

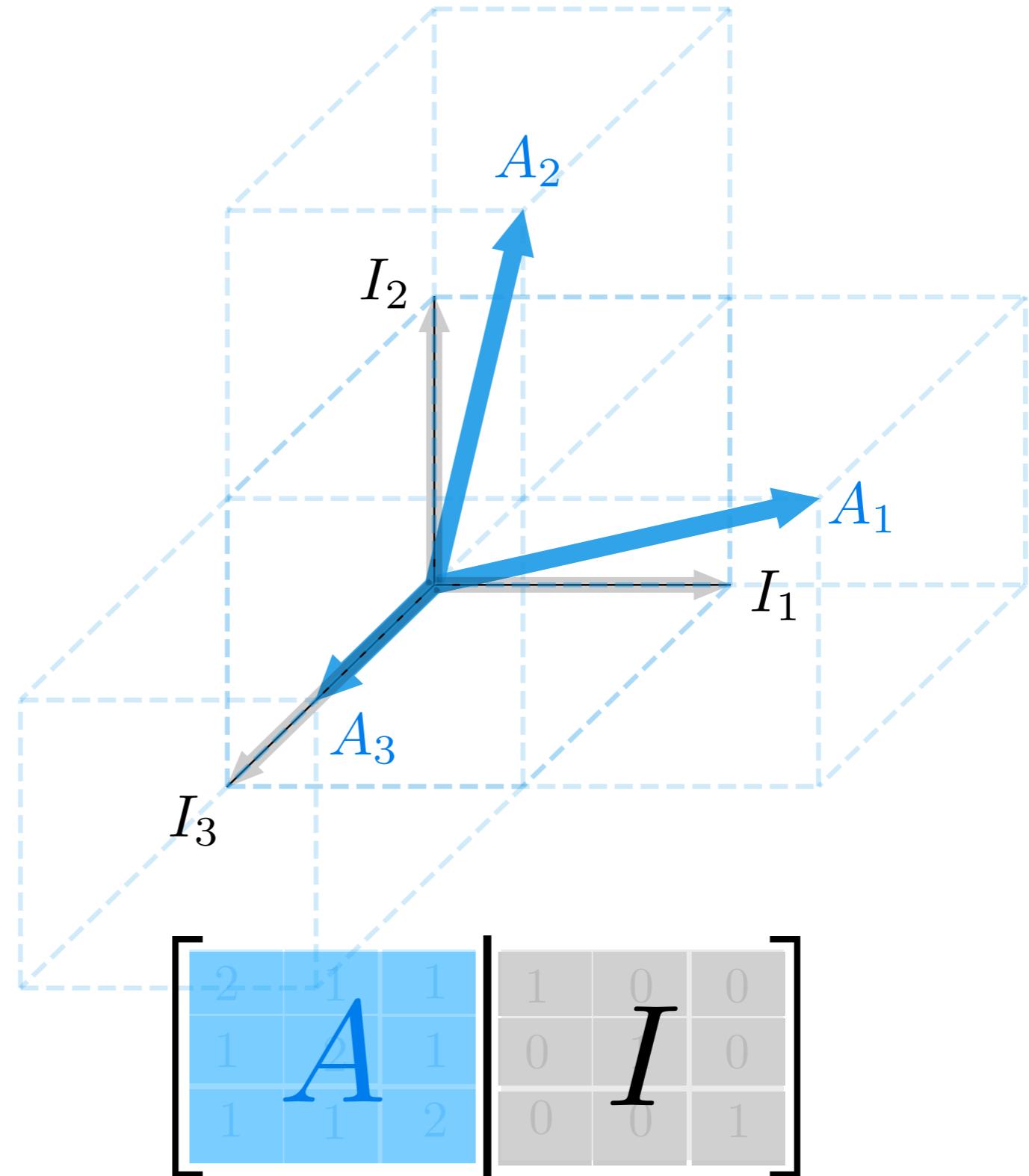
Gaussian Elimination: Elementary Row Operations (column geometry)

Linear Algebra

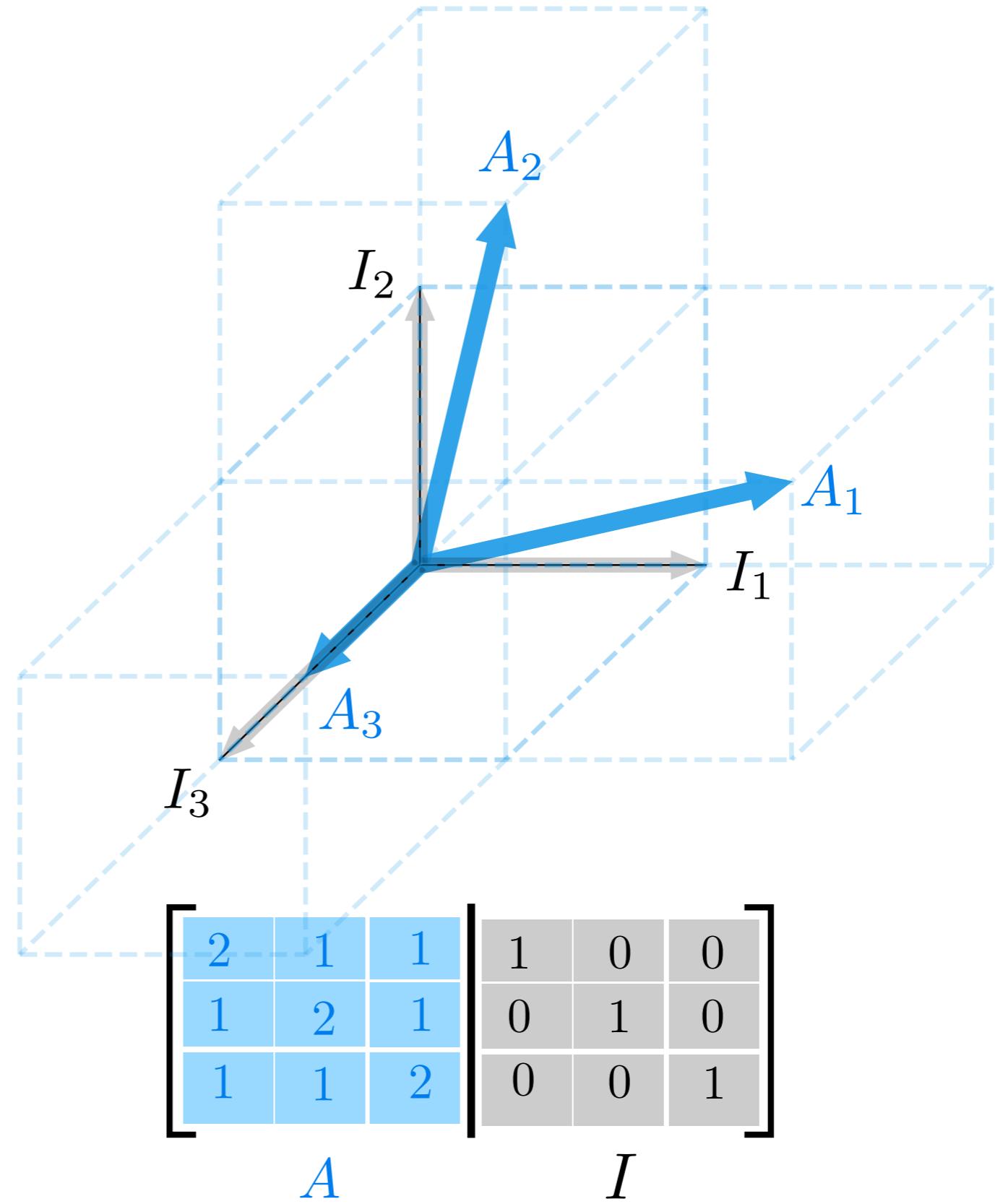
Major sources:

Winter 2022 - Dan Calderone

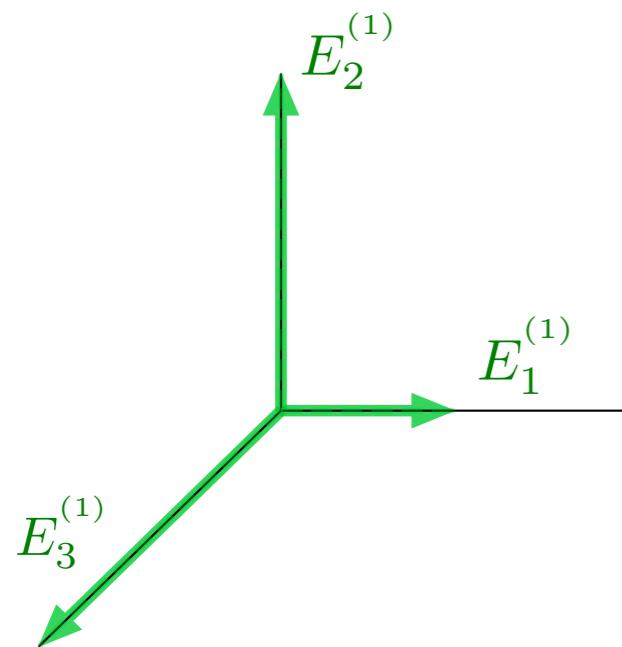
Augmented System



Augmented System



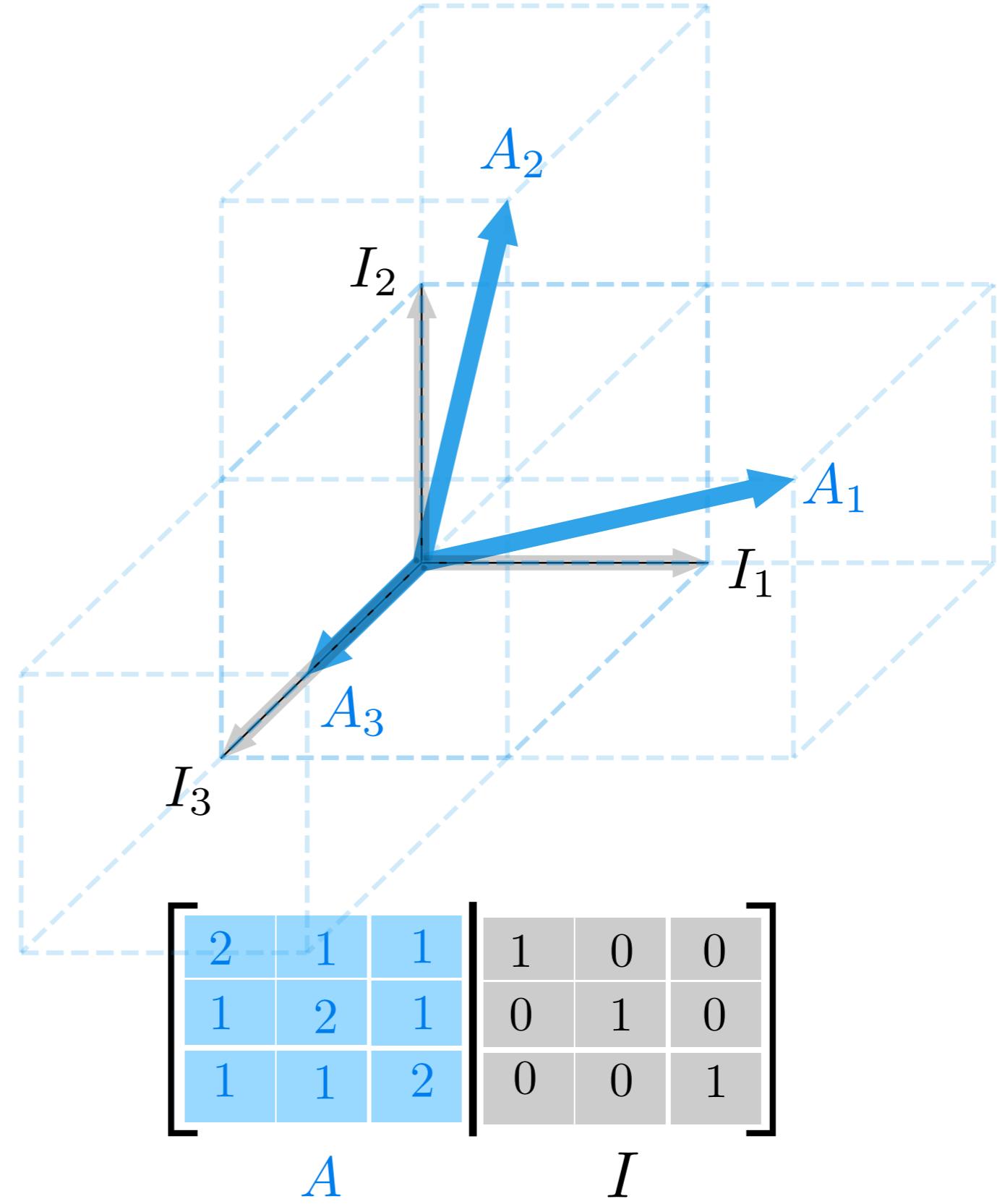
Elementary Matrix



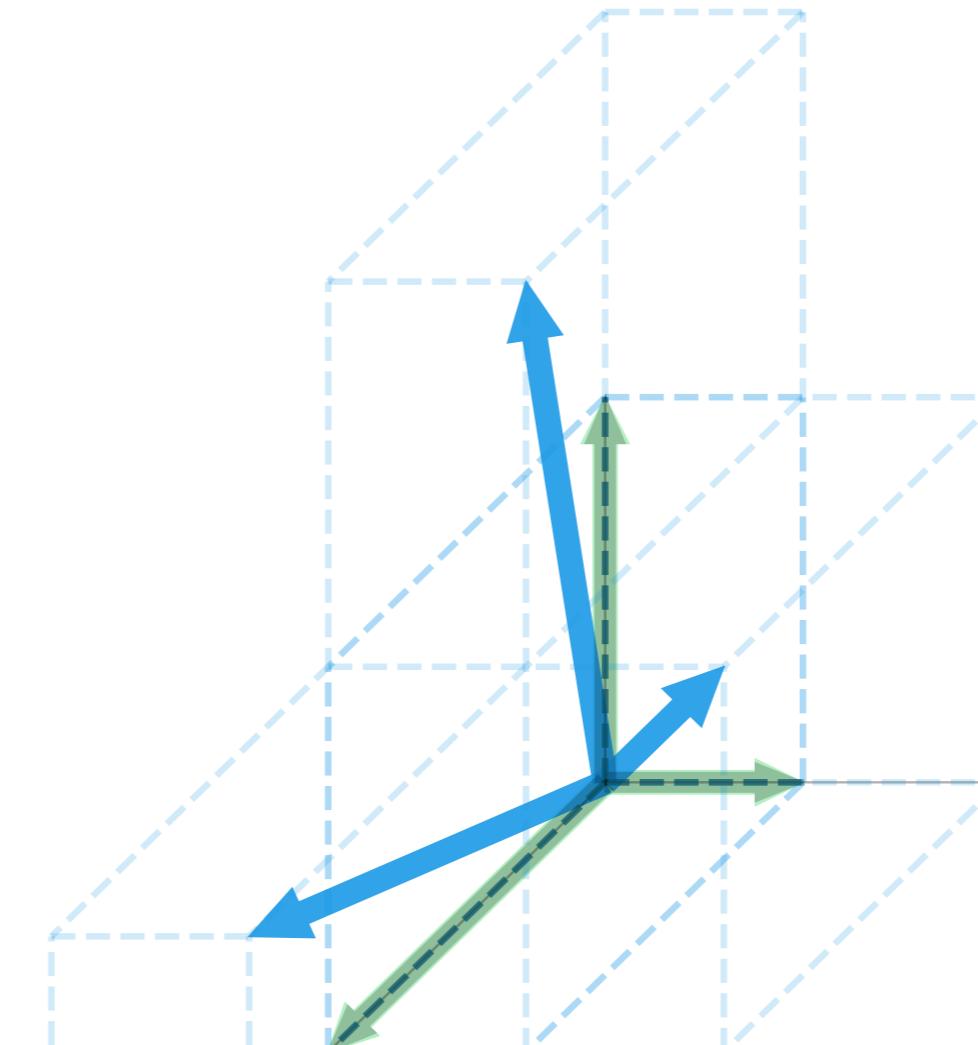
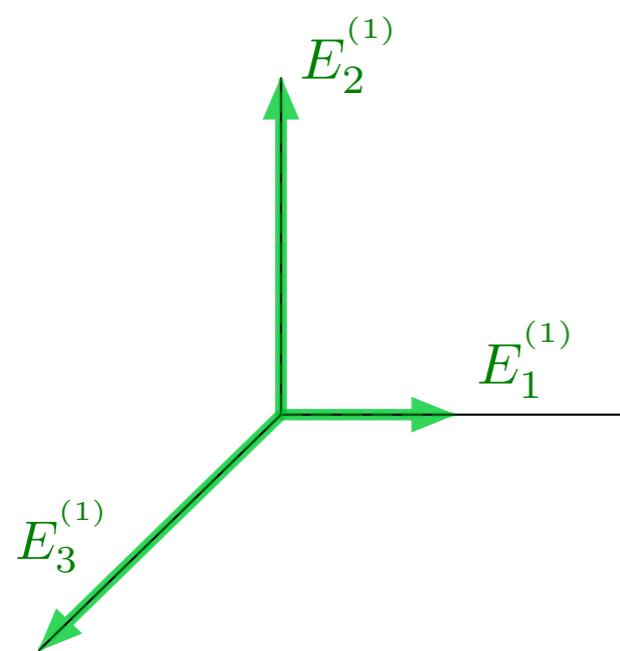
$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(1)}$

\times



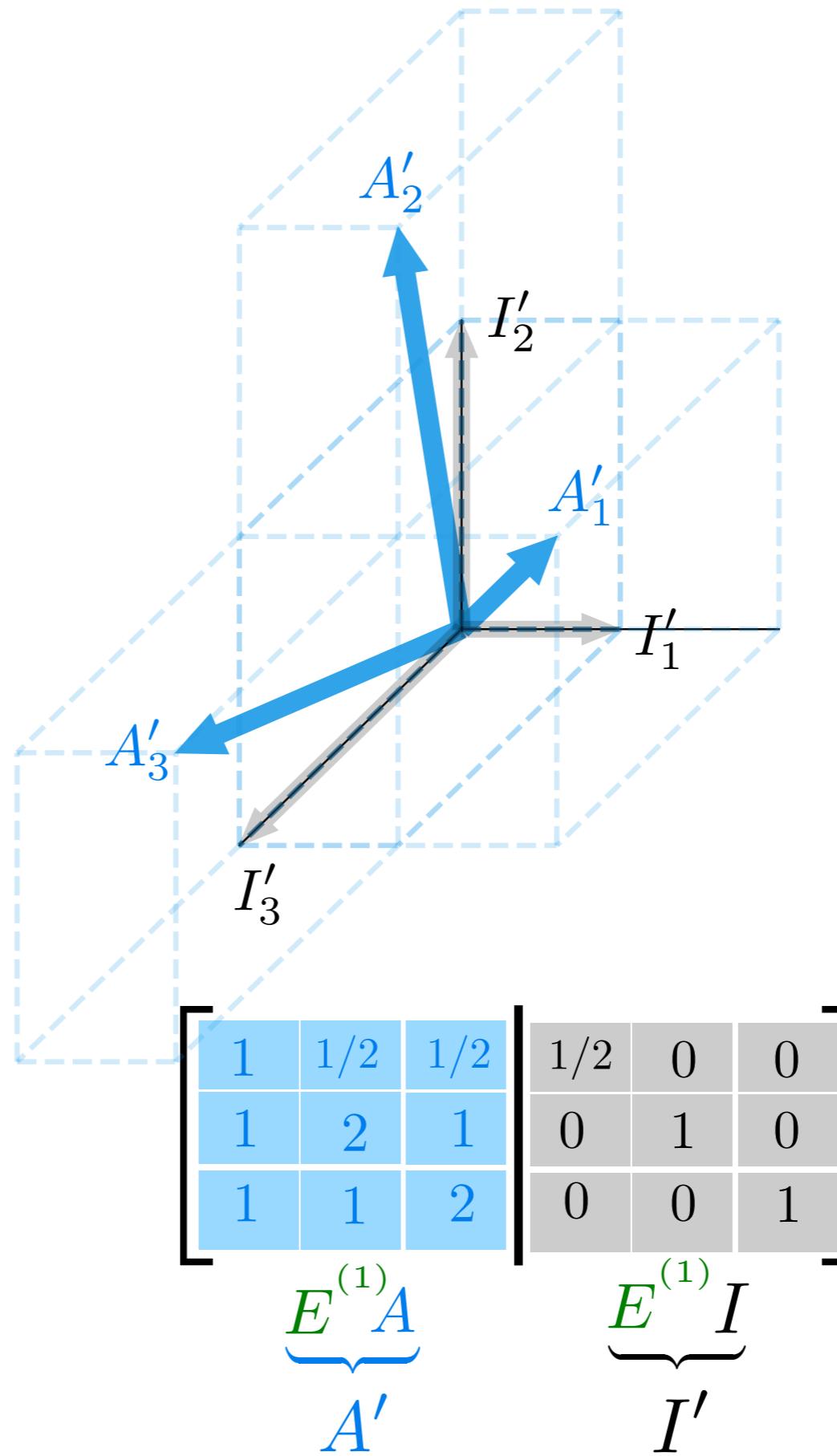
Row-reduction: operation 1...



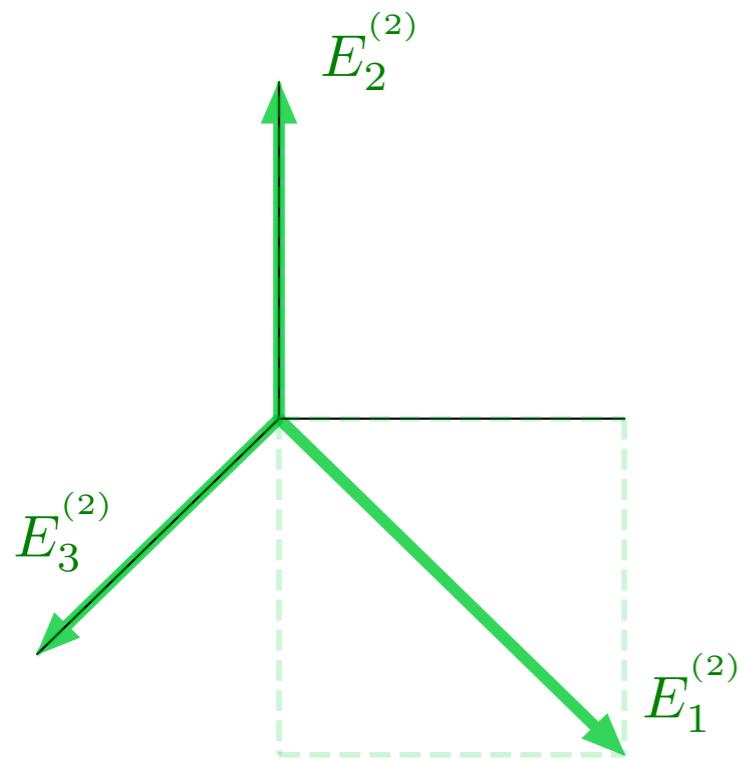
$$\left[\begin{array}{ccc} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

The matrices $E^{(1)}$, A , and I are shown as augmented matrices separated by a vertical bar. The matrix $E^{(1)}$ has green entries in the first two columns and a green identity matrix in the third column. The matrix A has blue entries. The matrix I is the identity matrix.

Row-reduction: operation 1...



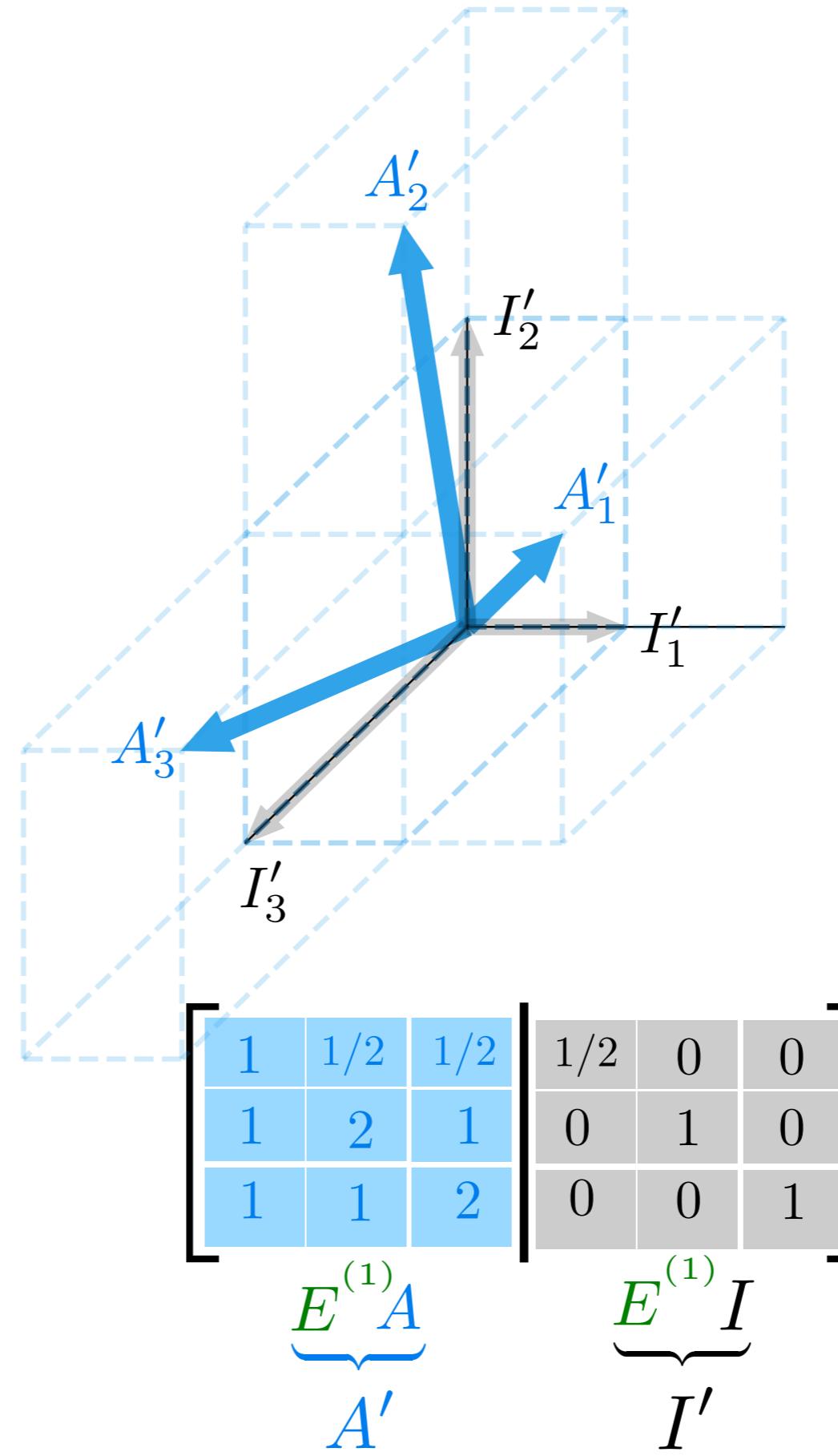
Row-reduction: operation 2...



$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(2)}$

\times



$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(1)} A$

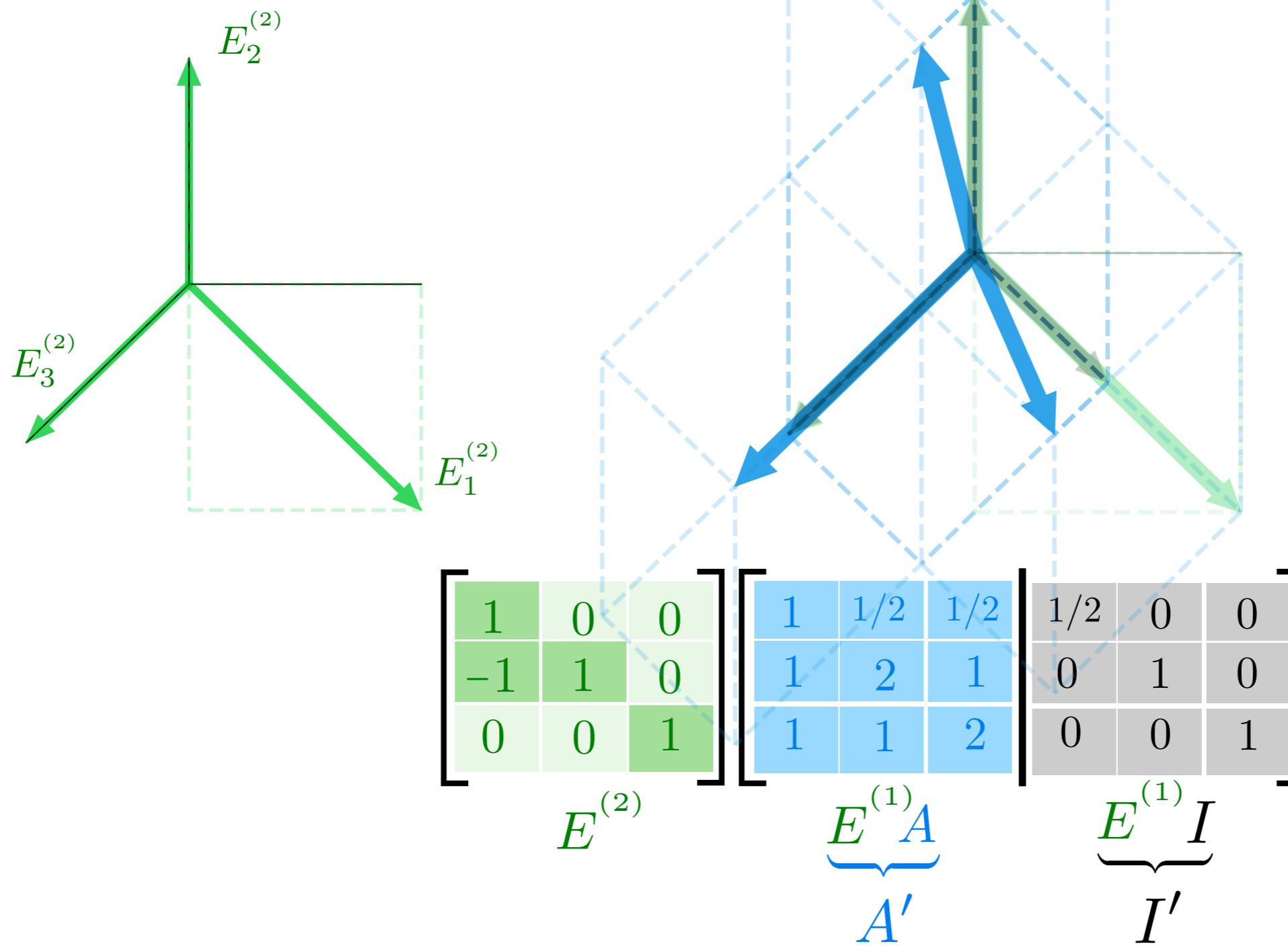
A'

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

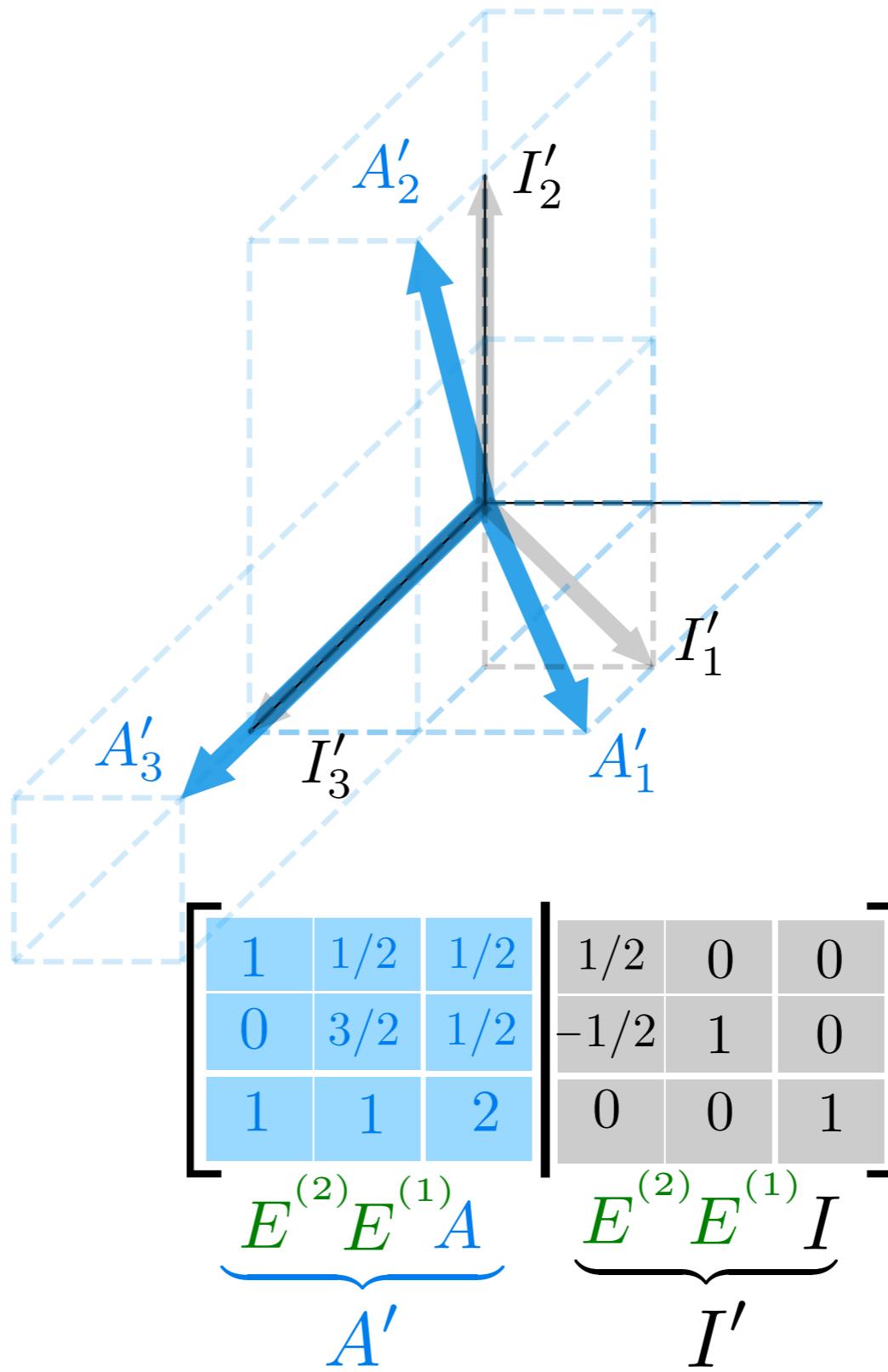
$E^{(1)} I$

I'

Row-reduction: operation 2...



Row-reduction: operation 2...



Row-reduction: operation 3...

Diagram illustrating the row-reduction operation 3... in a 3D space.

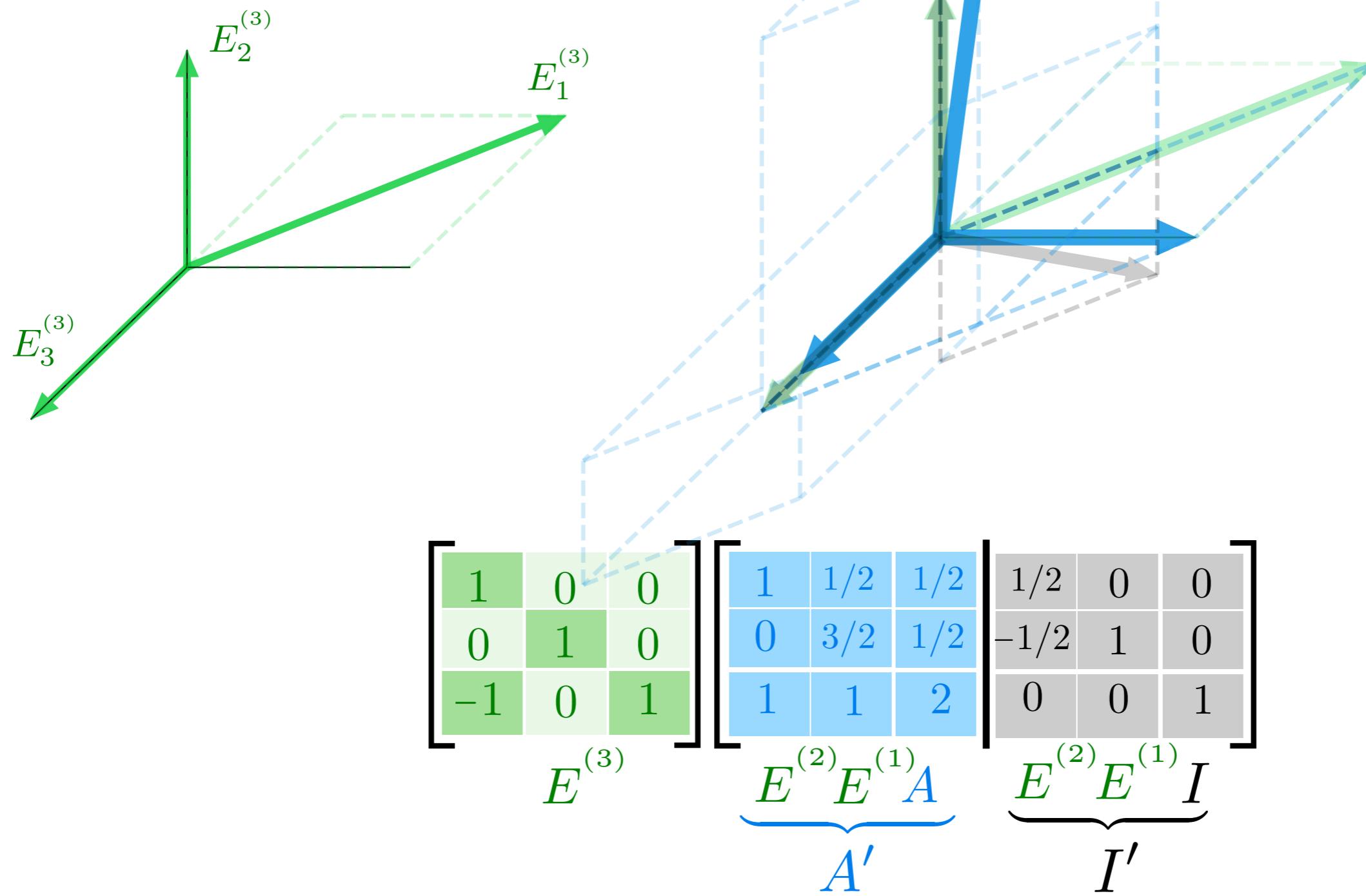
The diagram shows a coordinate system with three green unit vectors $E_1^{(3)}$, $E_2^{(3)}$, and $E_3^{(3)}$ forming a basis. A blue vector A is transformed by a matrix $E^{(2)}E^{(1)}A$ into a vector A' (represented by a blue arrow).

The transformation is shown as a series of projections onto dashed blue planes I'_1 , I'_2 , and I'_3 , which are defined by the columns of $E^{(2)}E^{(1)}I$. The resulting vector A' is shown as a blue arrow originating from the origin.

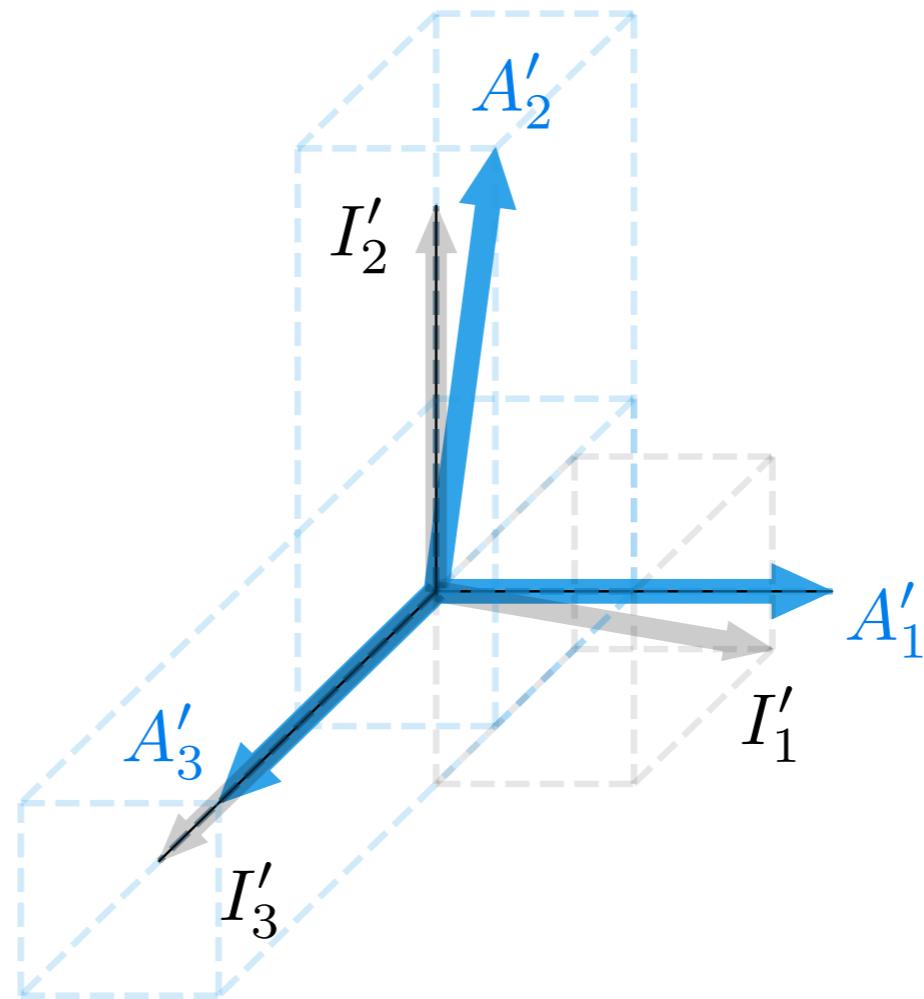
Below the diagram, the matrices are shown:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \times \underbrace{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 1 & 1 & 2 \end{bmatrix}}_{A'} \underbrace{\begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I'}$$

Row-reduction: operation 3...



Row-reduction: operation 3...

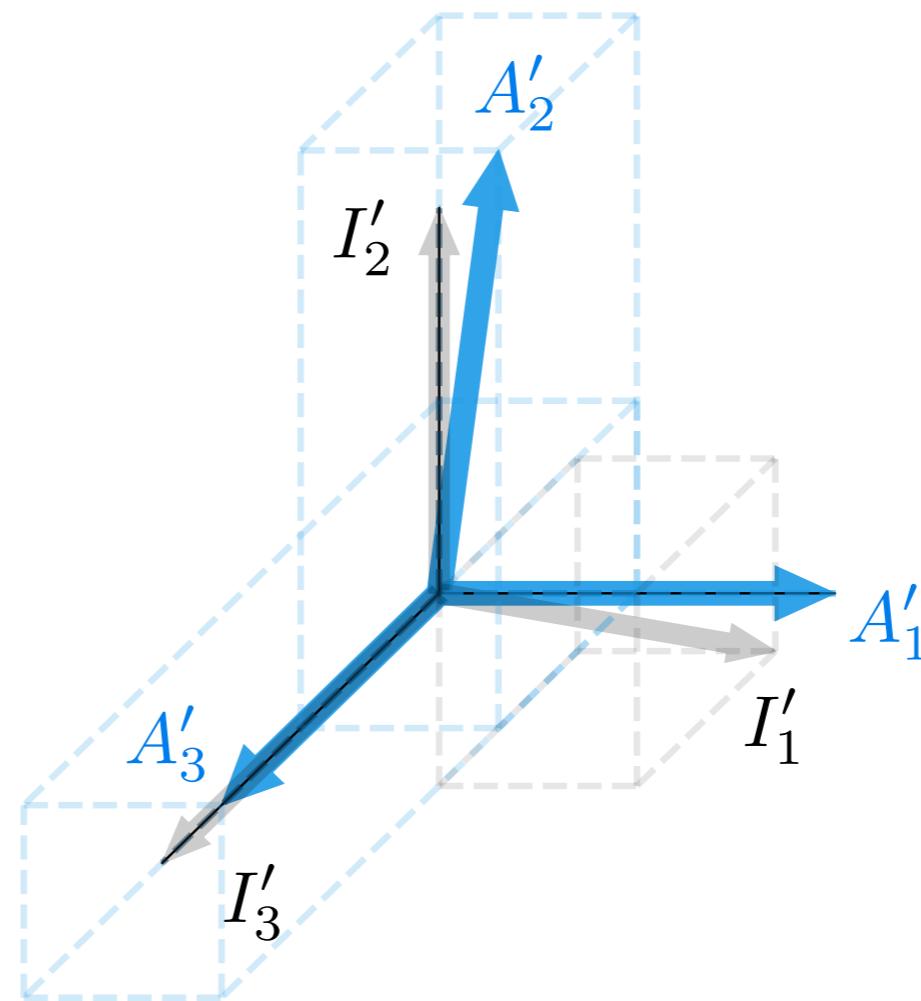
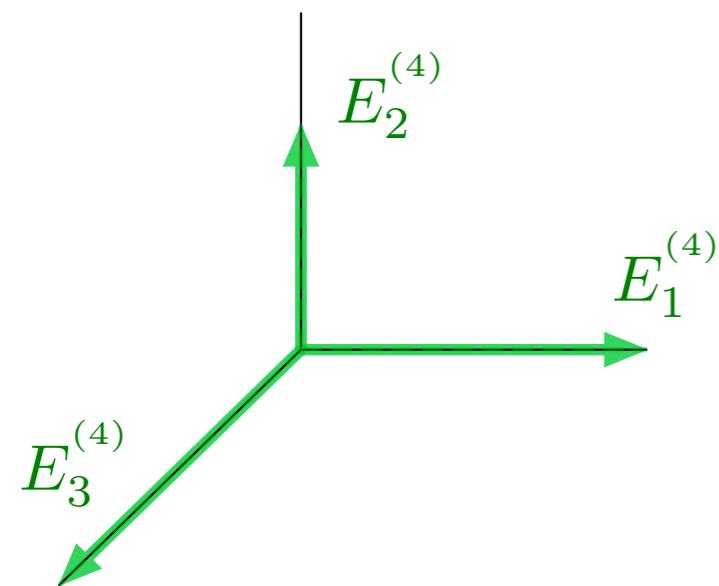


$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{array} \right]$$

$$\underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'}$$

$$\underbrace{E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 4...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(4)}$

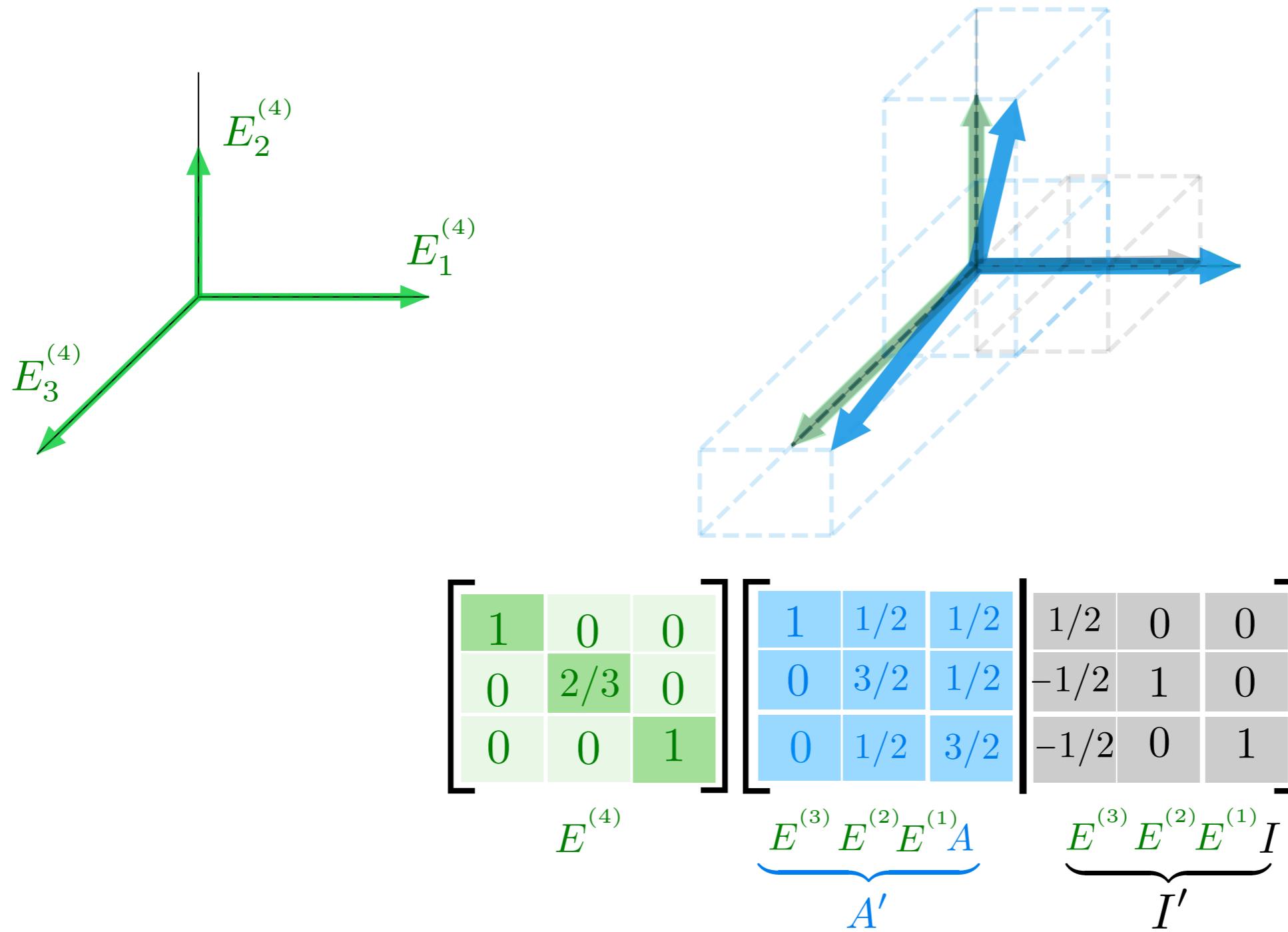
\times

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix}$$

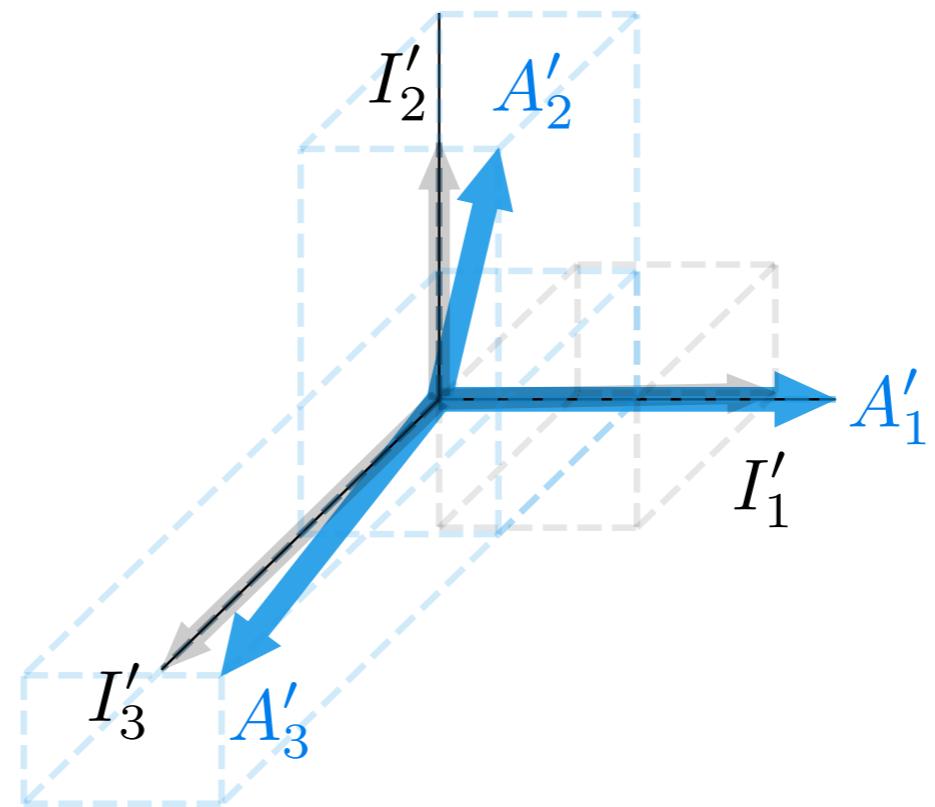
$\underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'}$

$\underbrace{E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

Row-reduction: operation 4...



Row-reduction: operation 4...



$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{array} \right]$$

$$\underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 5...

$E^{(5)}$

I'

A'

$E^{(5)}$

$E^{(4)} E^{(3)} E^{(2)} E^{(1)} A$

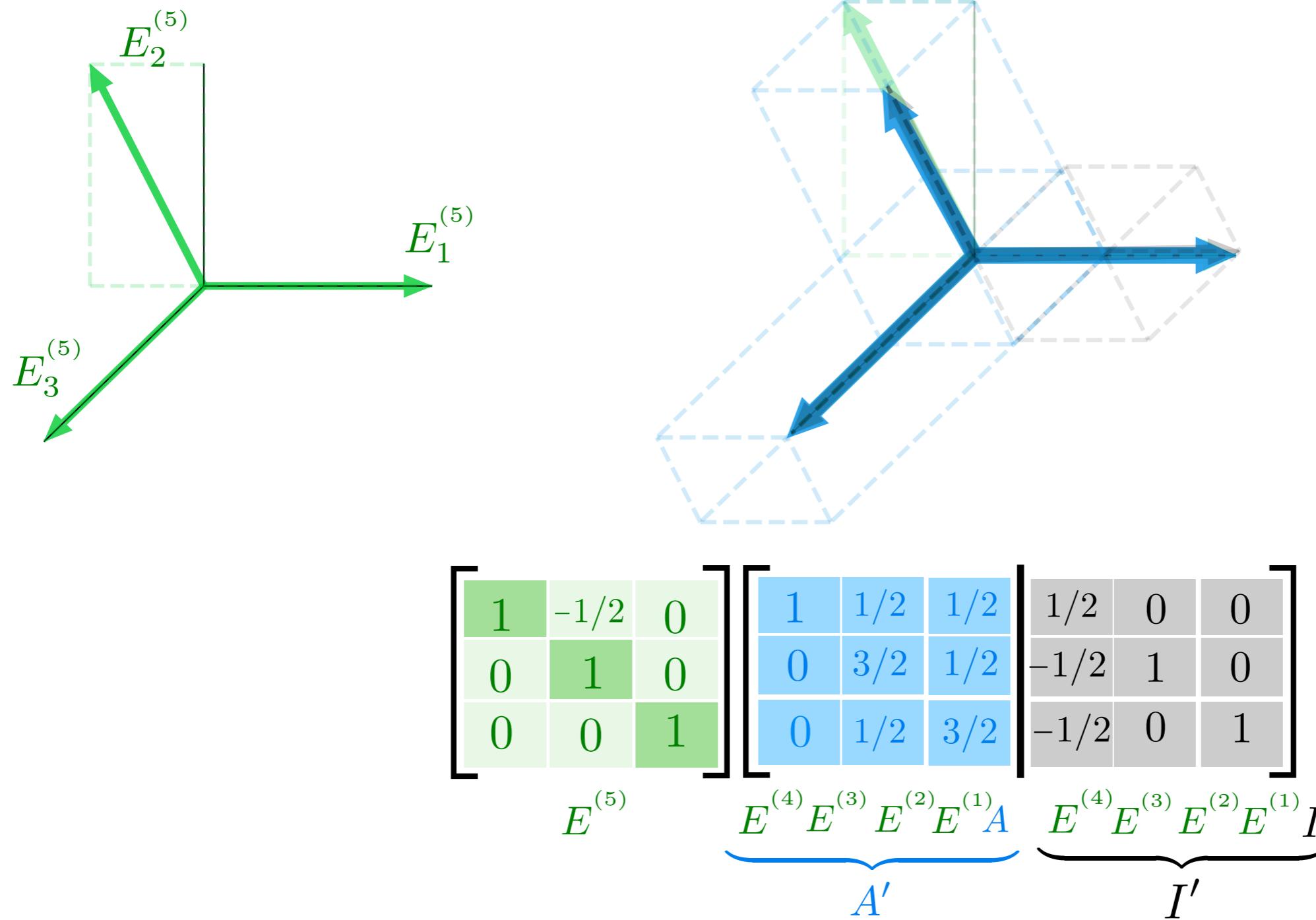
A'

$E^{(4)} E^{(3)} E^{(2)} E^{(1)} I$

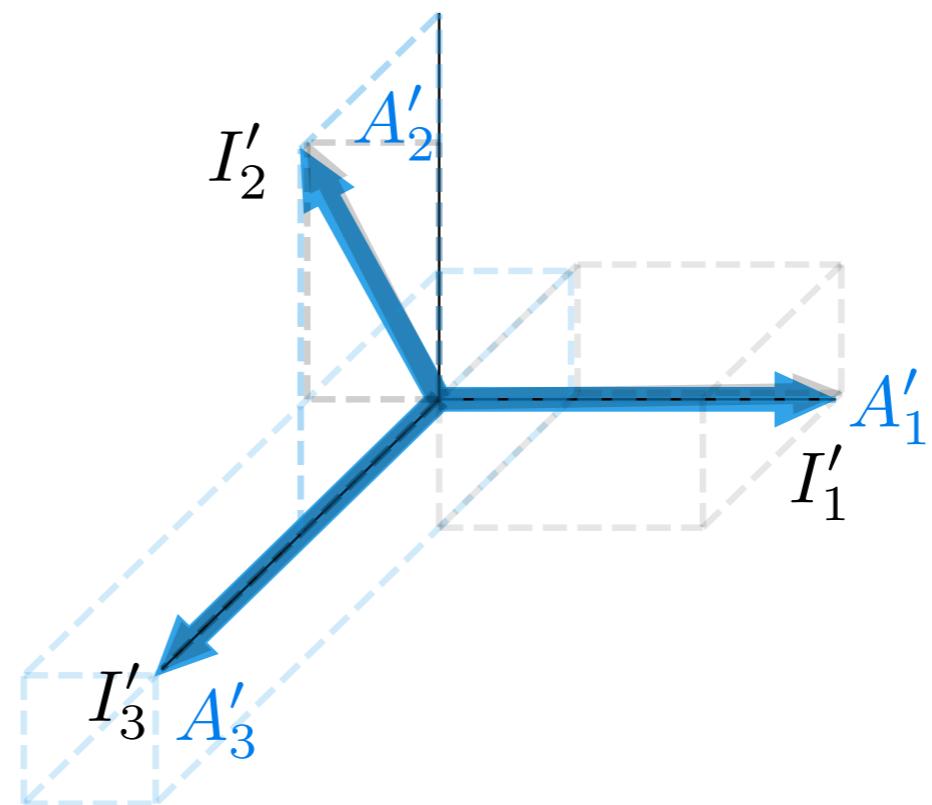
I'

1	-1/2	0	1/2	0	0
0	1	0	0	1	-1/2
0	0	1	1/2	0	-1/2

Row-reduction: operation 5...



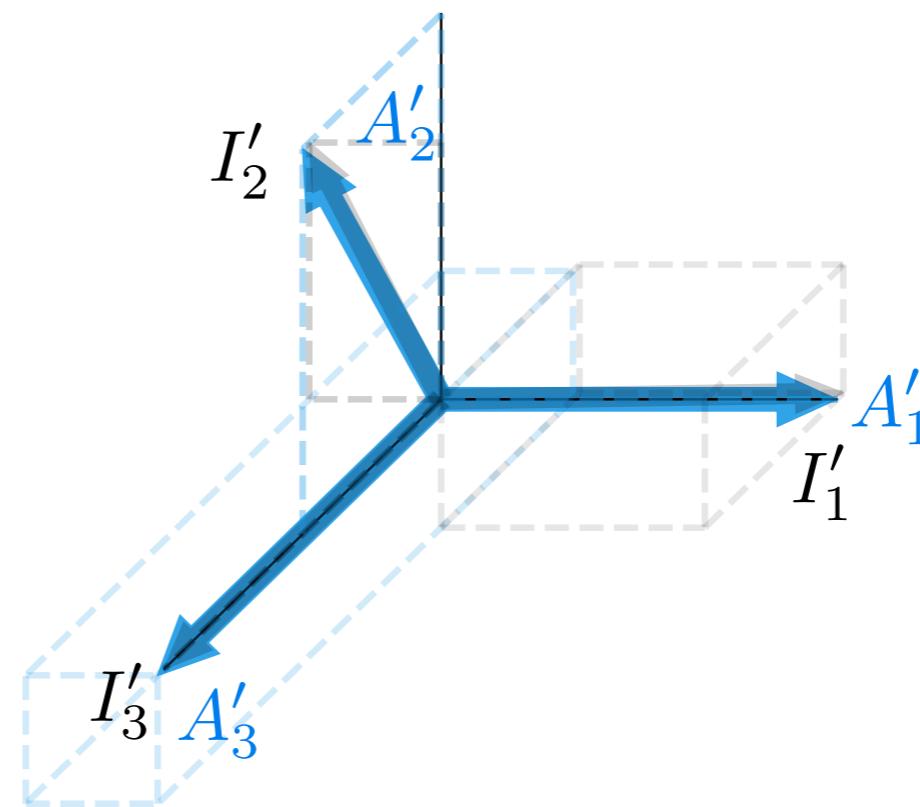
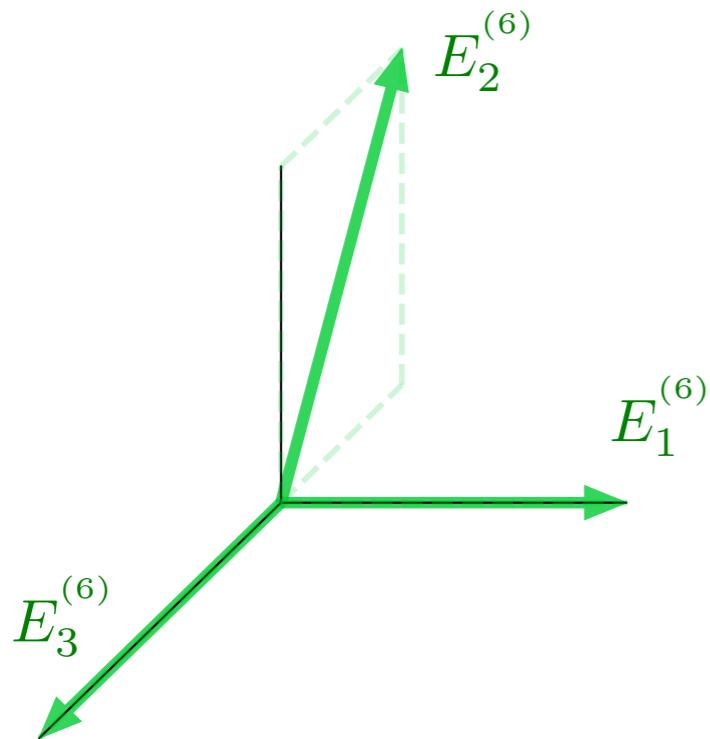
Row-reduction: operation 5...



$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{array} \right]$$

$$\underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 6...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$E^{(6)}$

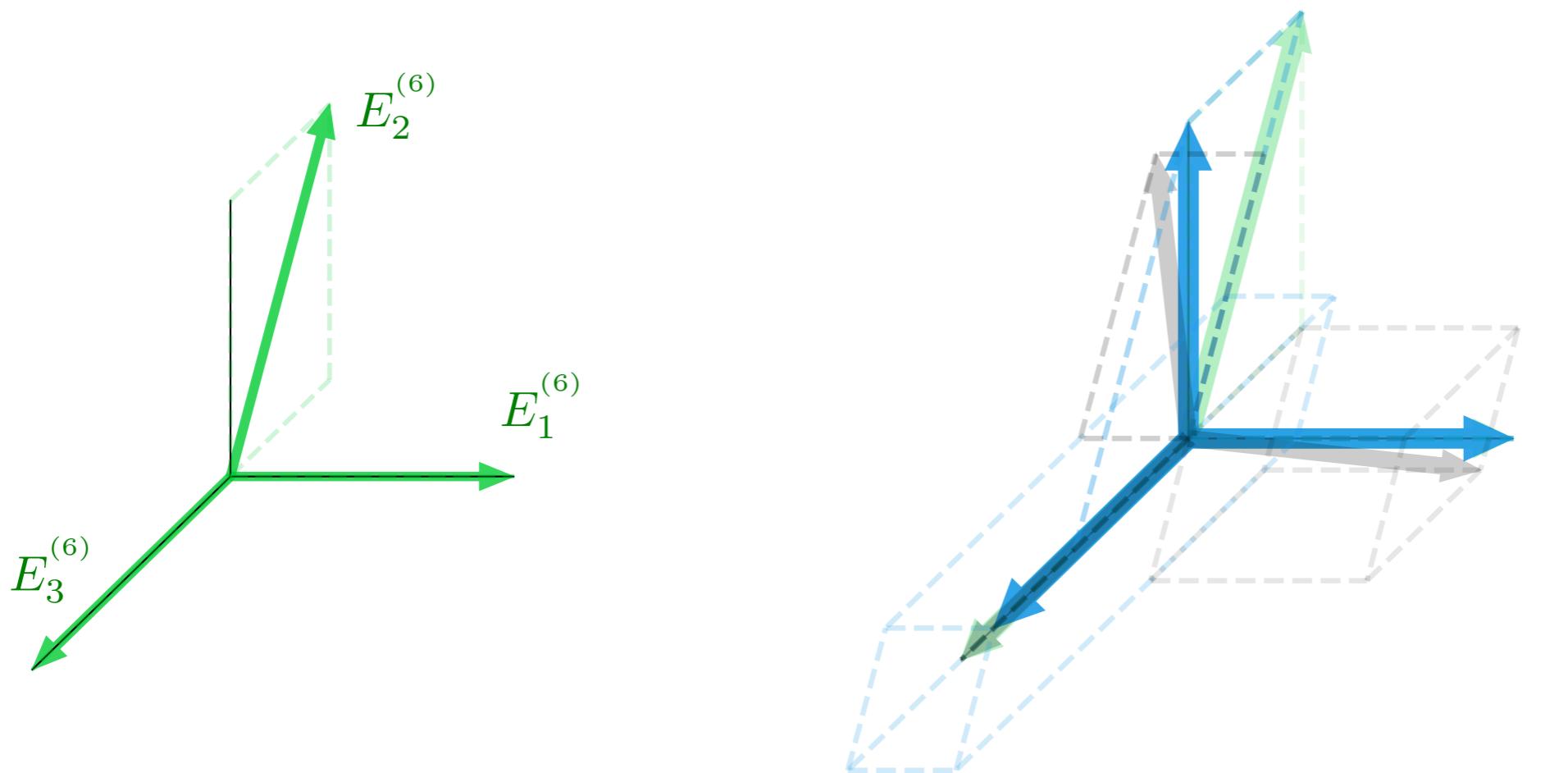
\times

$$\begin{array}{c|ccc|ccc} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{array}$$

$\underbrace{E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A}_{A'}$

$\underbrace{E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

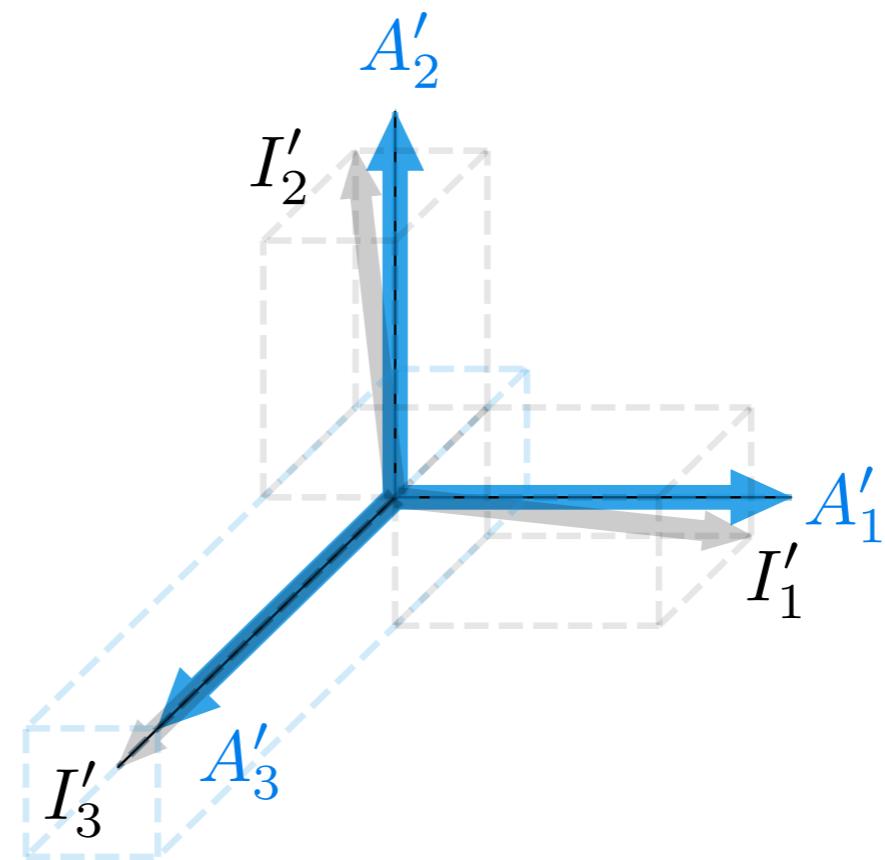
Row-reduction: operation 6...



$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{array} \right] \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 1/2 & 3/2 & 2/3 \end{array} \right]$$

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 1/2 & 3/2 \end{array} \right]}_{A'} \quad \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1 \end{array} \right]}_{I'}$$

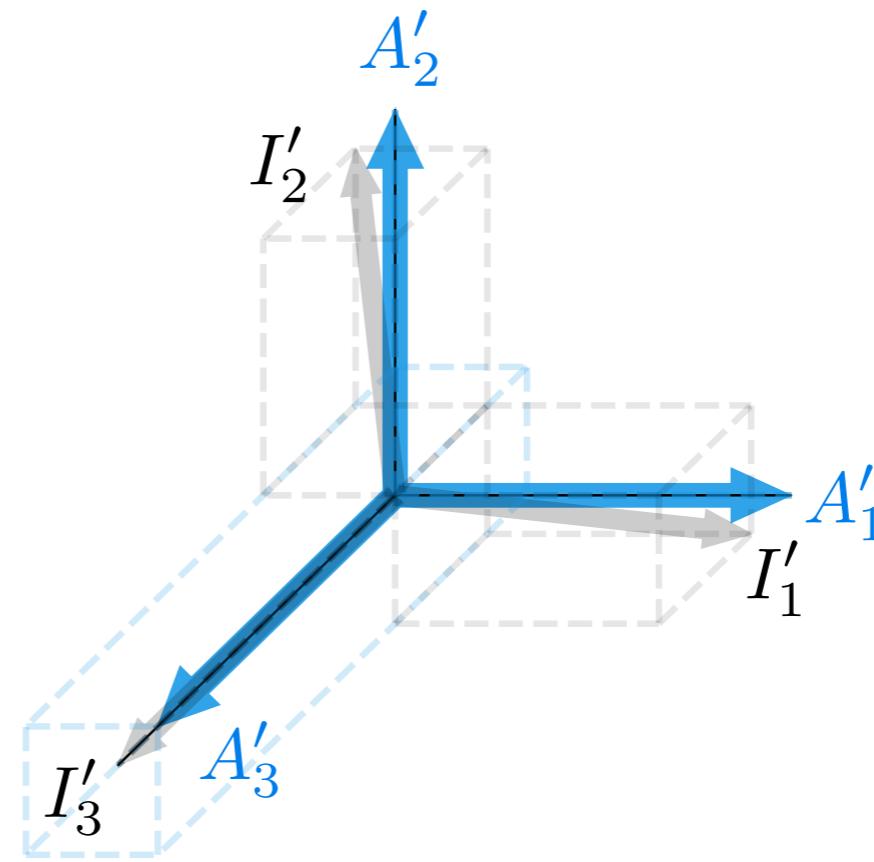
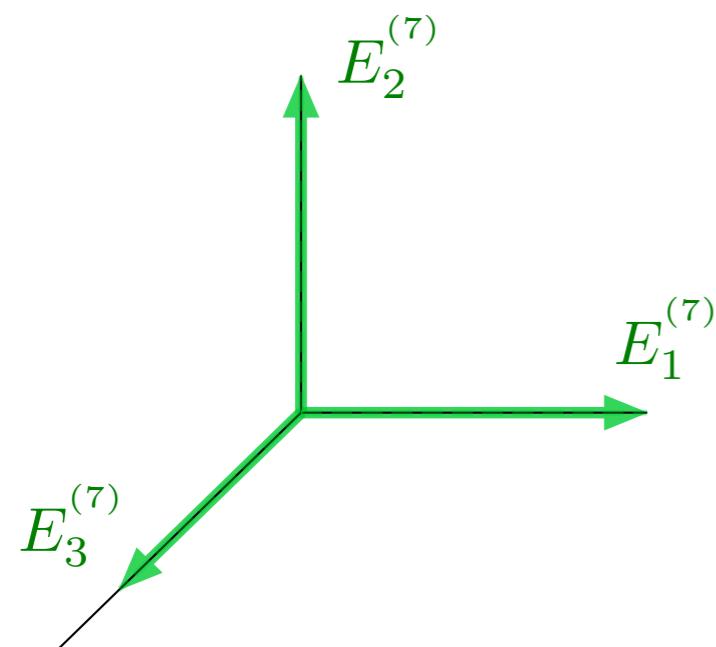
Row-reduction: operation 6...



$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{array} \right]$$

$$\underbrace{E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 7...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$E^{(7)}$

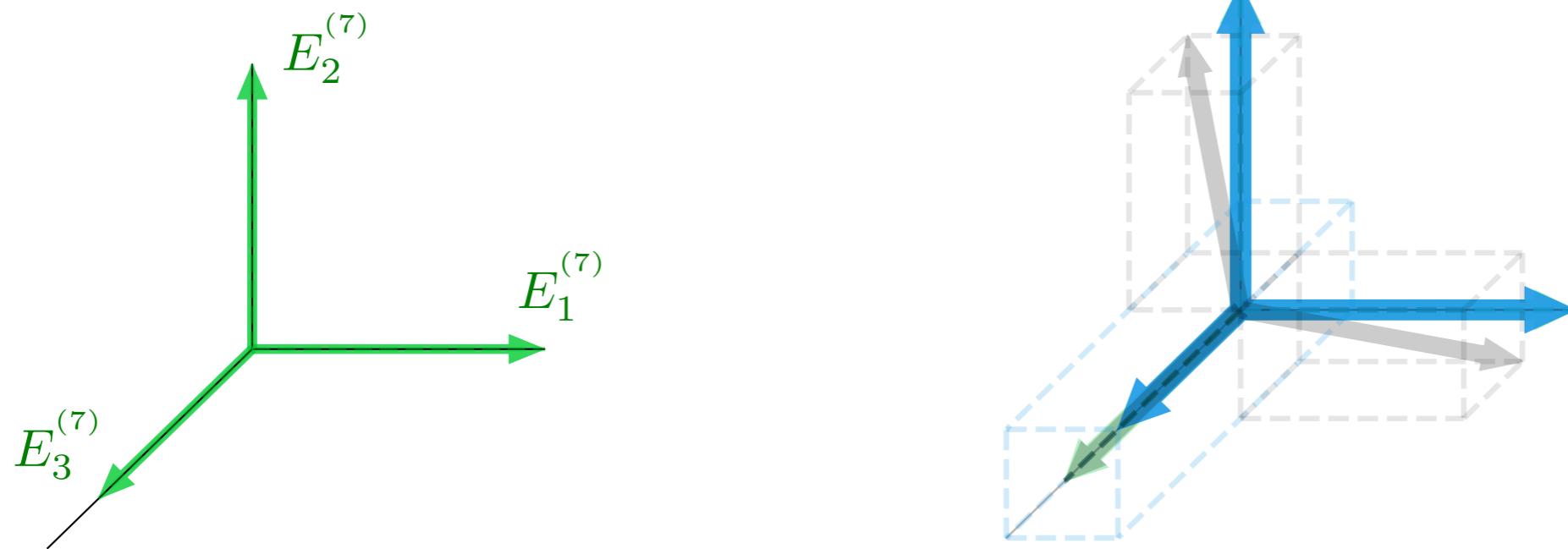
\times

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{array} \right]$$

$\underbrace{E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A}_{A'}$

$\underbrace{E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

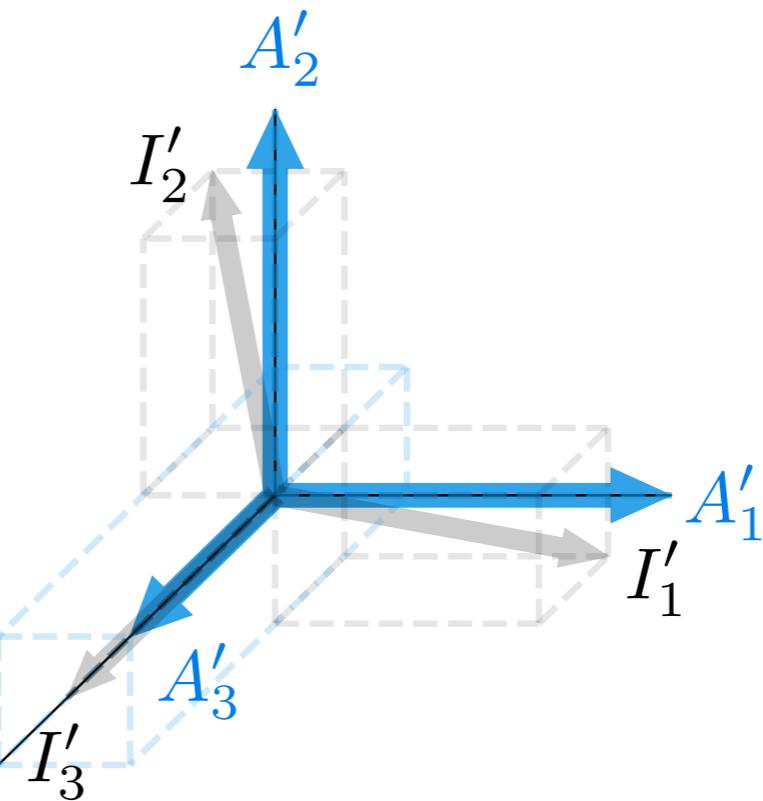
Row-reduction: operation 7...



$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array} \right] \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 4/3 & -1/3 \end{array} \right]$$

$E^{(7)}$
 $\underbrace{E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A}_{A'}$
 $\underbrace{E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

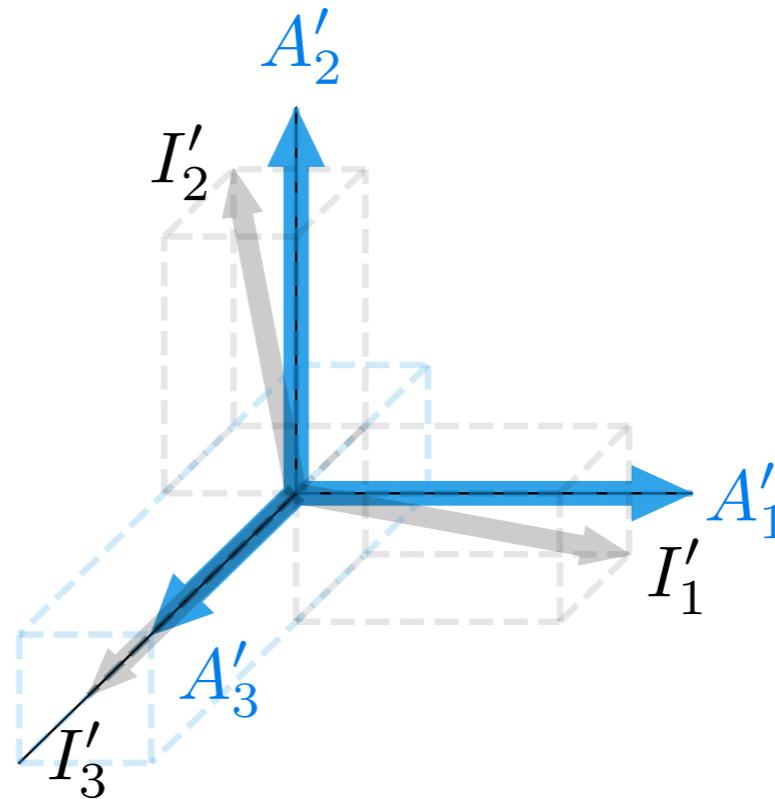
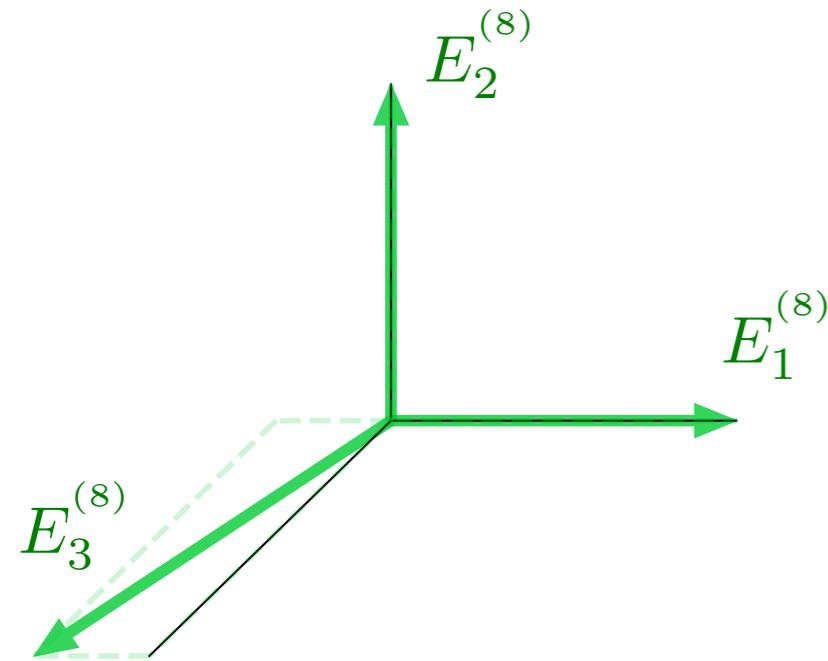
Row-reduction: operation 7...



$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array} \right]$$

$$\underbrace{E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 8...



$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(8)}$

\times

$$\begin{bmatrix} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

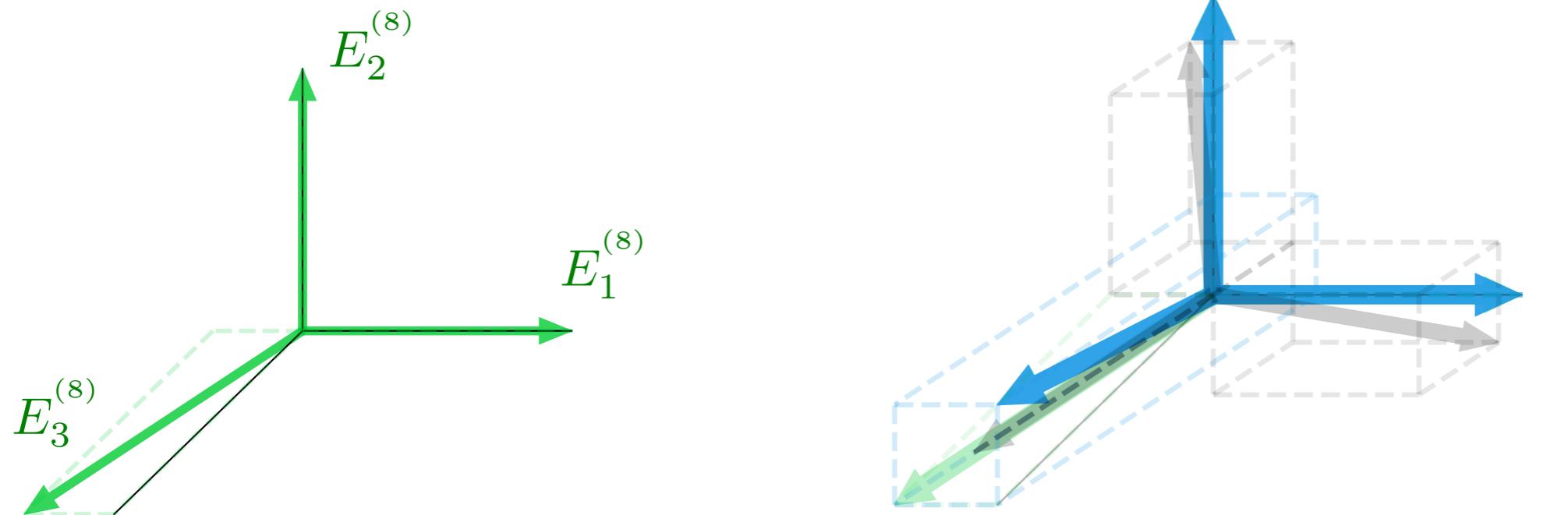
$E^{(7)} E^{(6)} E^{(5)} E^{(4)}$ $E^{(7)} E^{(6)} E^{(5)} E^{(4)}$

$\underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'}$

$E^{(3)} E^{(2)} E^{(1)} I$

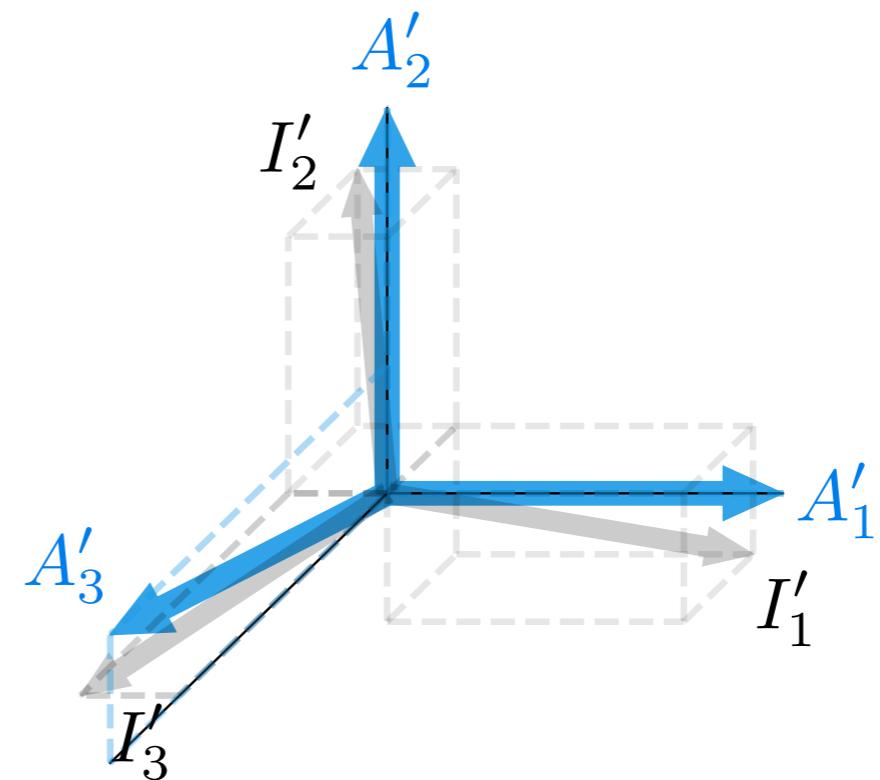
I'

Row-reduction: operation 8...



$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{(7)} E^{(6)} E^{(5)} E^{(4)} \underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'}} \underbrace{\begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}}_{E^{(7)} E^{(6)} E^{(5)} E^{(4)} \underbrace{E^{(3)} E^{(2)} E^{(1)} I}_{I'}}$$

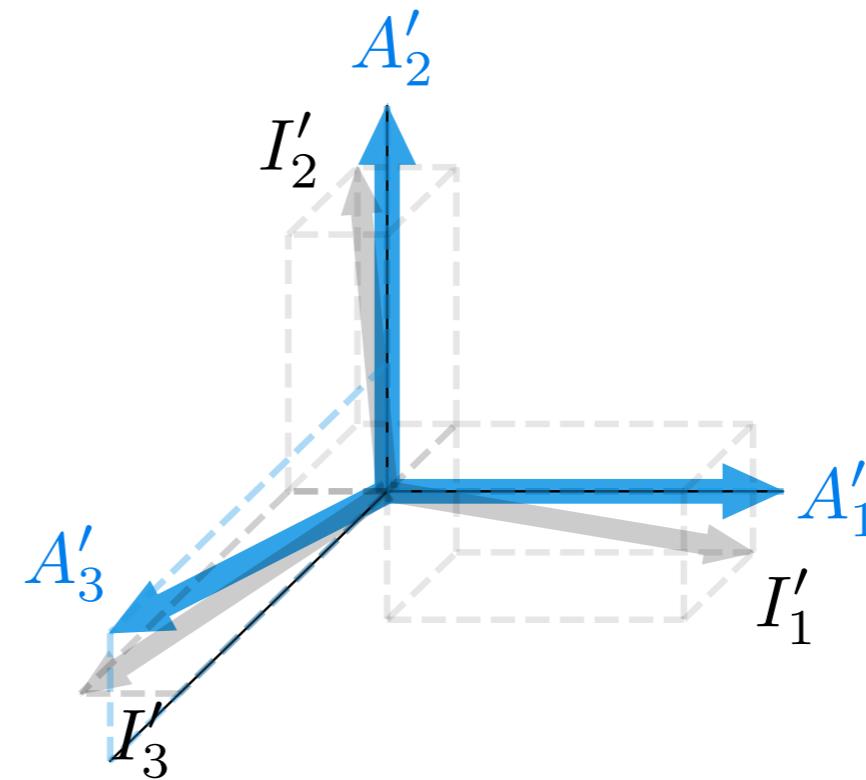
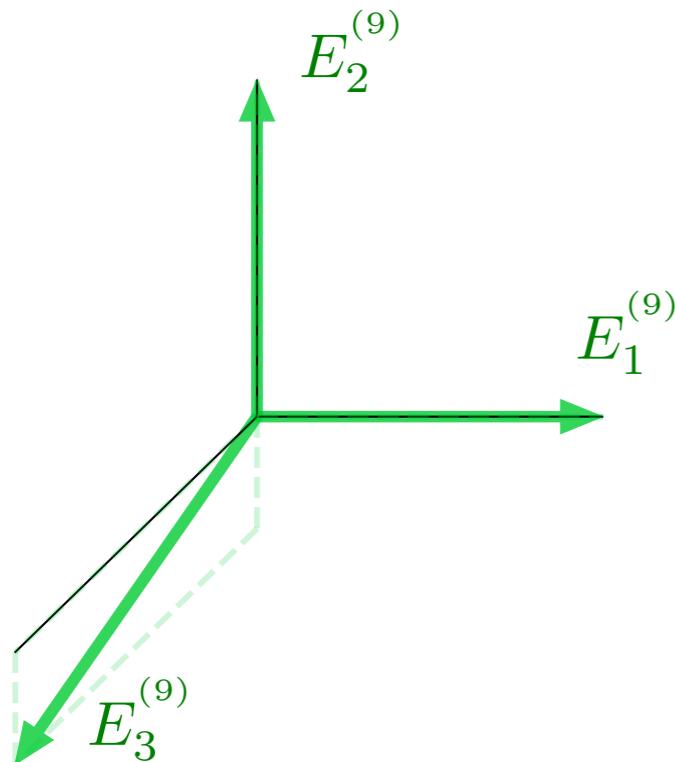
Row-reduction: operation 8...



$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array} \right]$$

$$\underbrace{\begin{matrix} E^{(8)}E^{(7)} \\ E^{(6)}E^{(5)}E^{(4)} \\ E^{(3)}E^{(2)}E^{(1)}A \end{matrix}}_{A'} \quad \underbrace{\begin{matrix} E^{(8)}E^{(7)} \\ E^{(6)}E^{(5)}E^{(4)} \\ E^{(3)}E^{(2)}E^{(1)}I \end{matrix}}_{I'}$$

Row-reduction: operation 9...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{(9)}$$

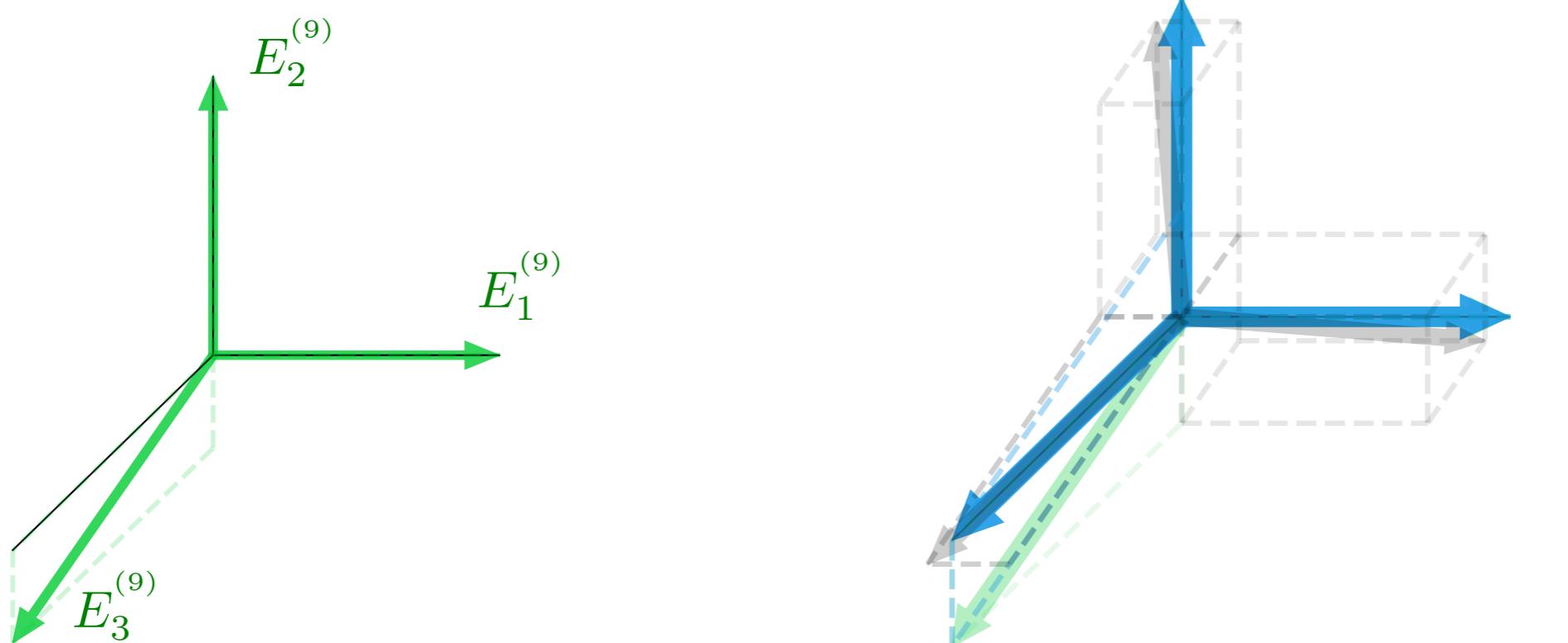
\times

$$\begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$E^{(8)} E^{(7)}$
 $E^{(6)} E^{(5)} E^{(4)}$
 $E^{(3)} E^{(2)} E^{(1)} A$

$E^{(8)} E^{(7)}$
 $E^{(6)} E^{(5)} E^{(4)}$
 $E^{(3)} E^{(2)} E^{(1)} I$

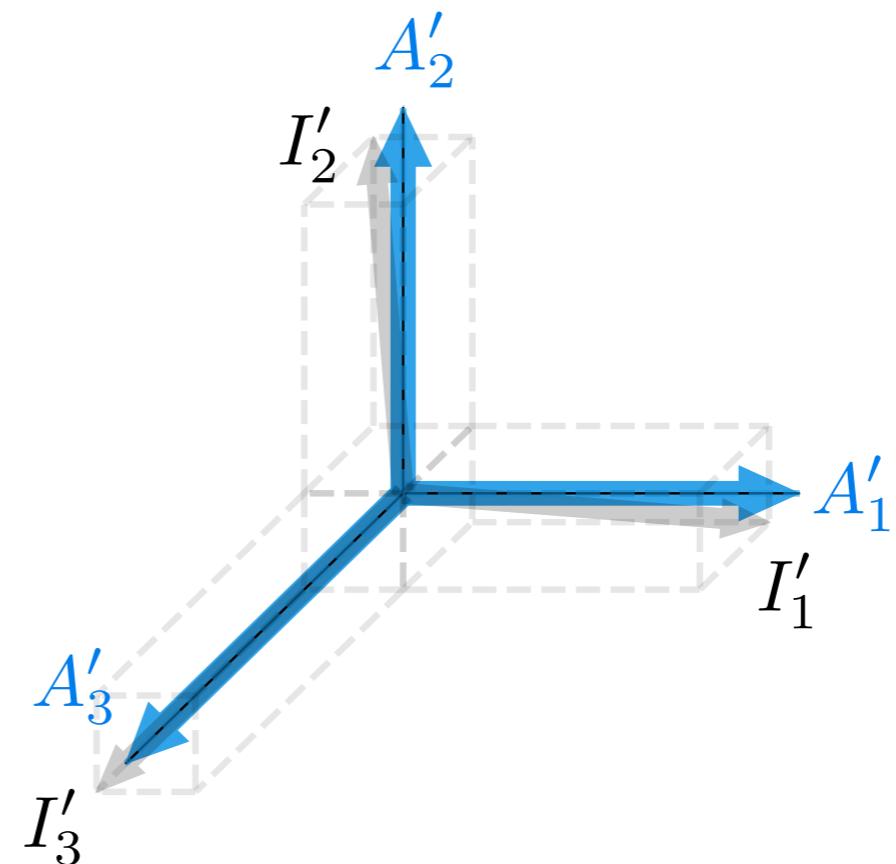
Row-reduction: operation 9...



$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 1 & -1/4 \end{array} \right]$$

$$\underbrace{\begin{array}{c} E^{(8)} \\ E^{(6)} \\ E^{(3)} \end{array}}_{A'} \underbrace{\begin{array}{c} E^{(7)} \\ E^{(5)} \\ E^{(2)} \\ E^{(1)} \end{array}}_A \quad
 \underbrace{\begin{array}{c} E^{(8)} \\ E^{(6)} \\ E^{(3)} \end{array}}_{I'} \underbrace{\begin{array}{c} E^{(7)} \\ E^{(5)} \\ E^{(2)} \\ E^{(1)} \end{array}}_I$$

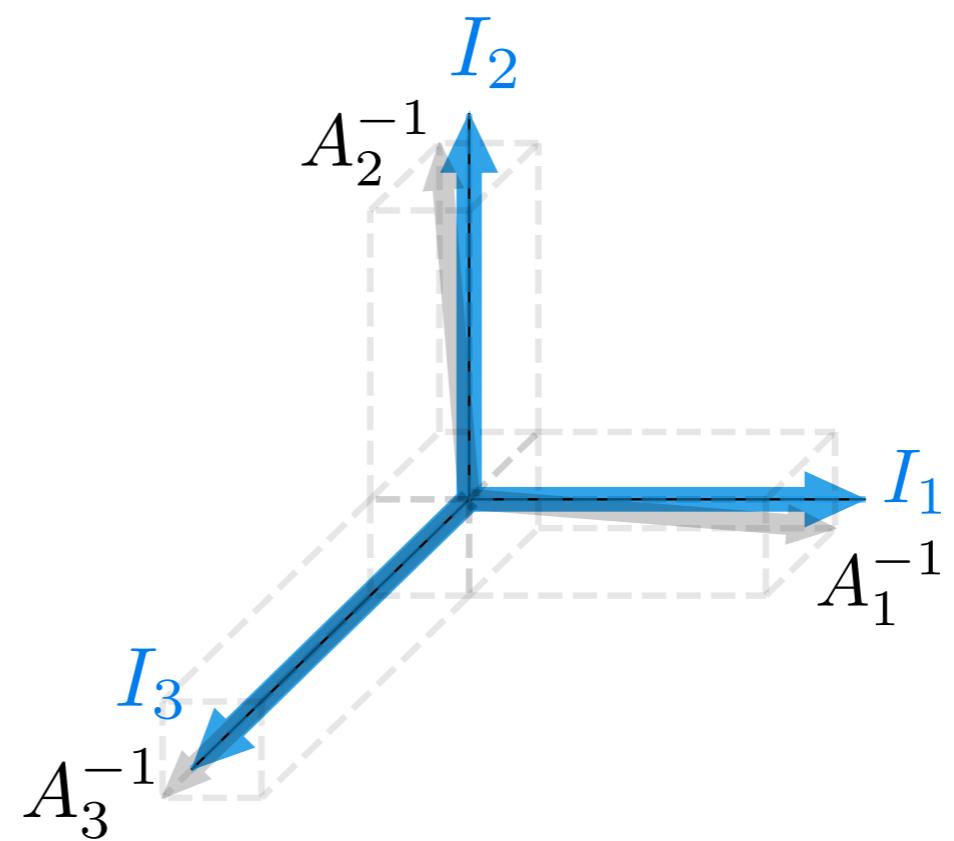
Row-reduction: operation 9...



final
system...

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array} \right]$$

$$\underbrace{\begin{matrix} E^{(9)} & E^{(8)} & E^{(7)} \\ E^{(6)} & E^{(5)} & E^{(4)} \\ E^{(3)} & E^{(2)} & E^{(1)} \end{matrix}}_{A'} \underbrace{\begin{matrix} E^{(9)} & E^{(8)} & E^{(7)} \\ E^{(6)} & E^{(5)} & E^{(4)} \\ E^{(3)} & E^{(2)} & E^{(1)} \end{matrix}}_I$$



final
system...

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array} \right] \xrightarrow{\quad A^{-1} \quad}$$