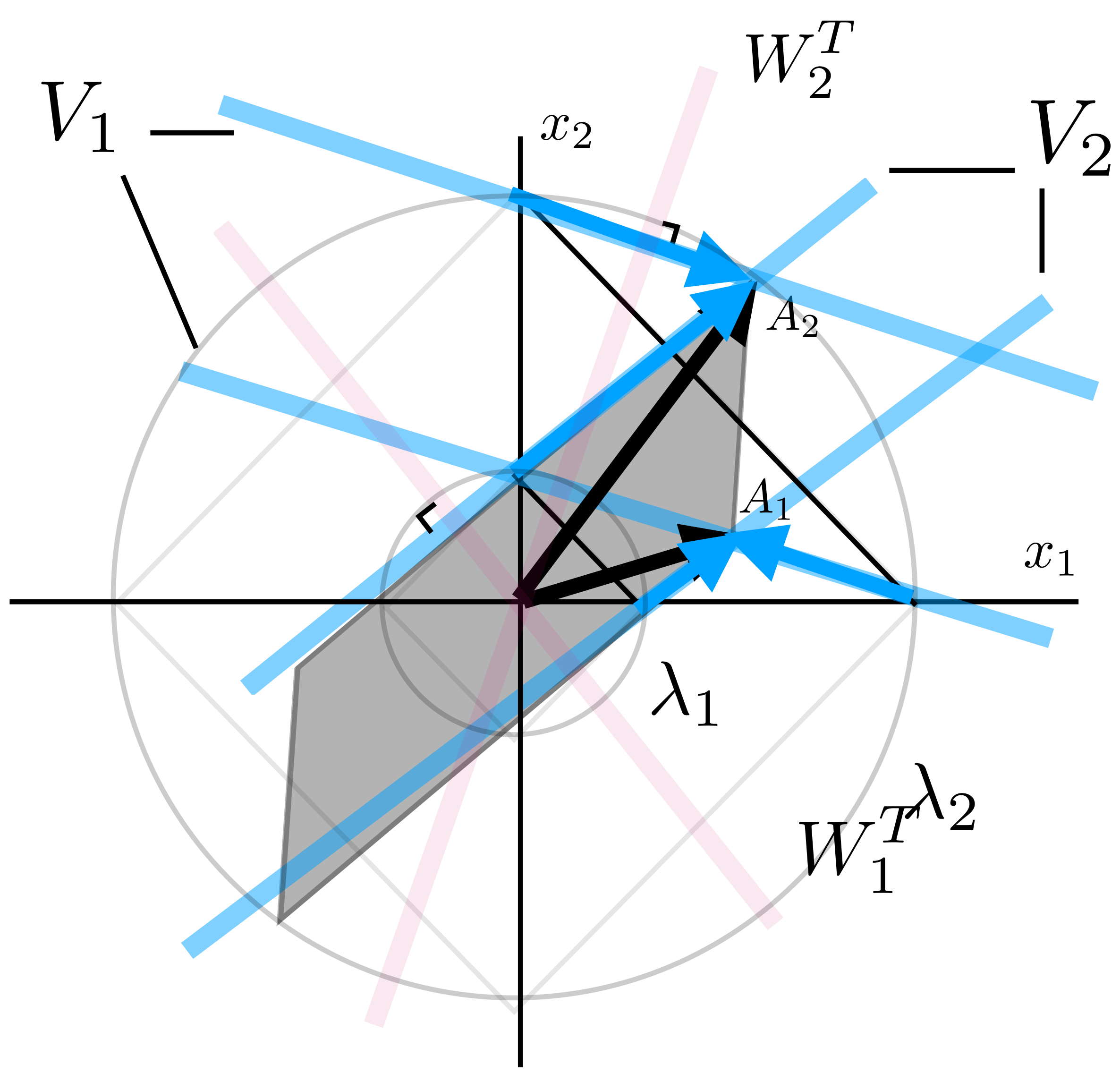

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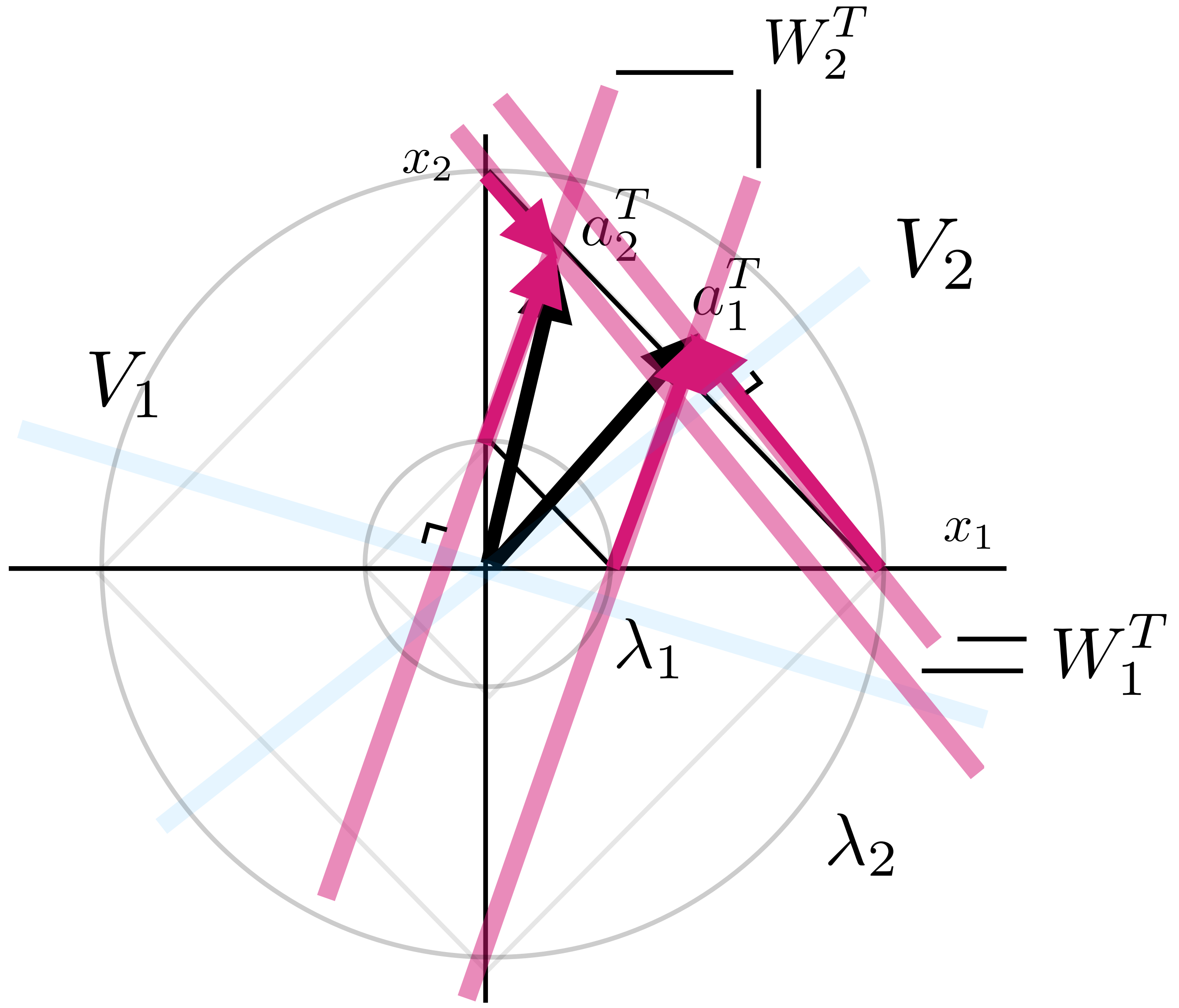

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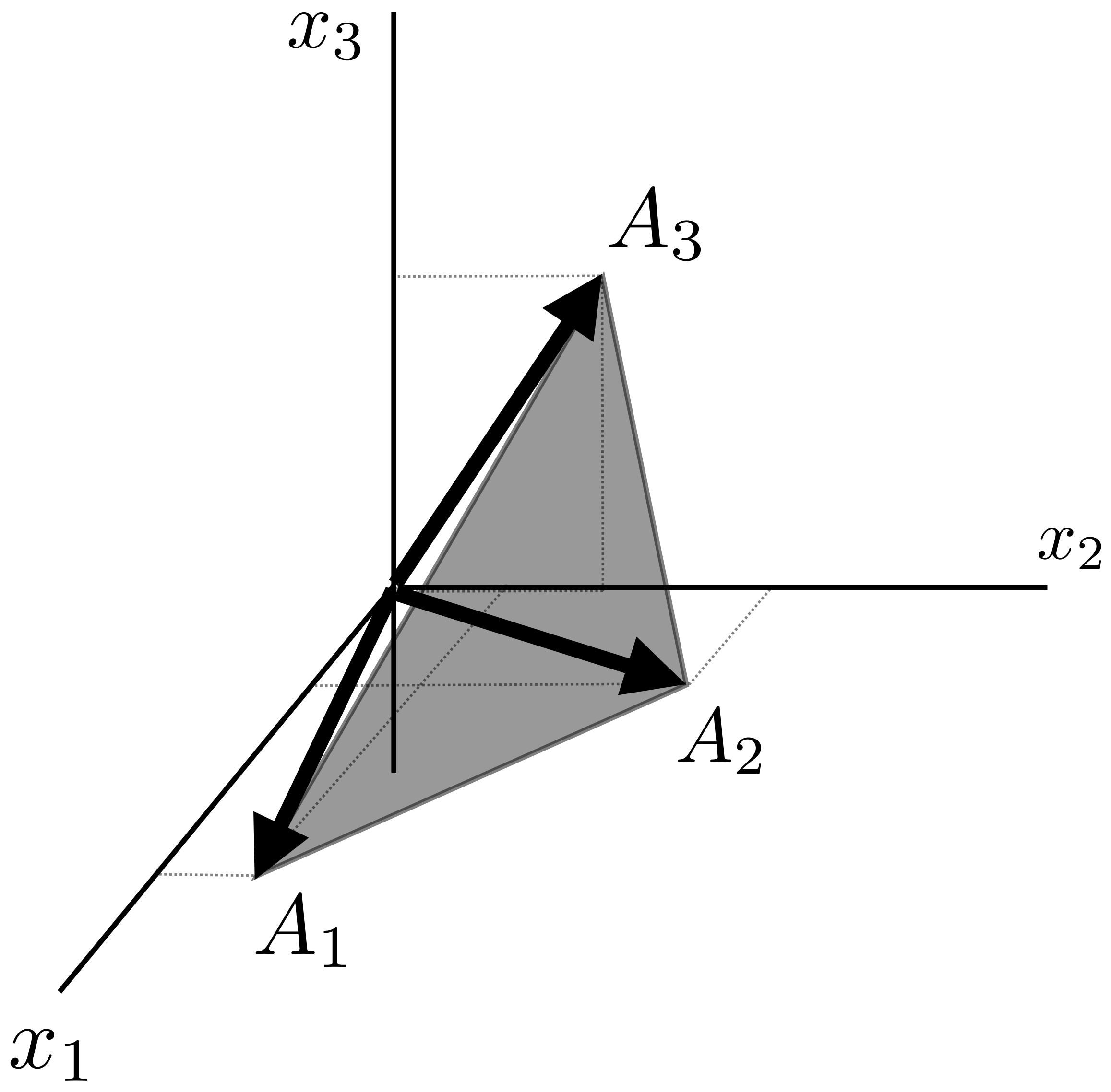

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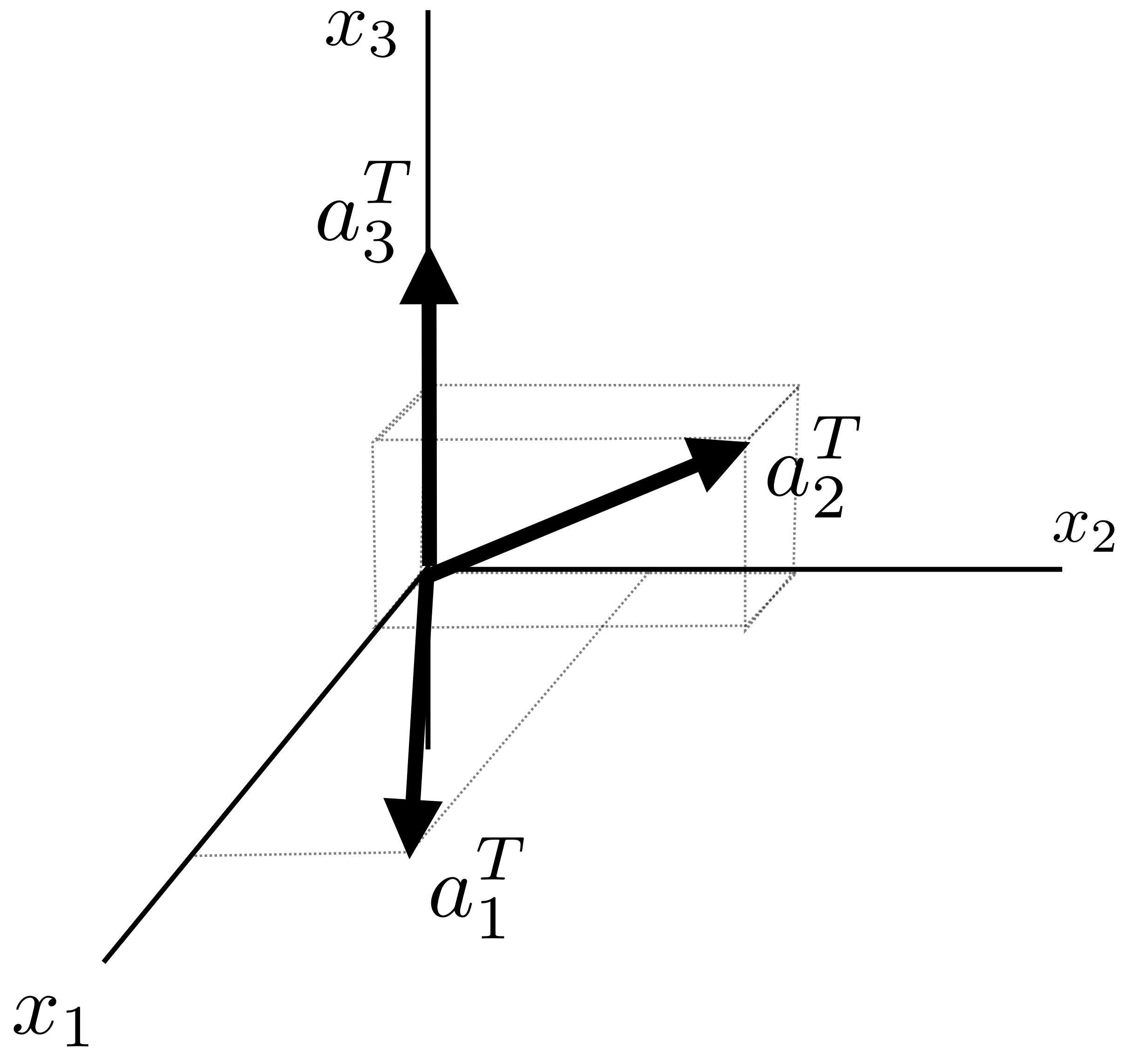

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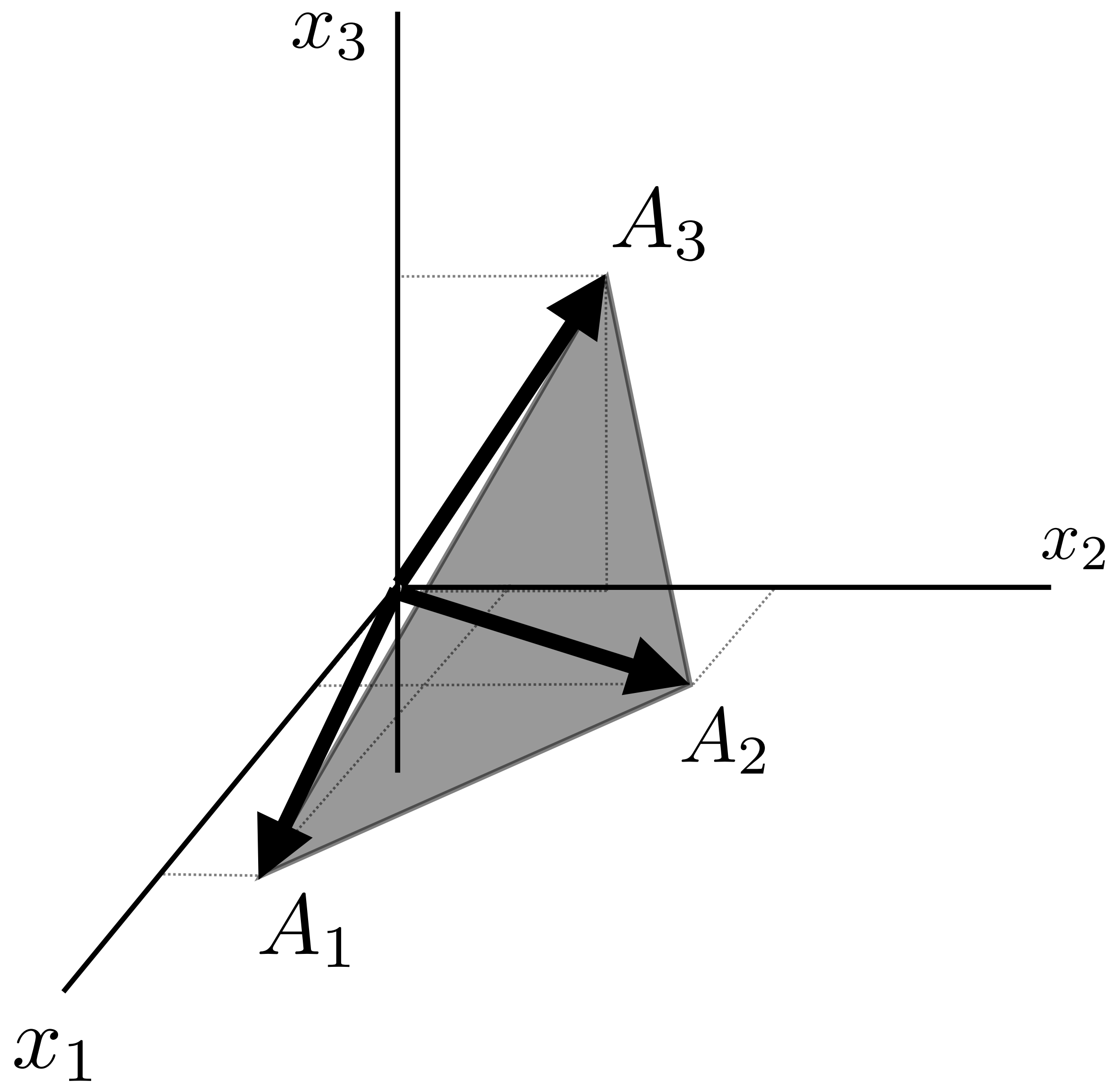

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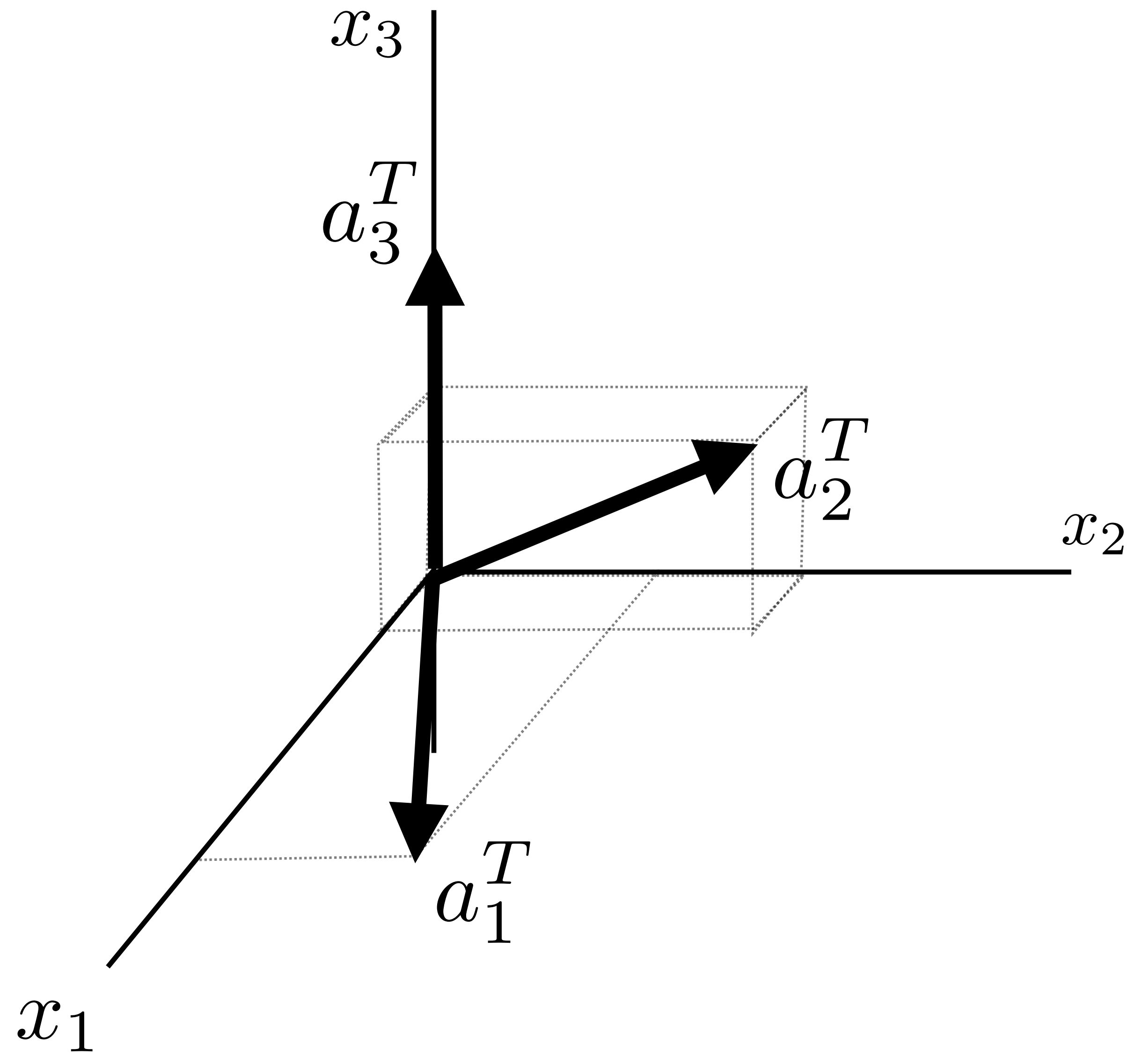

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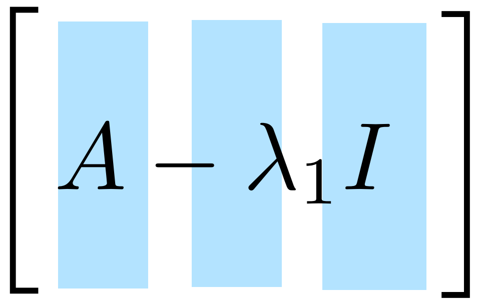

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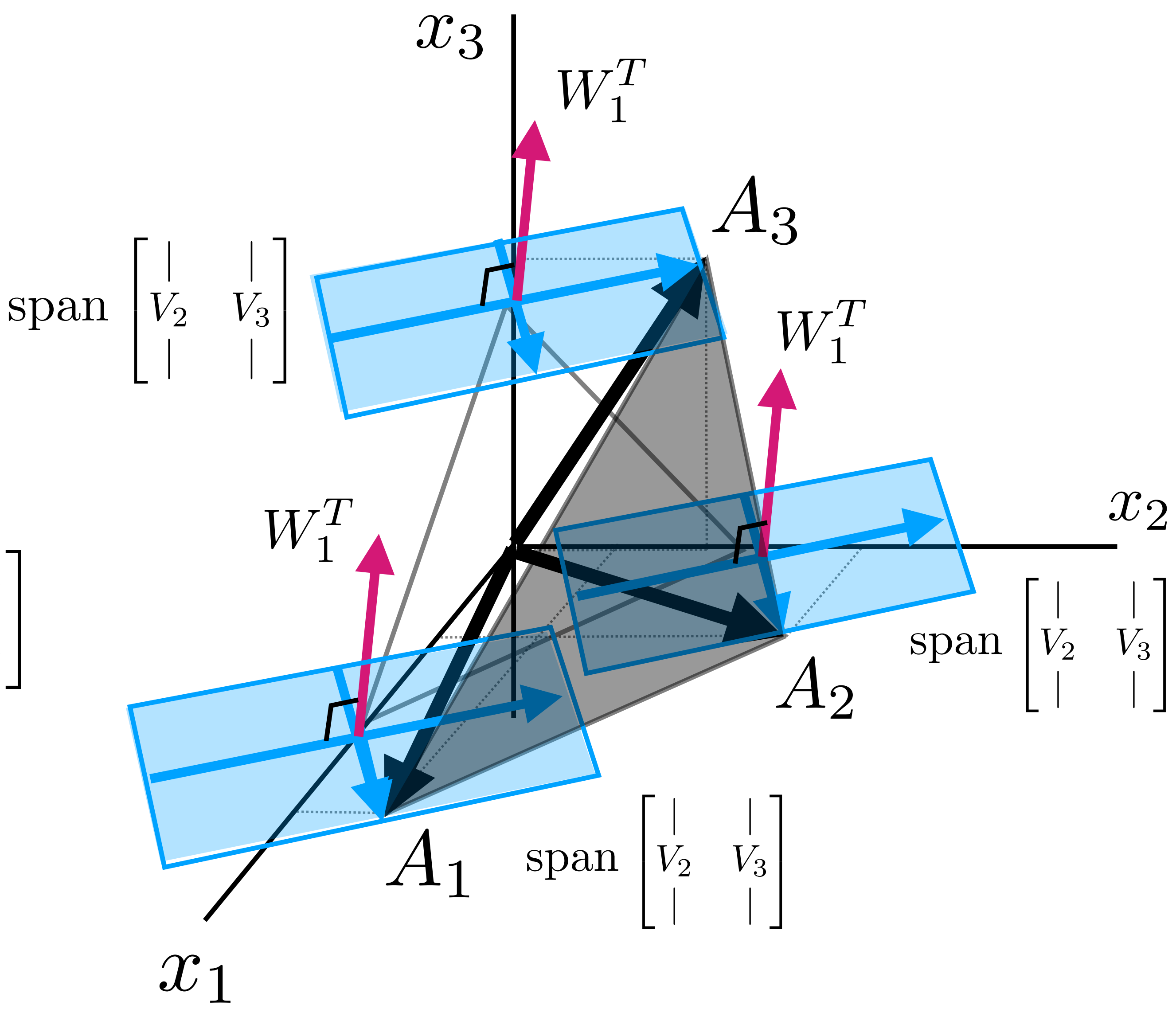

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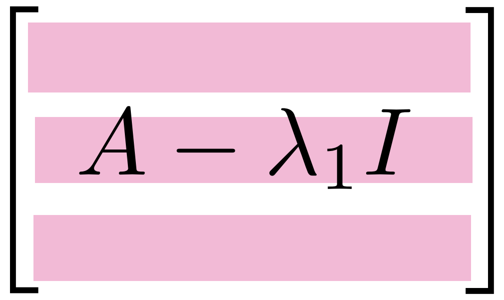

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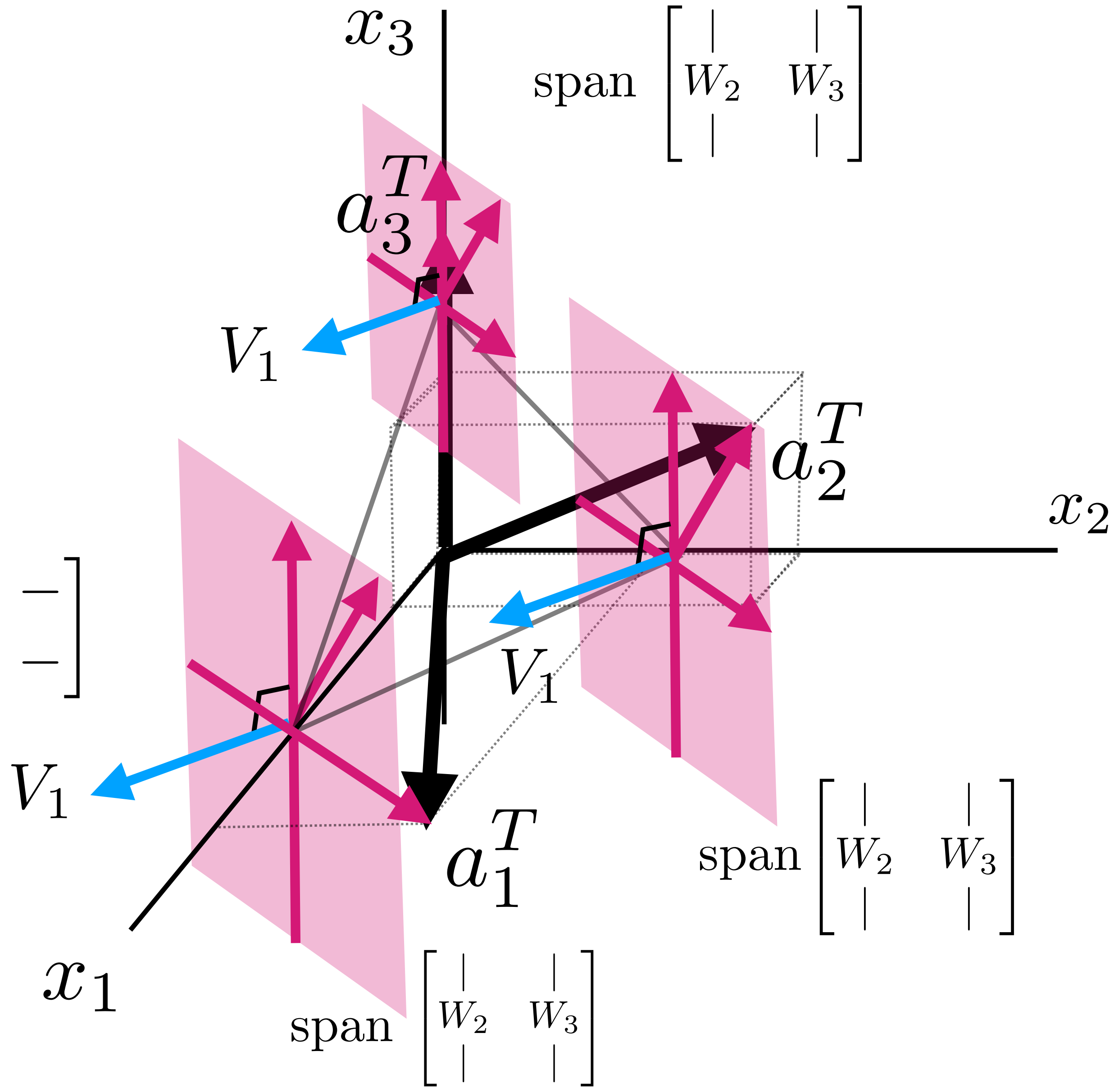

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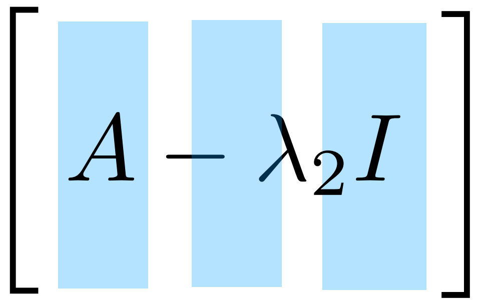

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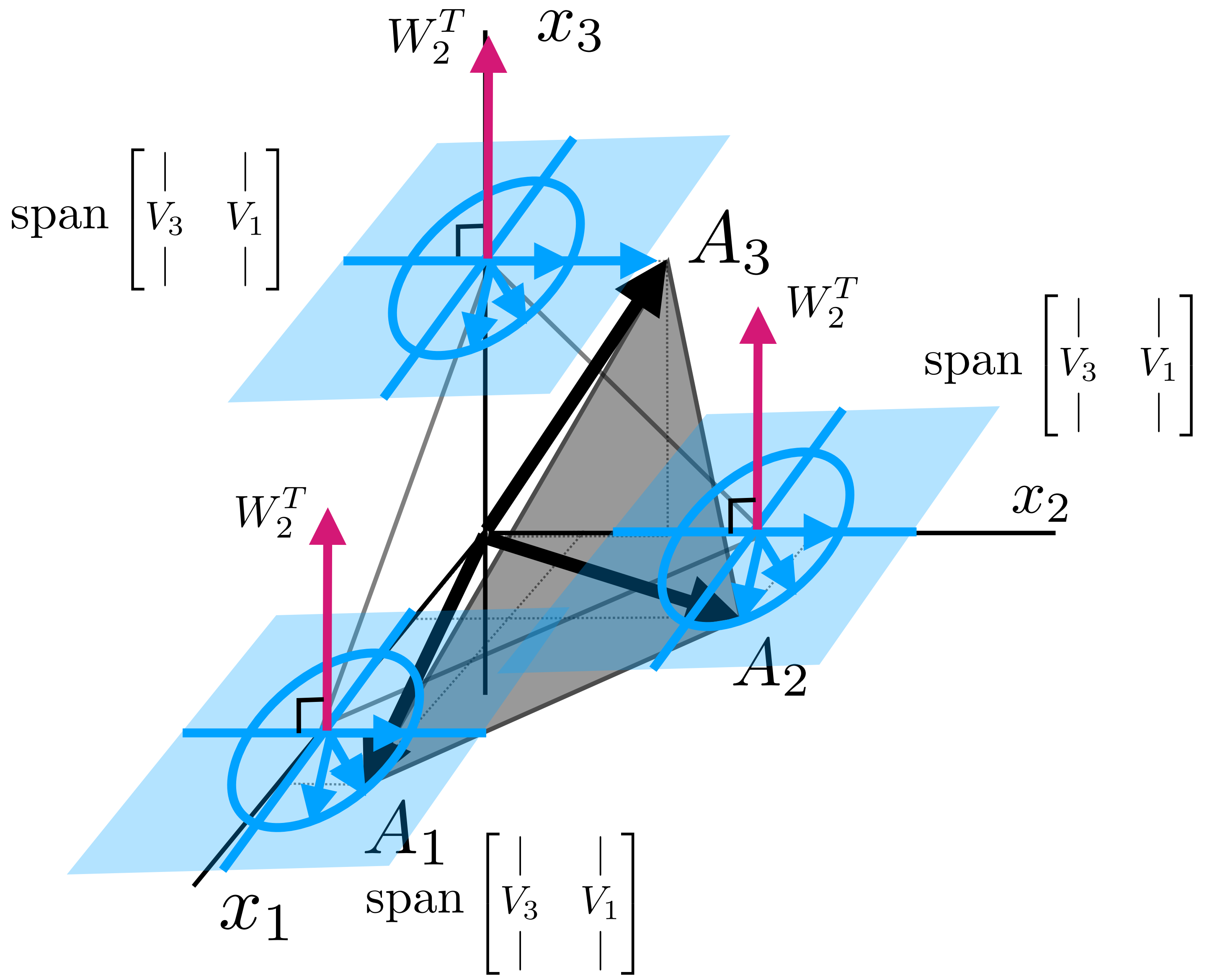

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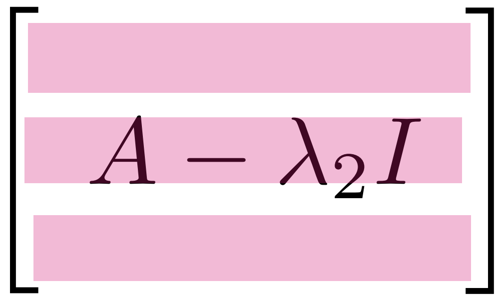

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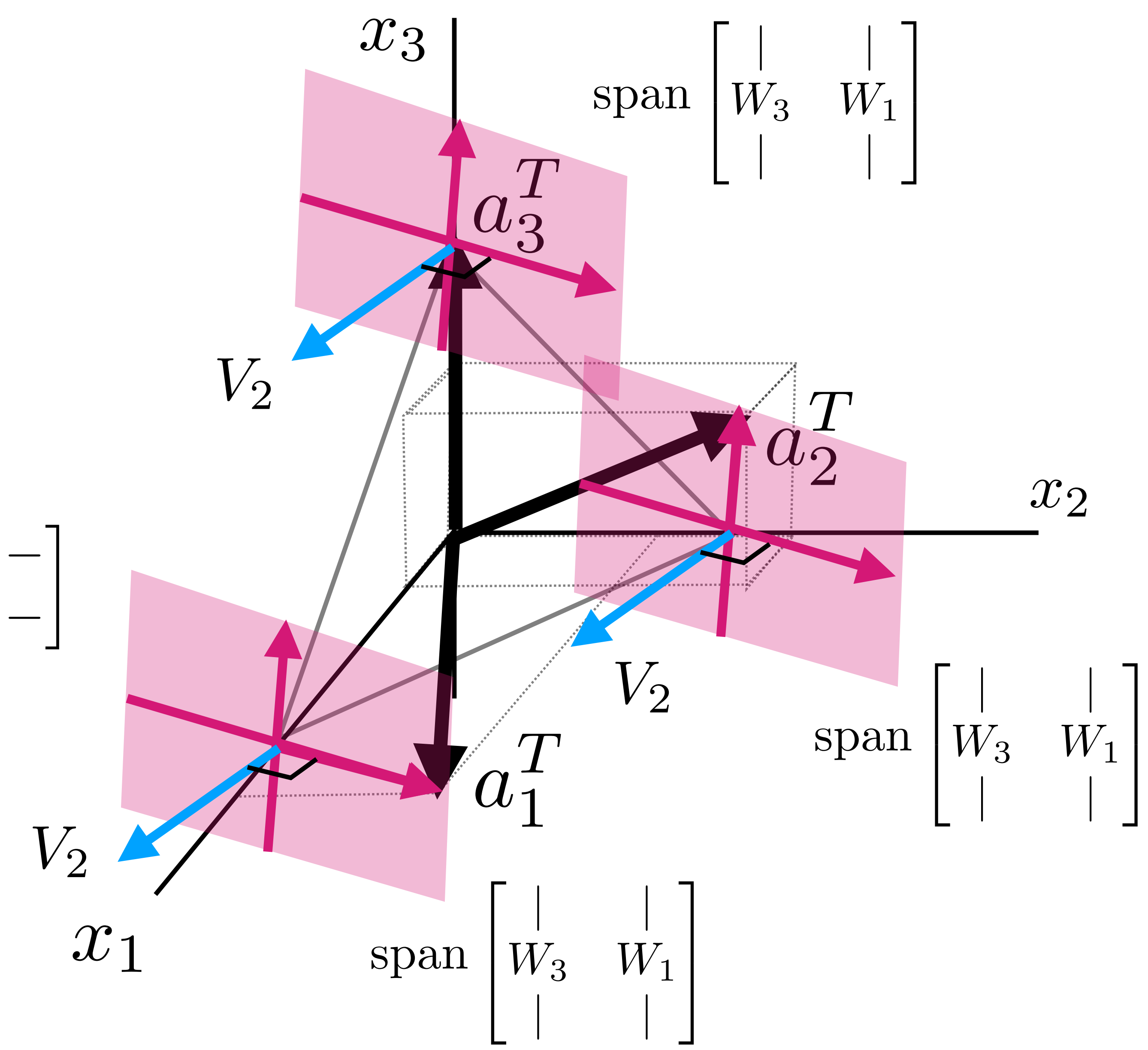

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For any eigenvalue $\lambda \in \mathbb{C}$

**Right Eigenvector:** $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

**Left Eigenvectors:** $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of $s$ where $A - sI$ drops rank…

**Characteristic Polynomial**

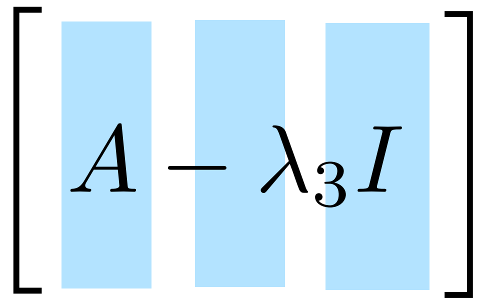
$sI - A$ drops rank only when

$$\text{char}_A(s) = \det(sI - A) \ = 0$$

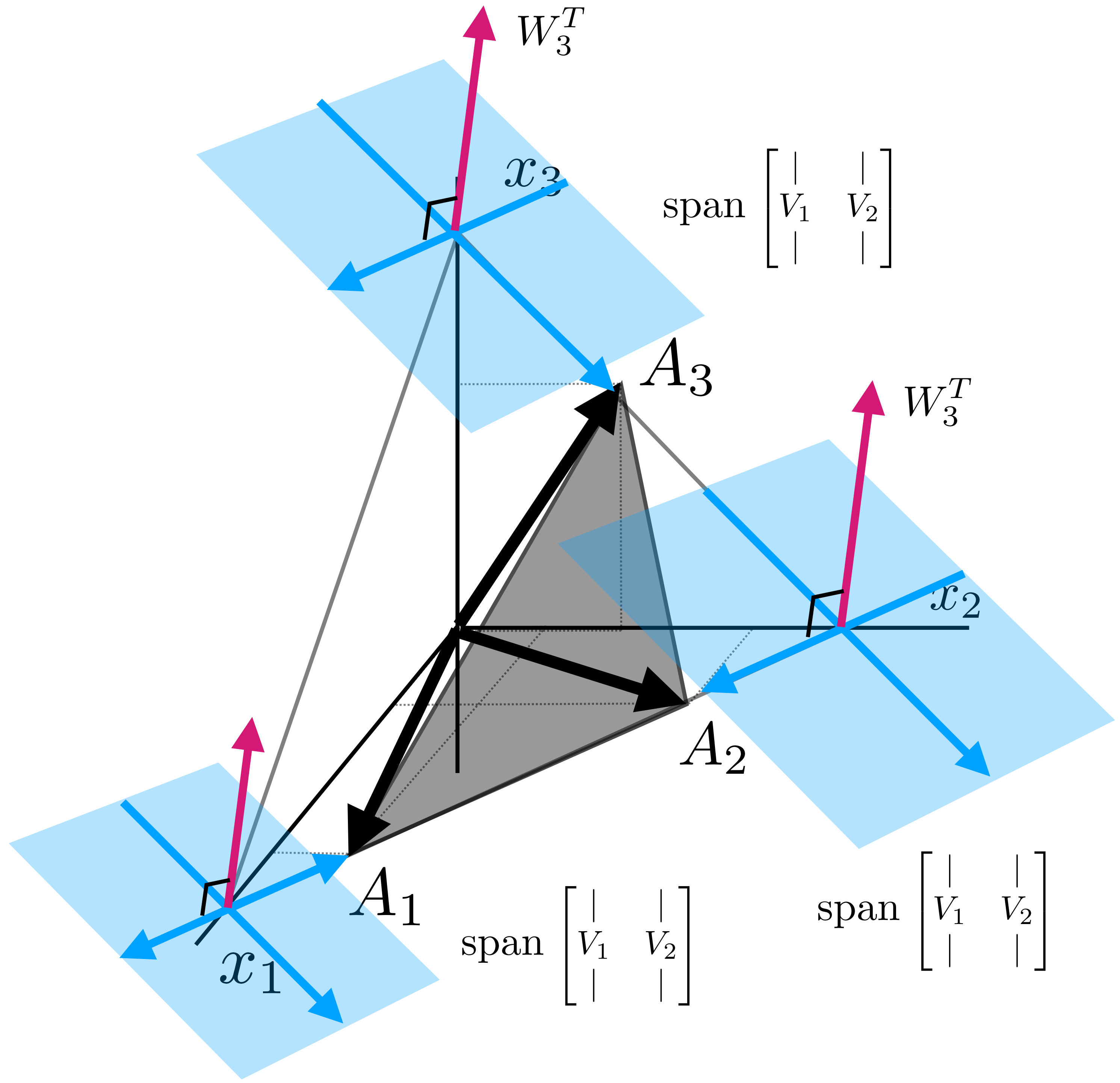
$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$

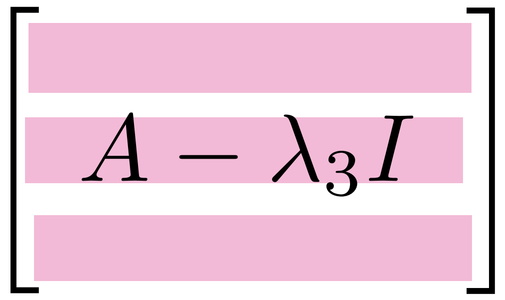
## COLUMN GEOMETRY



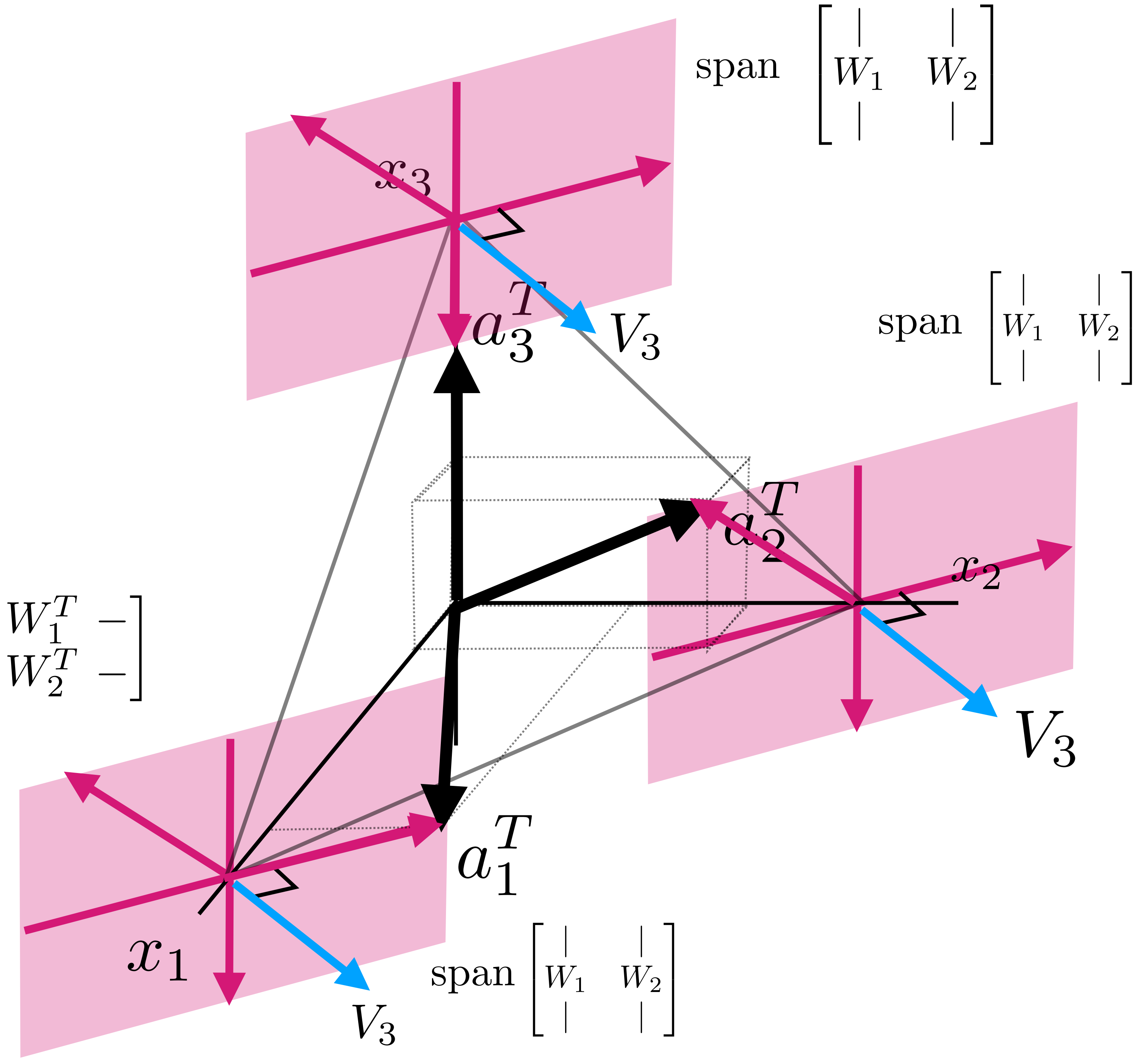
$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 0 \\ 0 & \lambda_2 - \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$



## ROW GEOMETRY



$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 0 \\ 0 & \lambda_2 - \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$



# Diagonalization

Square matrix: $A \in \mathbb{R}^{n\times n}$

Assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Right Eigenvectors:**

$$V = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \qquad AV = \begin{bmatrix} AV_1 \cdots AV_n \end{bmatrix} = \begin{bmatrix} V_1\lambda_1 \cdots V_n\lambda_n \end{bmatrix} = \begin{bmatrix} V_1 \cdots V_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{D} = VD$$

$$AV = VD$$

→ $A = VDV^{-1}$

**Left Eigenvectors:**

$$W = \begin{bmatrix} -W_1^*- \\ \vdots \\ -W_n^*- \end{bmatrix} \qquad WA = \begin{bmatrix} -W_1^*A- \\ \vdots \\ -W_n^*A- \end{bmatrix} = \begin{bmatrix} -\lambda_1 W_1^*- \\ \vdots \\ -\lambda_n W_n^*- \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{D} \begin{bmatrix} -W_1^*- \\ \vdots \\ -W_n^*- \end{bmatrix} = DW$$

$$WA = DW$$

→ $A = W^{-1}DW$

Assuming V & W are chosen with compatible orderings and lengths of columns/rows…

$$V^{-1} = W$$

# Diagonalization

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

## Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\substack{\textbf{Right}\\ \textbf{eigen-}\\ \textbf{vectors}}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-}\\ \textbf{values}\\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\substack{\textbf{Left}\\ \textbf{eigen-}\\ \textbf{vectors}}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

**Dyadic**
**Expansion**

$$V^{-1}V = \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} W_1^*V_1 & \cdots & W_1^*V_n \\ \vdots & & \vdots \\ W_n^*V_1 & \cdots & W_n^*V_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

…from off diagonal terms $W_j^* V_i = 0 \quad j \neq i$

$V_i$ orthogonal to all other $W_j$

…from diagonal terms $W_i^* V_i = 1$

$V_i, W_i$ can be scaled so that $W_i^* V_i = 1$

# Diagonalization - Similarity Transform

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

A is similar to a diagonal matrix

**Diagonalization**

$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big]$$

$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}}$$

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big] \big[- \quad W_i^* \quad -\big]$$

Sum of rank-1 matrices

**Dyadic Expansion**

…transform into eigenvector coordinates

$$x = Vx' \qquad y = Vy'$$

$$\begin{aligned} y &= Ax \\ Vy' &= AVx' \\ y' &= V^{-1}AVx' \\ y' &= V^{-1}VDV^{-1}Vx' \\ y' &= Dx' \end{aligned}$$

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \vdots \\ \lambda_n x_n' \end{bmatrix}$$

# Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigen-values}\text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

Interpretation of Matrix Multiplication $Ax$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$$

transforming into eigen-vector coords

# Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} & D & \end{bmatrix} \begin{bmatrix} & V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

Interpretation of Matrix Multiplication $Ax$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix} \text{ transforming into eigen-vector coords}}}_{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix} \text{ Scaling each coord by eigenvalue}}$$

# Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \begin{bmatrix} \; V \; \end{bmatrix} \begin{bmatrix} \; D \; \end{bmatrix} \begin{bmatrix} \; V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

Interpretation of Matrix Multiplication $Ax$

$$\begin{bmatrix} \; A \; \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix} \text{ transforming into eigen-vector coords}}}_{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix} \text{ Scaling each coord by eigenvalue}}}_{V_1 \lambda_1 W_1^* x + \cdots + V_n \lambda_n W_n^* x \text{ Transforming back into regular coordinates}}$$

# Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \begin{bmatrix} \ V \ \end{bmatrix}\begin{bmatrix} \ D \ \end{bmatrix}\begin{bmatrix} \ V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \ A \ \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} \ A \ \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

**If x is an eigenvector...**

Interpretation of Matrix Multiplication $AV_i$

$$\begin{bmatrix} \ A \ \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}$$

Orthogonal to all other left eigenvectors

# Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \begin{bmatrix} \; V \; \end{bmatrix} \begin{bmatrix} \; D \; \end{bmatrix} \begin{bmatrix} \; V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

**If x is an eigenvector…**

Interpretation of Matrix Multiplication $AV_i$

$$\begin{bmatrix} \; A \; \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ Orthogonal to all other left eigenvectors}}}_{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \text{ Scaled by specific eigenvalue}}$$

# Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \begin{bmatrix} \ V \ \end{bmatrix}\begin{bmatrix} \ D \ \end{bmatrix}\begin{bmatrix} \ V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \ A \ \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigenvectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigenvalues} \text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigenvectors}} =$$

$$\begin{bmatrix} \ A \ \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

**If x is an eigenvector…**

Interpretation of Matrix Multiplication $AV_i$

$$\begin{bmatrix} \ A \ \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}}_{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}}}_{\lambda_i V_i}$$

Orthogonal to all other left eigenvectors

Scaled by specific eigenvalue

Select out that specific eigenvector

# Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\operatorname{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\operatorname{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Sum of rank-1 matrices

**Dyadic Expansion**

**Permutation Matrix** $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

**Ex.** $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

**Shuffling eigenvalues and eigenvectors**

← Order of sum does not matter…

# Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Permutation Matrix** $P \in \mathbb{R}^{n\times n}$

Shuffle columns (or rows) of identity…

**Ex.** $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

**Diagonalization**

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}\begin{bmatrix} P \end{bmatrix}\begin{bmatrix} P^T \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} P \end{bmatrix}\begin{bmatrix} P^T \end{bmatrix}\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

**Shuffling eigenvalues and eigenvectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

← Order of sum does not matter…

# Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Permutation Matrix** $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

**Ex.** $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

**Diagonalization**

$$A = \left[ \; V \; \right] \left[ \; D \; \right] \left[ \; V^{-1} \right]$$

$$\left[ \; A \; \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{} = \boxed{\begin{bmatrix} | & & & | \\ V_1 & & \cdots & V_n \\ | & & & | \end{bmatrix} \left[ \; P \; \right]} \boxed{\left[ \; P^T \; \right] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \left[ \; P \; \right]} \boxed{\left[ \; P^T \; \right] \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}$$

**Right eigen-vectors**

**Eigen-values** (on diagonal)

**Left eigen-vectors**

**Shuffling eigenvalues and eigenvectors**

$$\left[ \; A \; \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[ \lambda_i \right] \left[ - \quad W_i^* \quad - \right]$$

Sum of rank-1 matrices

**Dyadic Expansion**

←

Order of sum does not matter…

# Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Permutation Matrix** $P \in \mathbb{R}^{n \times n}$

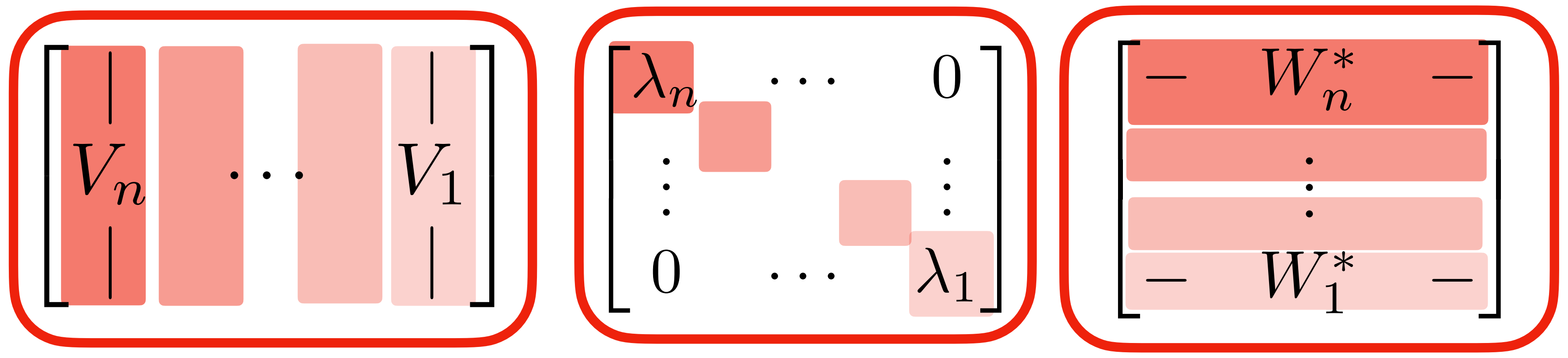
Shuffle columns (or rows) of identity…

**Ex.** $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

**Diagonalization**

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$



$$\begin{bmatrix} | & & | \\ V_n & \cdots & V_1 \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_n^* & - \\ & \vdots & \\ - & W_1^* & - \end{bmatrix}$$

**Shuffling eigenvalues and eigenvectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

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$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Sum of
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**Dyadic**
**Expansion**

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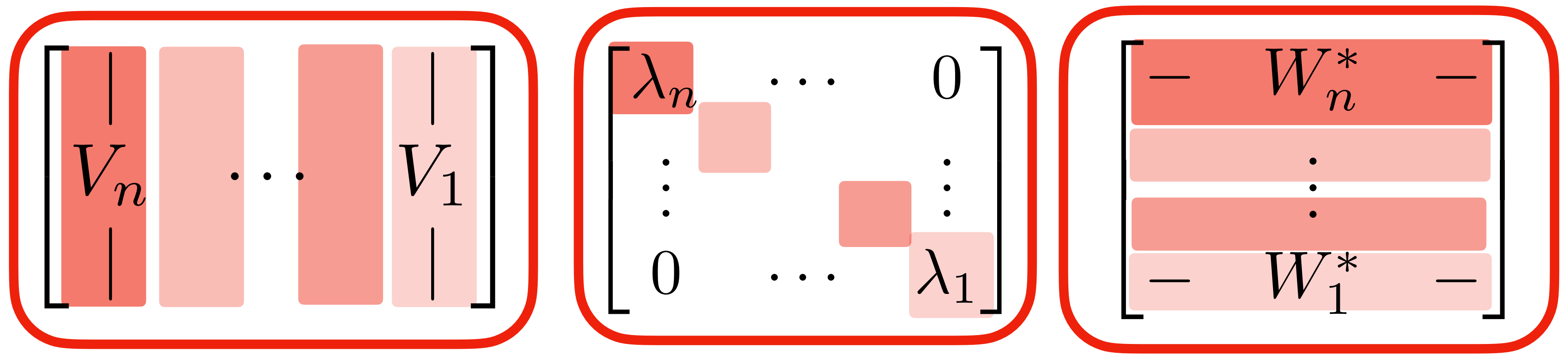

**Shuffling eigenvalues and eigenvectors**

← Order of sum does not matter…

# Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

$$A = \big[\ V\ \big]\big[\ D\ \big]\big[\ V^{-1}\big]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{} =$$

**Right eigen-vectors**

**Eigen-values** (on diagonal)

**Left eigen-vectors**

diagonal matrices commute…

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \frac{\gamma_1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \frac{\gamma_n}{\gamma_n} \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

**Scaling eigenvectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

**Dyadic Expansion**

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Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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**Scaling eigenvectors**

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**Right eigenvectors**

**Eigenvalues** (on diagonal)

**Left eigenvectors**

**Scaling eigenvectors**

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

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# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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**Complex Conjugate Pairs:** $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & V_2 \\ | & | \end{matrix}} & \boxed{\begin{matrix} | & | \\ V_3 & V_4 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} \gamma_2 e^{-i\phi_2} & 0 \\ 0 & \gamma_2 e^{i\phi_2} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & W_2^* & - \end{matrix}} \\ \boxed{\begin{matrix} - & W_3^* & - \\ - & W_4^* & - \end{matrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

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**Eigenvectors:** $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

*can be conjugate pairs*

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

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**Eigenvectors:** $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

*can be conjugate pairs*

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1}\frac{z}{z} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1}\frac{z'}{z'} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

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Note: *may differ by any complex scalars* $z, z' \in \mathbb{C}$

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# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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*can be conjugate pairs*

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where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

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Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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*can be conjugate pairs*

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ \begin{matrix} - & \bar{W}_n^* & - \end{matrix} \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

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# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

## Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

## Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

**Eigenvectors:** $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

*can be conjugate pairs*

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

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$$\underset{\text{Real}}{\uparrow} \quad \underset{\text{Imag}}{\uparrow}$$

(scaled by $\sqrt{2}$ )

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similarly…

$$\begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}}_{U^*} \begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}$$

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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*can be conjugate pairs*

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Note: $U \in \mathbb{C}^{2 \times 2}$

*unitary*

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Note: $U \in \mathbb{C}^{2 \times 2}$ $\quad U^*U = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = I$

*unitary*

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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Also: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}}$

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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$$U^* \qquad U \qquad U^* \qquad U$$

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$$\left[ \; A \; \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[ \lambda_i \right] \left[ - \quad W_i^* \quad - \right]$$

**Complex Conjugate Pairs:** $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \vdots & \qquad & \vdots \end{matrix} & \ddots & \vdots \\ \begin{matrix} 0 & \qquad & 0 \end{matrix} & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $\qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**Eigenvalues:** $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

**Diagonalization**

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$$\left[\; A \;\right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\substack{\textbf{Right}\\ \textbf{eigen-}\\ \textbf{vectors}}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-}\\ \textbf{values}\\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\substack{\textbf{Left}\\ \textbf{eigen-}\\ \textbf{vectors}}} = \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n \times n}} \begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \vdots & \quad & \vdots \end{matrix} & \ddots & \vdots \\ \begin{matrix} 0 & \quad & 0 \end{matrix} & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & {w_1'}^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

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$$\Big[\; A \;\Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big] \big[- \quad W_i^* \quad -\big]$$

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n \times n}} \underbrace{\begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix}}_{D' \in \mathbb{R}^{n \times n}} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & {w_1'}^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

*Real Expansion…*

*Block diagonal…*

***Pseudo-diagonalization***

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**GEOMETRY**

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$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big] = \left[\boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \ \cdots \ \begin{matrix} | \\ V_n \\ | \end{matrix}\right] \begin{bmatrix} \boxed{\quad \gamma R_\phi \quad} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

$$\underbrace{\Big[\ A\ \Big] = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

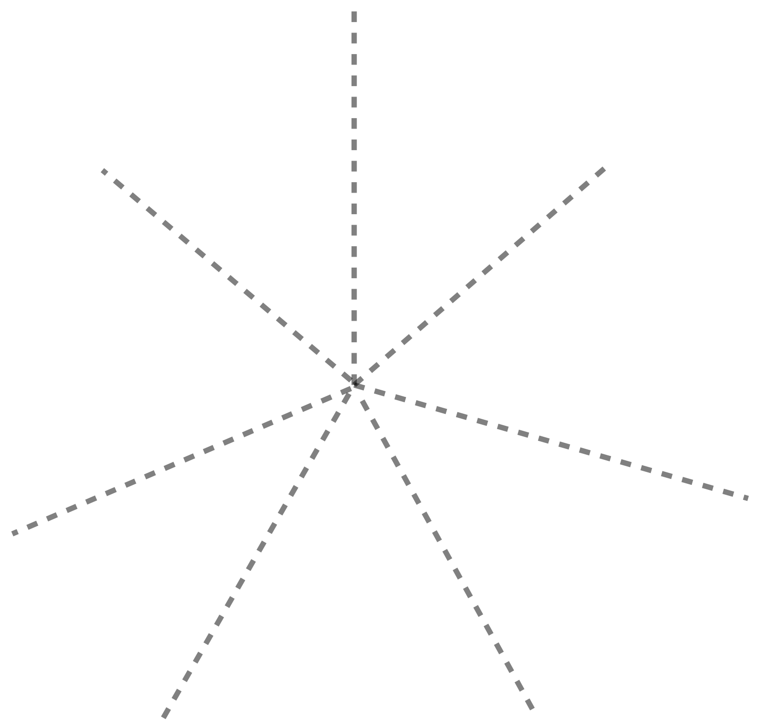
**Right eigen-vectors**

**Eigen-values** (on diagonal)

**Left eigen-vectors**

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- \quad W_i^* \quad -]$$

$\mathbb{R}^n$



# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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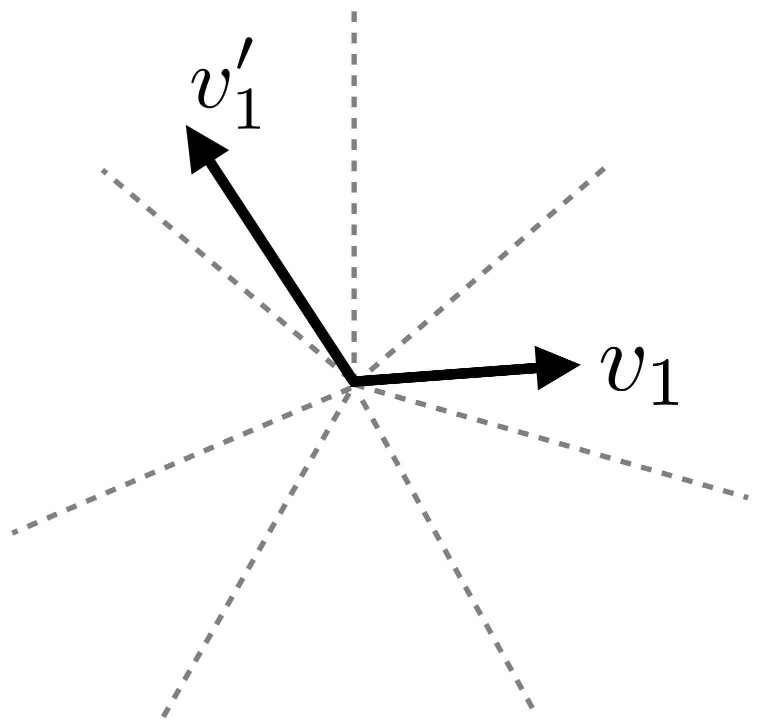
$\mathbb{R}^n$



# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big]$$

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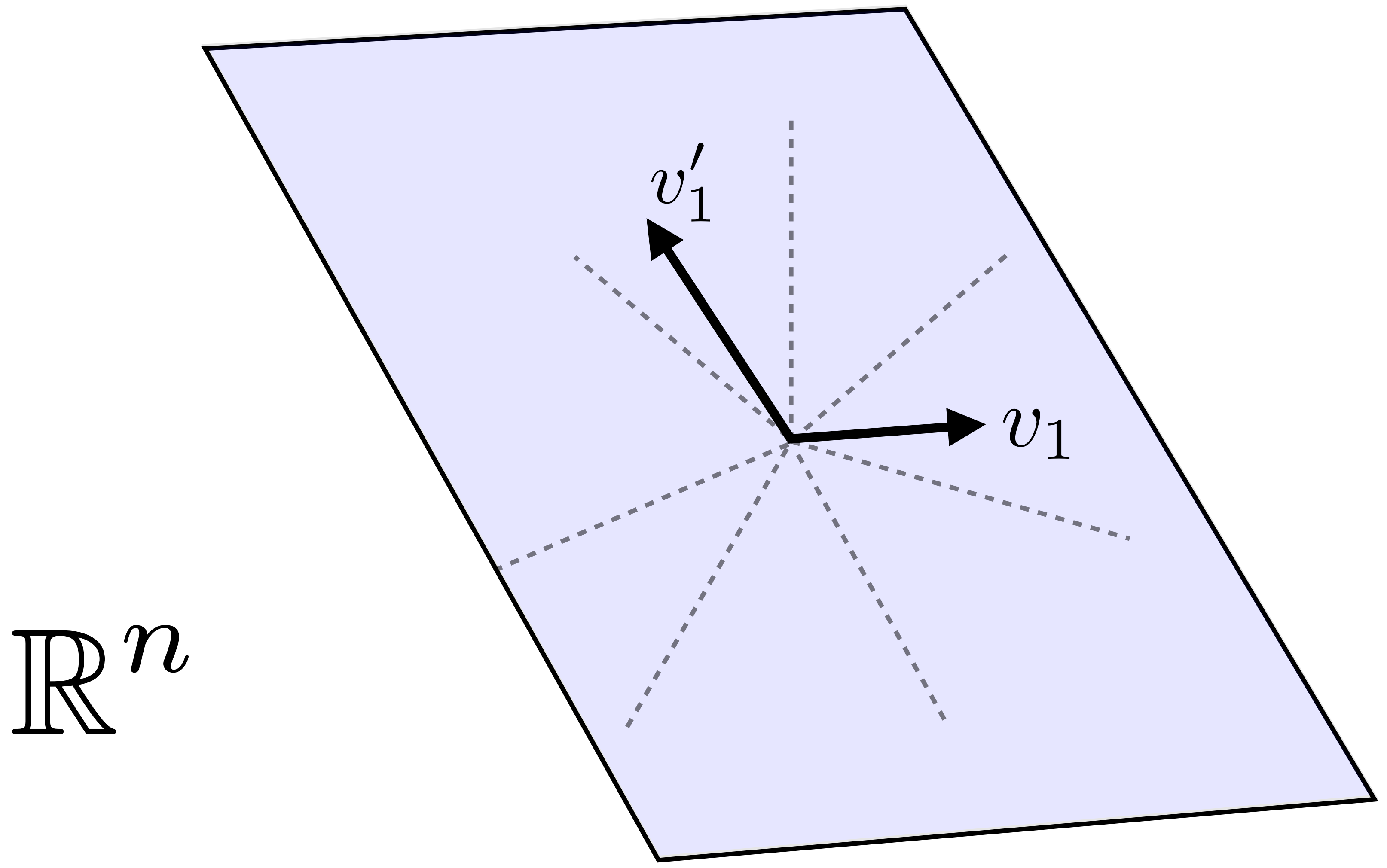
$$\underbrace{\qquad\qquad}_{\text{Right eigen-vectors}} \quad \underbrace{\qquad\qquad}_{\text{Eigen-values (on diagonal)}} \quad \underbrace{\qquad\qquad}_{\text{Left eigen-vectors}}$$

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \Big[\lambda_i\Big] \Big[- \quad W_i^* \quad -\Big]$$

**Complex Conjugate Pairs:** $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\quad \gamma R_\phi \quad} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \qquad \vdots & \ddots & \vdots \\ 0 \qquad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$



$$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$$ 2D

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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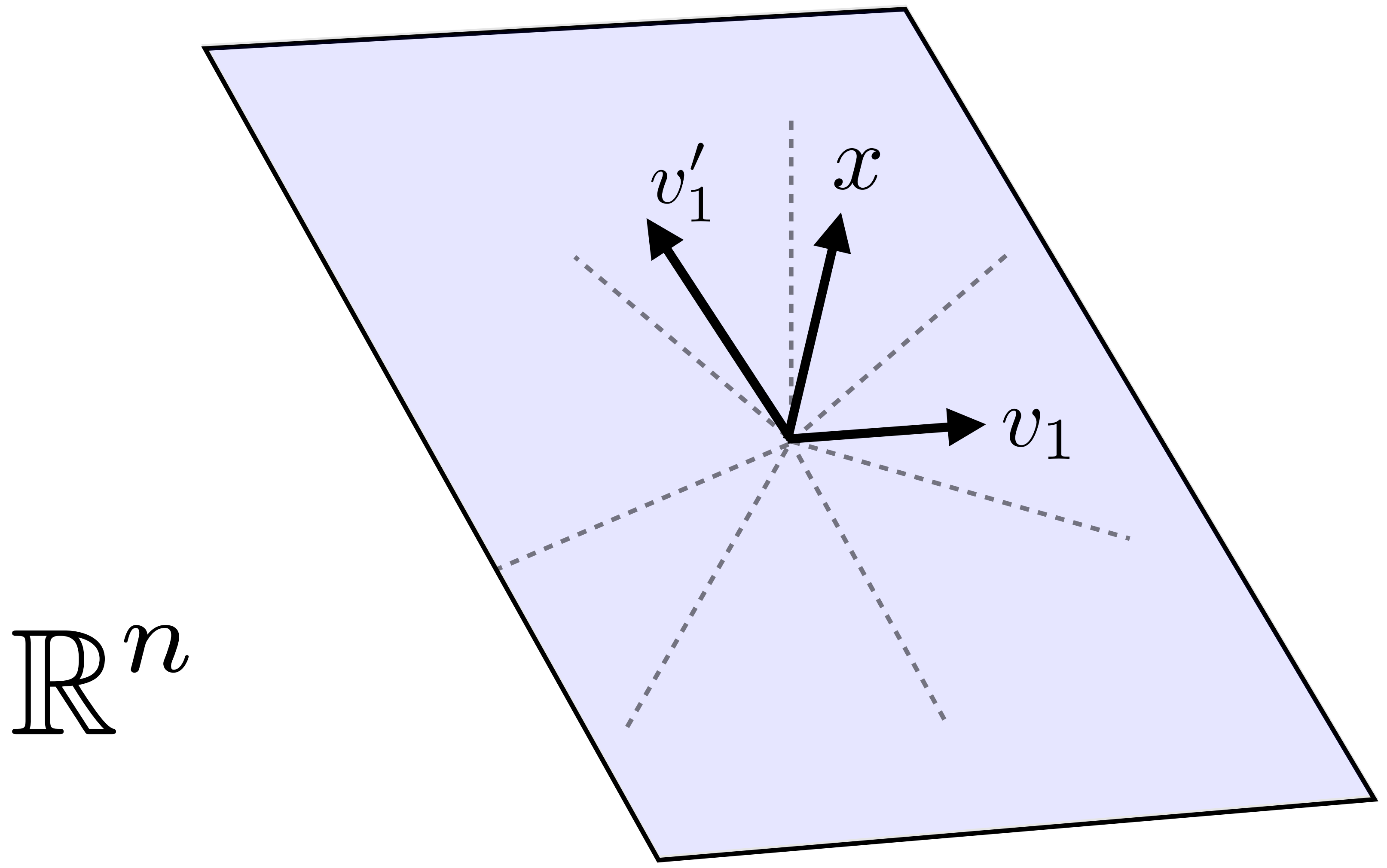
$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

vector in plane of rotation



$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big] \big[- \quad W_i^* \quad -\big]$$

$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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vector in plane of rotation

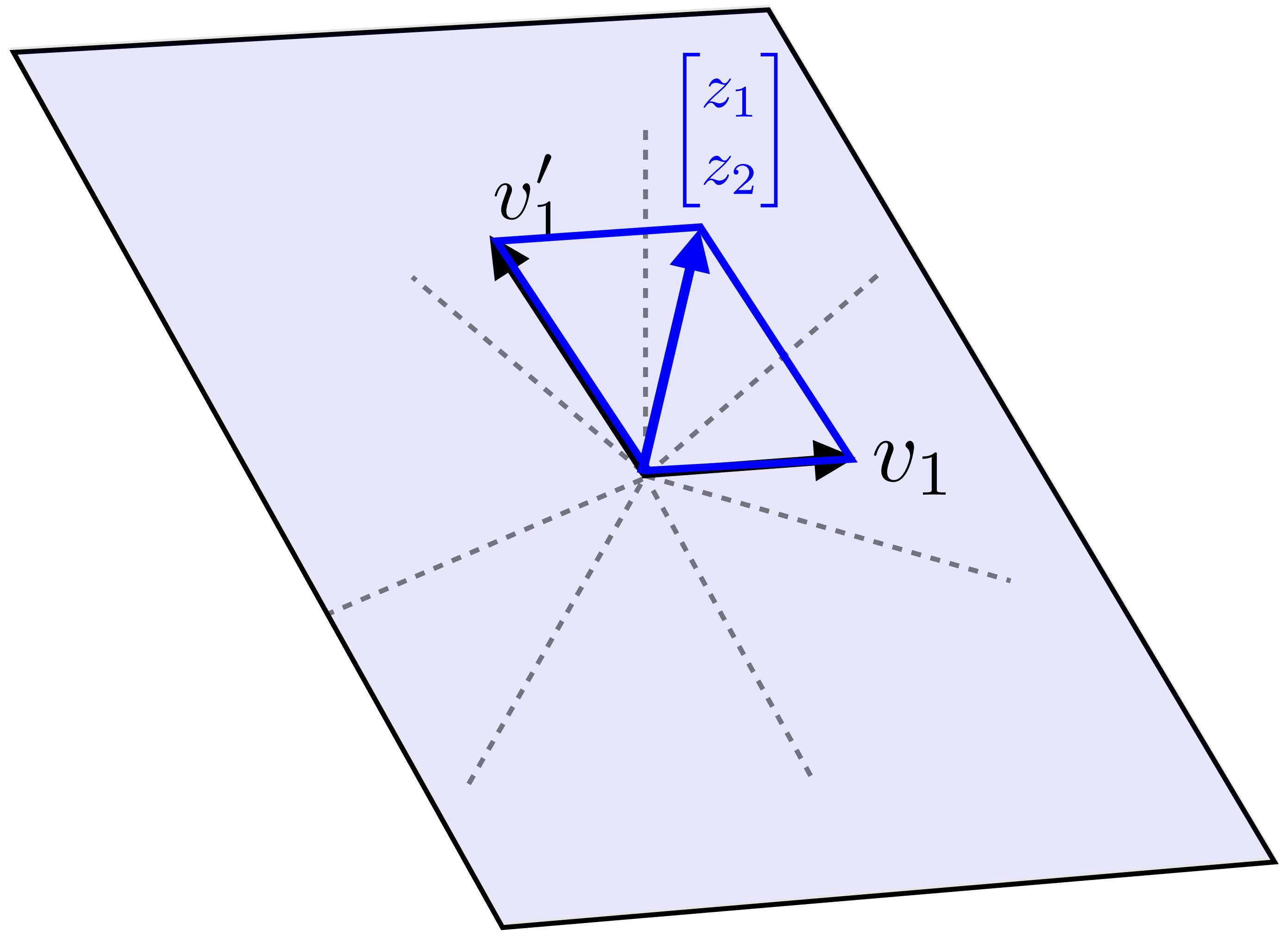
$$\begin{bmatrix} \; A \; \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

**Right eigen-vectors**

**Eigen-values** (on diagonal)

**Left eigen-vectors**

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

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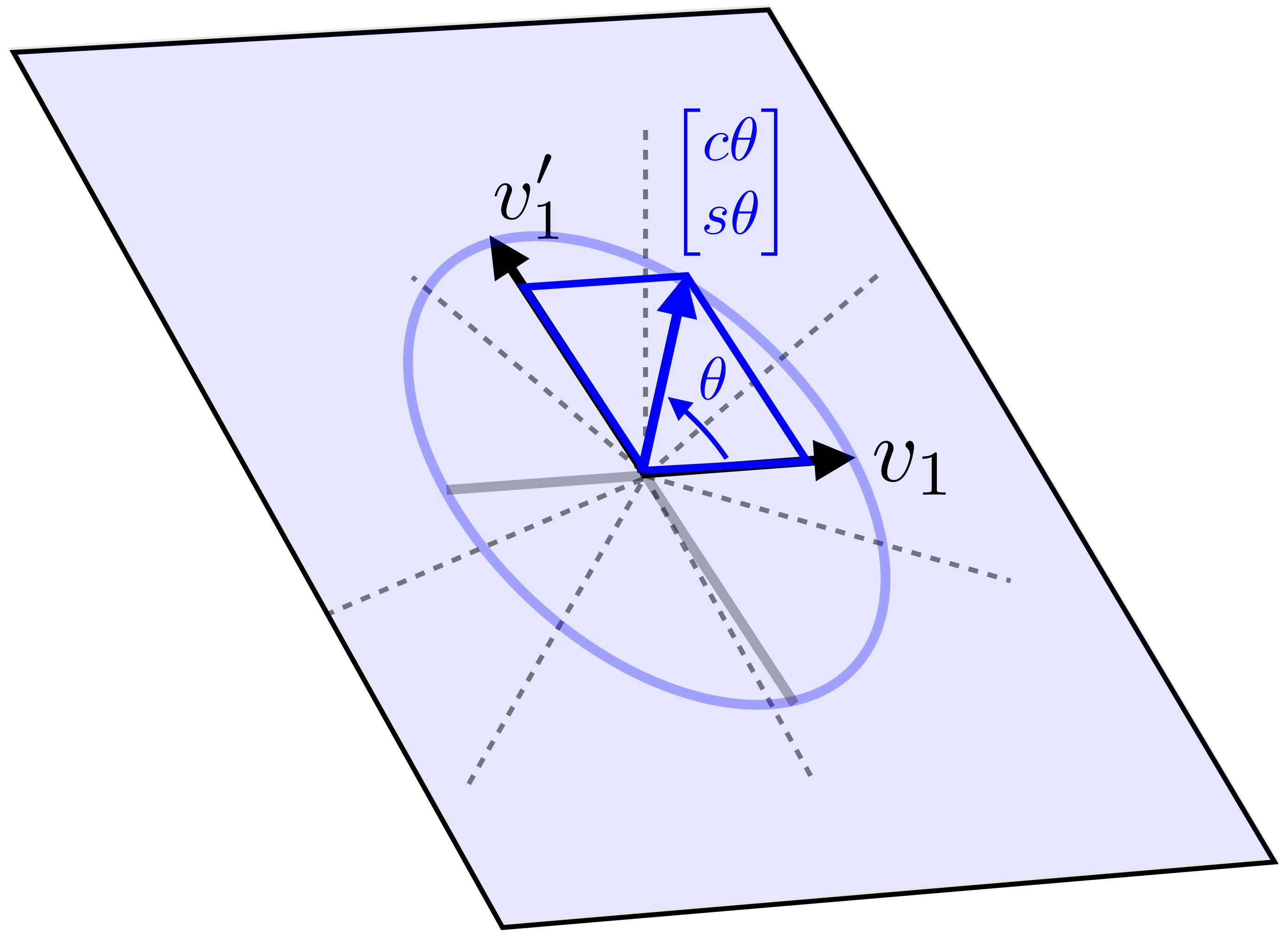
$$\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}\begin{bmatrix} c\theta \\ s\theta \end{bmatrix}^{|z|_2}$$

↑ vector in plane of rotation

$\mathbb{R}^n$



$$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \quad \text{2D}$$

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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**Eigen-values** (on diagonal)

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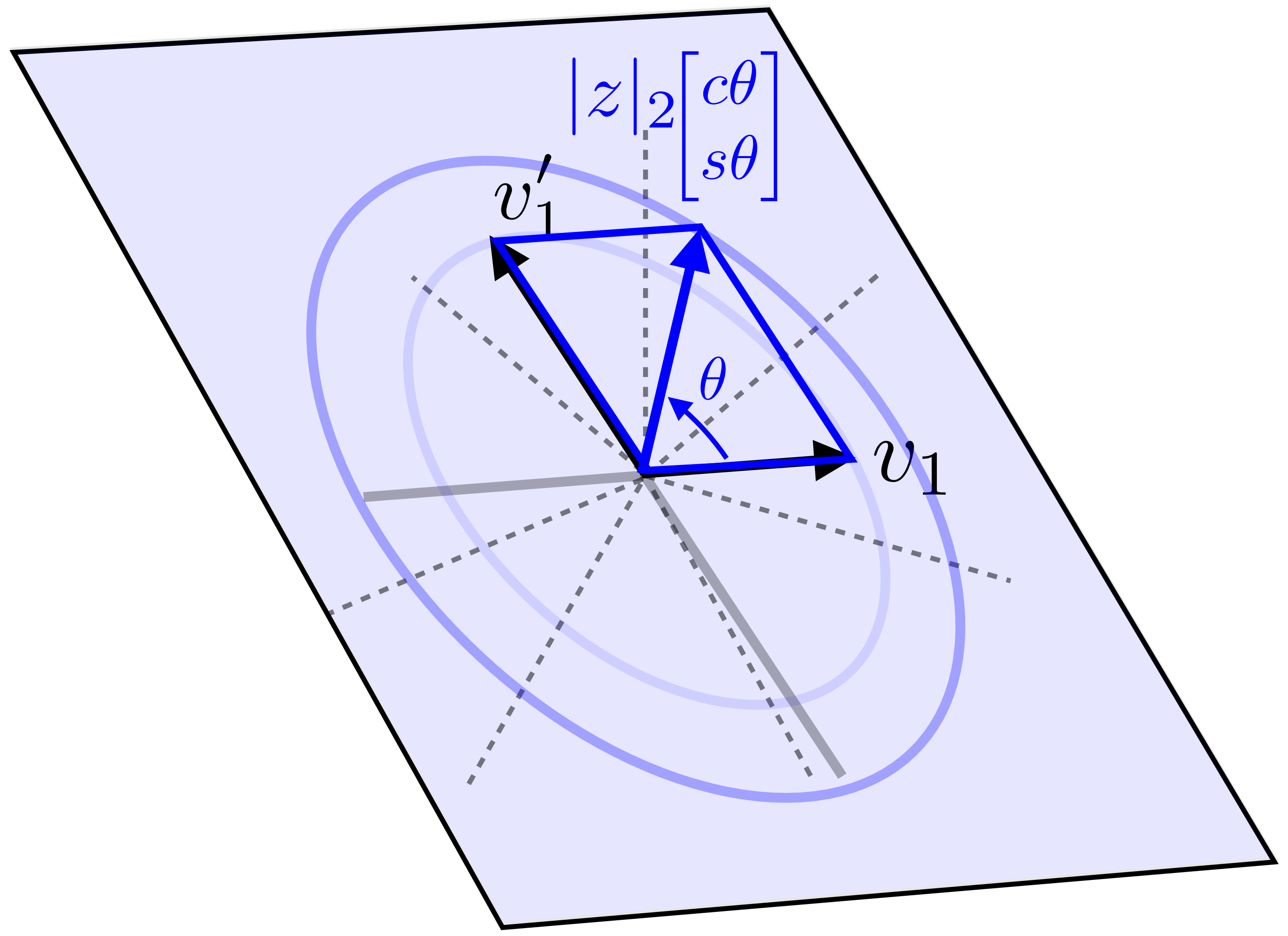
$$\left[\boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \cdots \begin{matrix} | \\ V_n \\ | \end{matrix}\right]\begin{bmatrix} \boxed{\gamma R_\phi} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}\begin{bmatrix} c\theta \\ s\theta \end{bmatrix}^{|z|_2}$$

$\mathbb{R}^n$



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$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

**Right eigen-vectors**

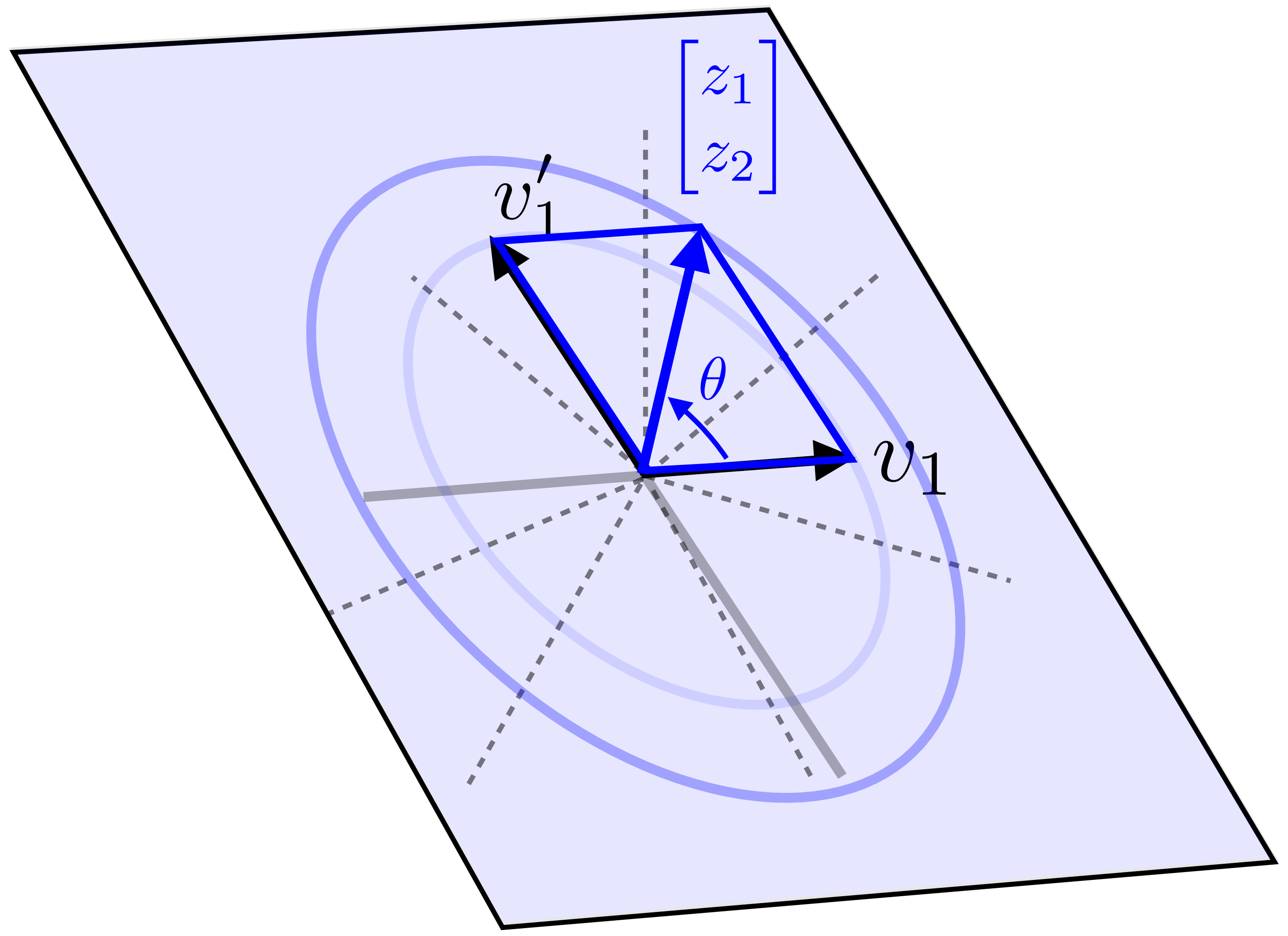
**Eigen-values**
(on diagonal)

**Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

**Complex Conjugate Pairs:** $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

# Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

**GEOMETRY**

**Eigenvalues:** $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

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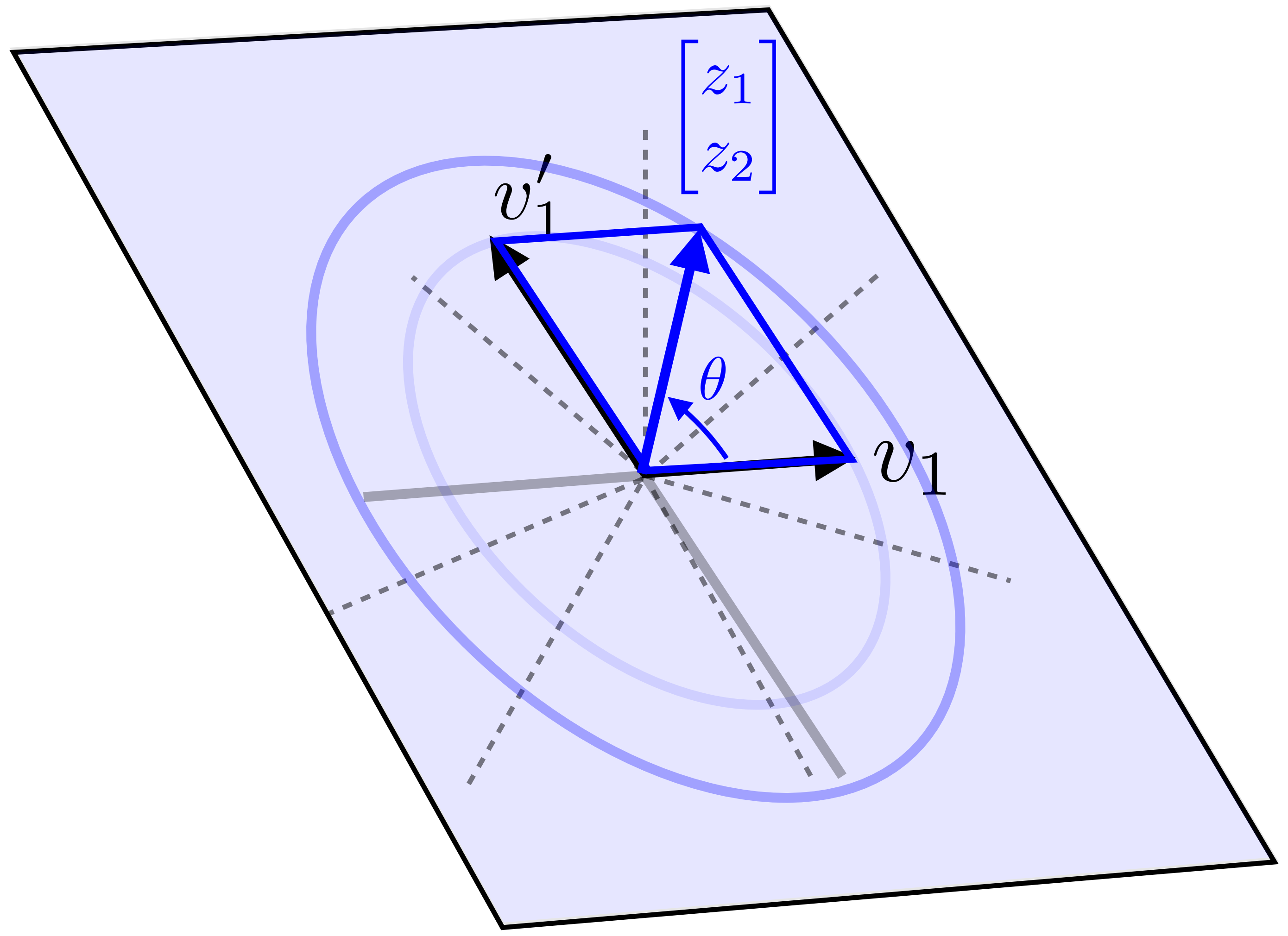
$\mathbb{R}^n$



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**Eigen-values** (on diagonal)

**Left eigen-vectors**

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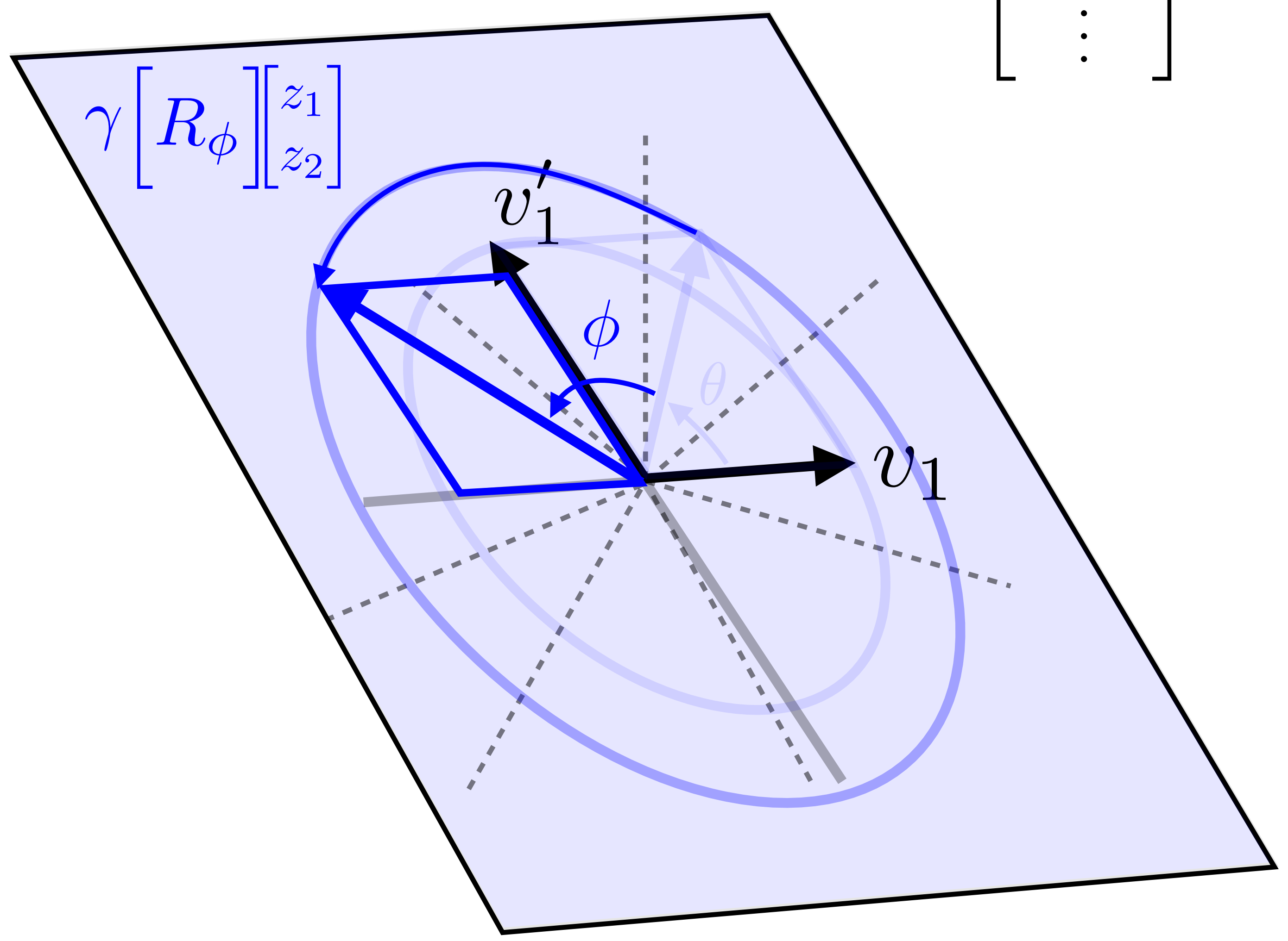
$$\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma = 1$

$\mathbb{R}^n$



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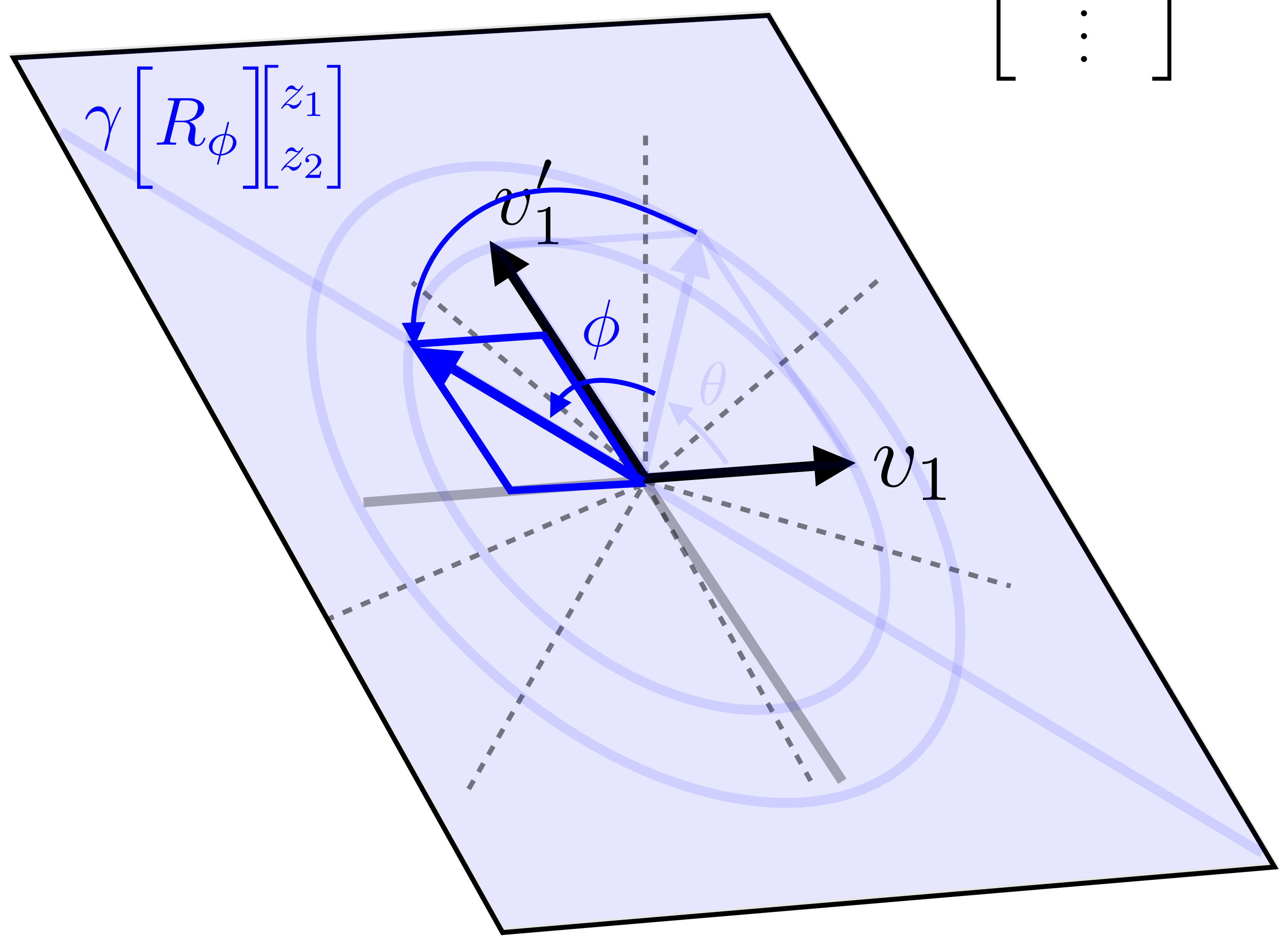
$$\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma < 1$

$\mathbb{R}^n$



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$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big] =$$

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**Right eigen-vectors**

**Eigen-values** (on diagonal)

**Left eigen-vectors**

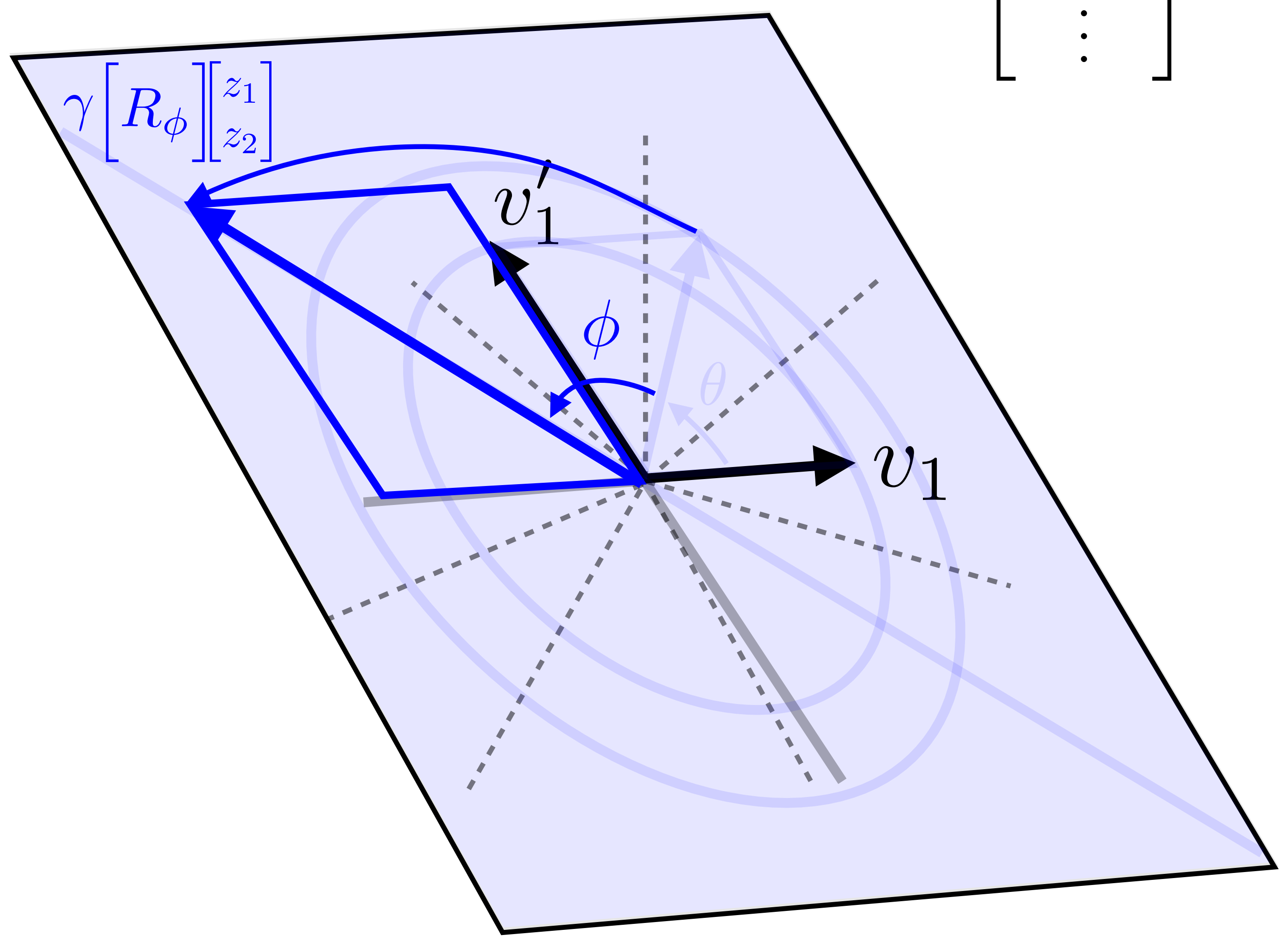
$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \Big[\lambda_i\Big]\Big[- \quad W_i^* \quad -\Big]$$

**Complex Conjugate Pairs:** $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \cdots \begin{matrix} | \\ V_n \\ | \end{matrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma > 1$

$\mathbb{R}^n$



$$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$$ 2D

# Diagonalization: Complex eigenvalues

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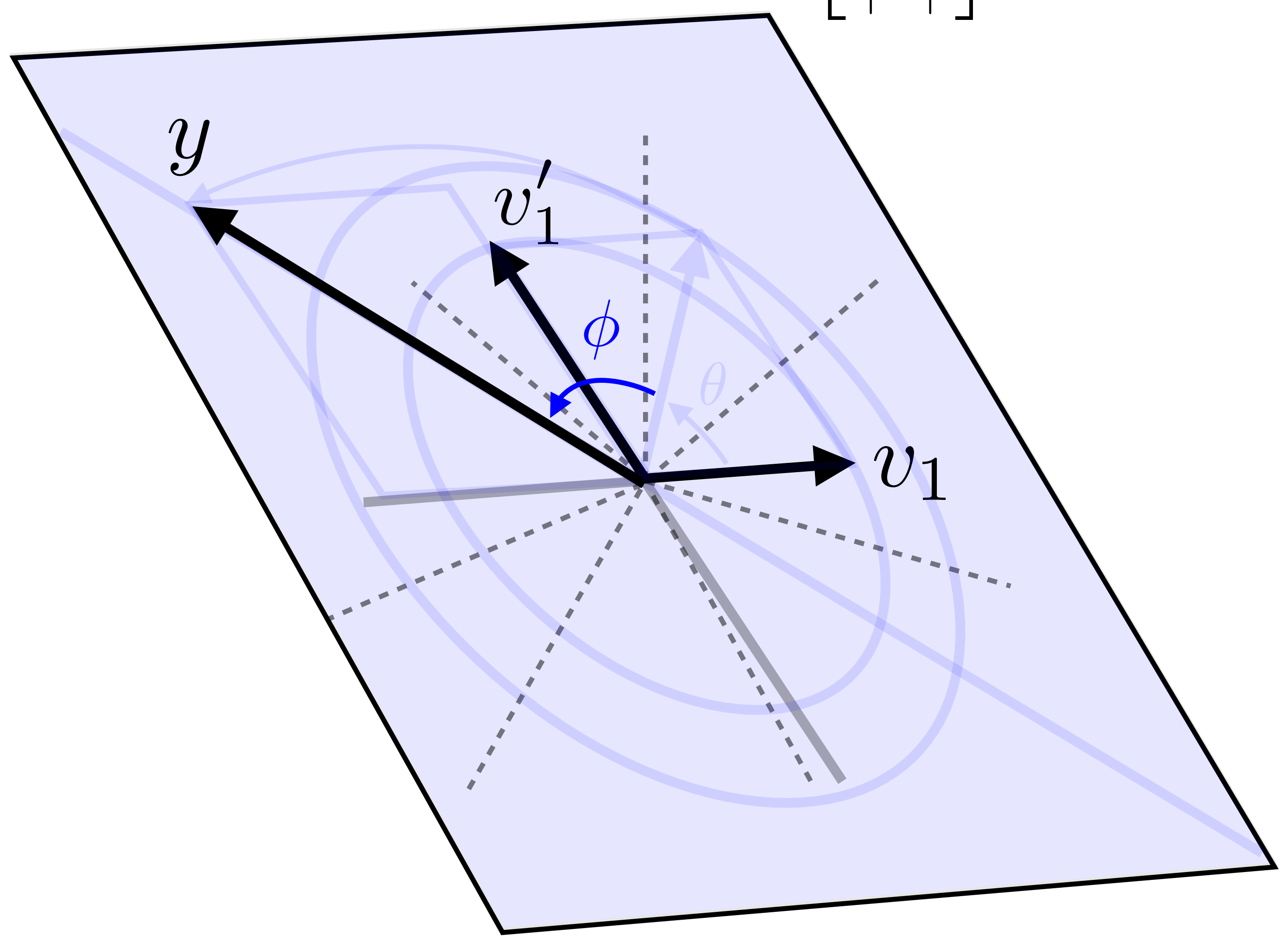

$\mathbb{R}^n$

$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

# Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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**Diagonalization** $A = VDV^{-1}$

**Powers of A**

$$A^k = VDV^{-1} \times VDV^{-1} \times \cdots \times VDV^{-1}$$
$$= VDV^{-1}VDV^{-1} \cdots VDV^{-1}$$
$$= VD^kV^{-1}$$

$$A^k = VD^kV^{-1}$$

$$= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}$$

**Polynomials of A**

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1}s^{k-1} + \alpha_{k-2}s^{k-2} + \cdots + \alpha_1 s + \alpha_0 1$

plugging in A…

$$\Psi(A) = \alpha_k A^k + \alpha_{k-1}A^{k-1} + \alpha_{k-2}A^{k-2} + \cdots + \alpha_1 A + \alpha_0 I$$
$$= \alpha_k VD^kV^{-1} + \alpha_{k-1}VD^{k-1}V^{-1} + \alpha_{k-2}VD^{k-2}V^{-1} + \cdots + \alpha_1 VDV^{-1} + \alpha_0 VV^{-1}$$
$$= V\Big(\alpha_k D^k + \alpha_{k-1}D^{k-1} + \alpha_{k-2}D^{k-2} + \cdots + \alpha_1 D + \alpha_0 I\Big)V^{-1}$$

$$\Psi(A) = V\Psi(D)V^{-1}$$

$$= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}$$

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**Spectral Mapping Theorem** for f(s) analytic

$\lambda \in \mathrm{eig}(A)$ ➡ $f(\lambda) \in \mathrm{eig}\big(f(A)\big)$

$A, f(A)$ have the same eigenvectors

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$A, f(A)$ have the same eigenvectors

$$A^k = VD^kV^{-1}$$

$$= \begin{bmatrix} & & \\ & V & \\ & & \end{bmatrix} \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} & & \\ & V^{-1} & \\ & & \end{bmatrix}$$

**Specific Useful Case: Matrix Exponential**

$$e^{At} = I + At + \tfrac{1}{2!}A^2t^2 + \tfrac{1}{3!}A^3t^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \tfrac{1}{k!}A^kt^k$$

**Derivative:** $\frac{d}{dt}\Big(e^{At}\Big) = Ae^{At}$

- can see from polynomial definition
- related to definition of e

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$$

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$$
\begin{aligned}
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&= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}
\end{aligned}
$$

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&= \sum_{k=0}^{\infty} \tfrac{1}{k!}A^k t^k
\end{aligned}
$$

$$
\begin{aligned}
e^{At} &= Ve^{Dt}V^{-1} \\
&= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}
\end{aligned}
$$