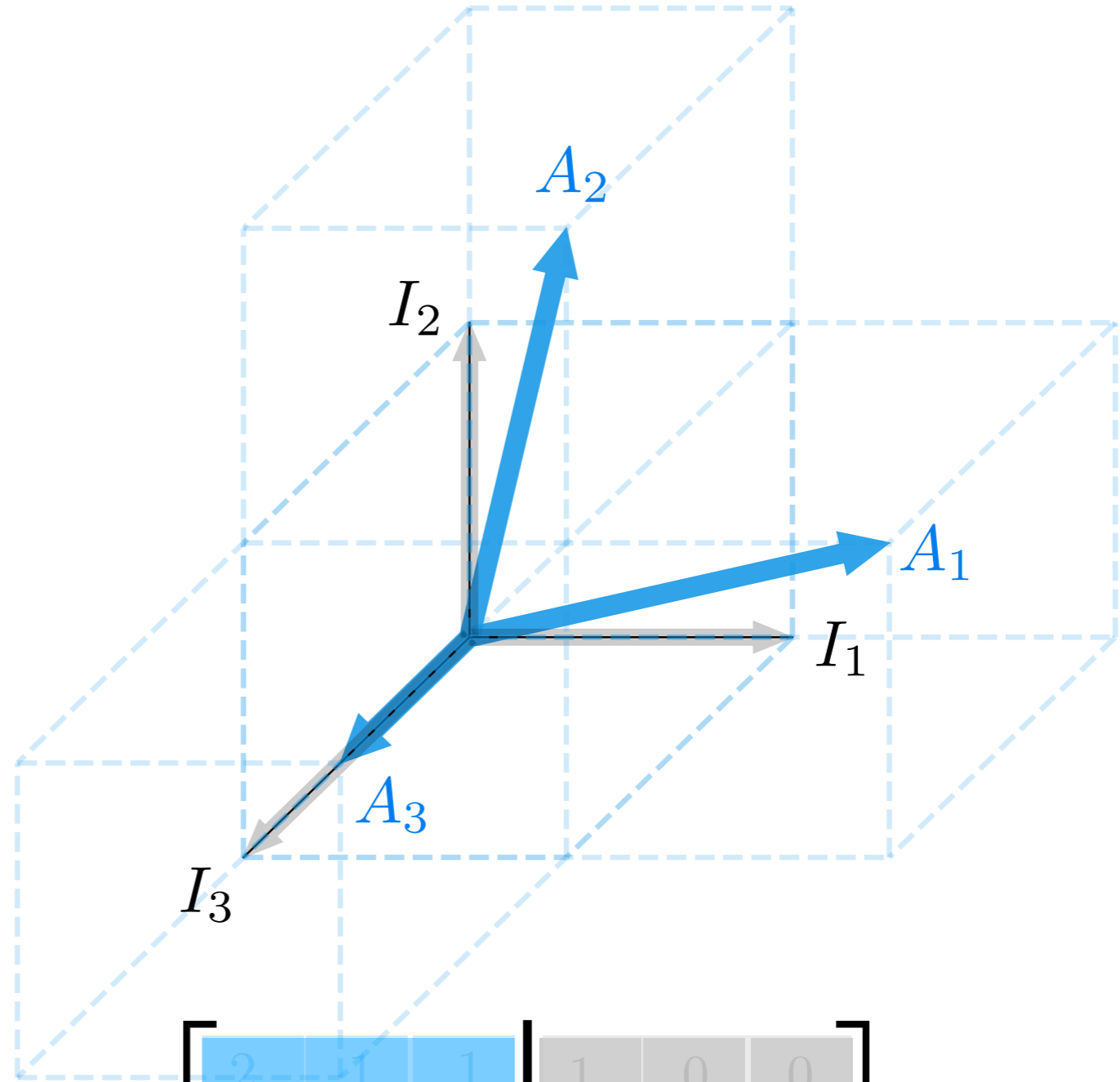


Gaussian Elimination: Elementary Row Operations (column geometry)

Linear Algebra

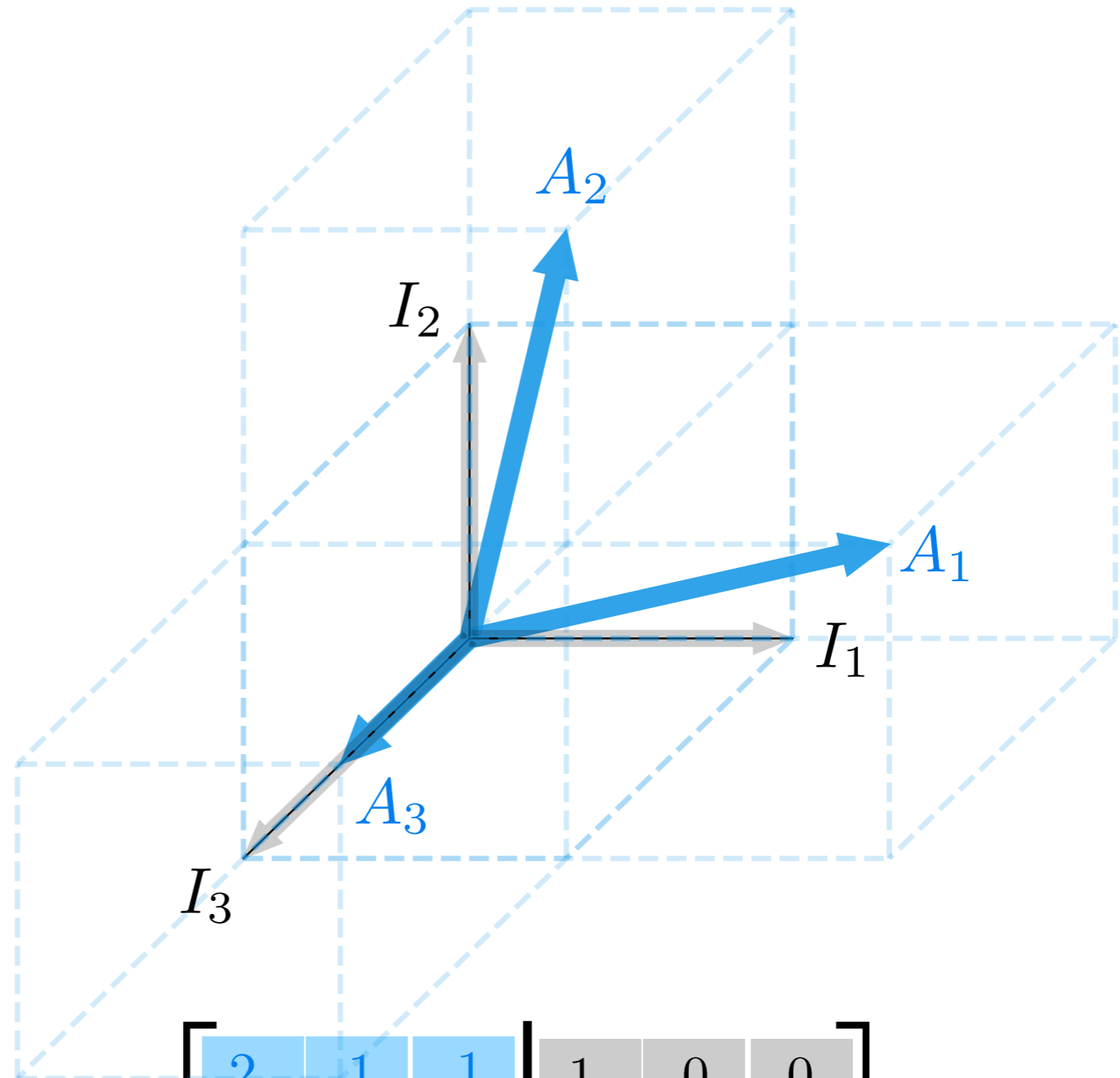
Major sources:

Winter 2022 - Dan Calderone



Augmented
System

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

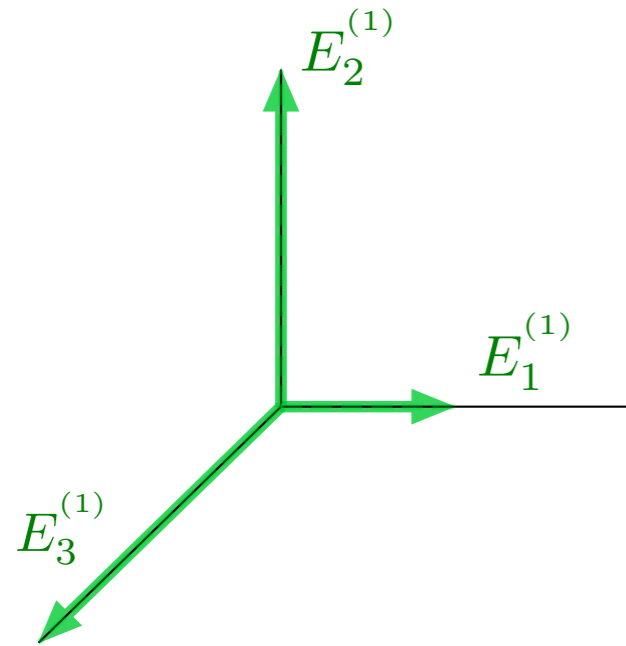


**Augmented
System**

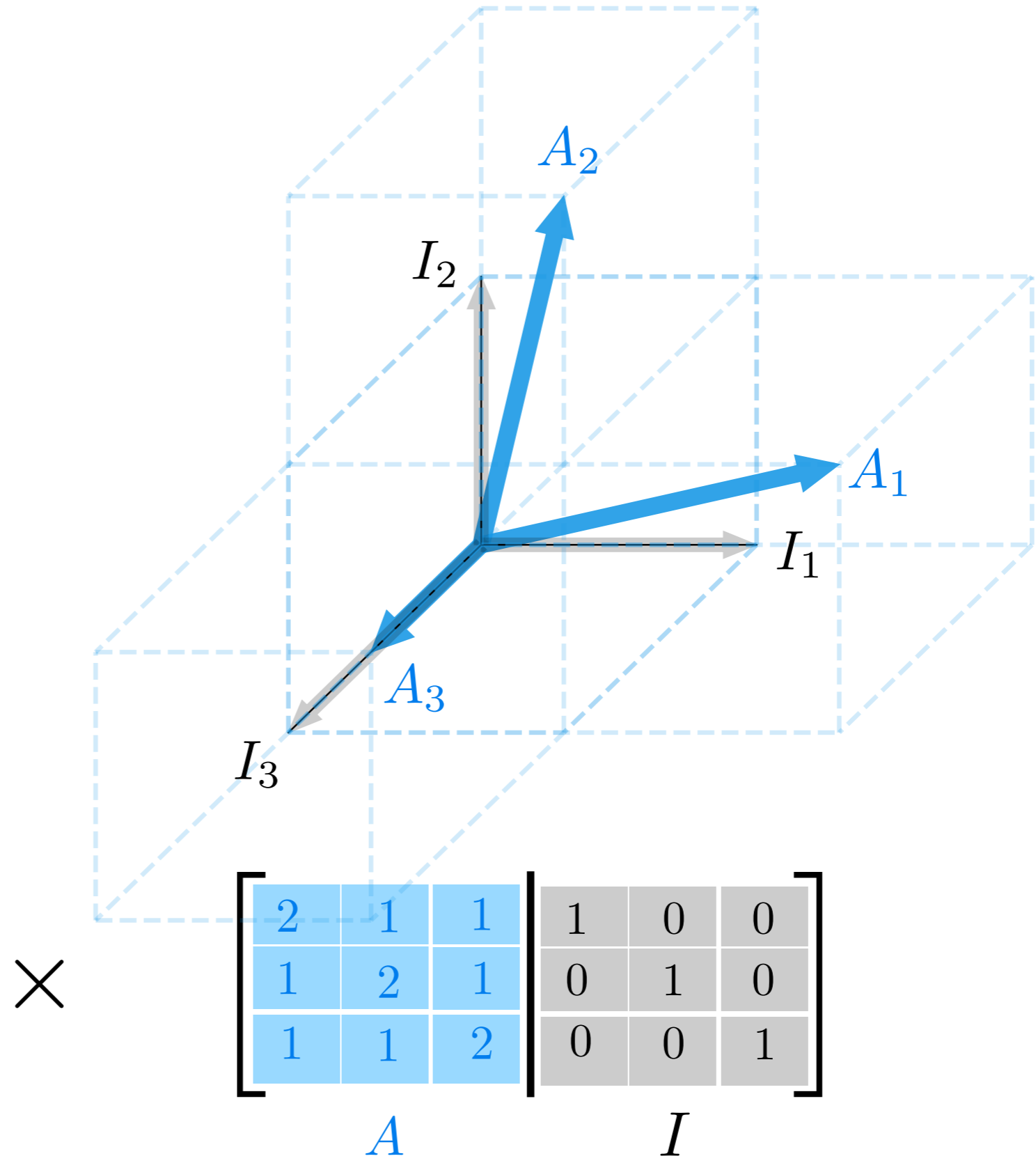
$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

A I

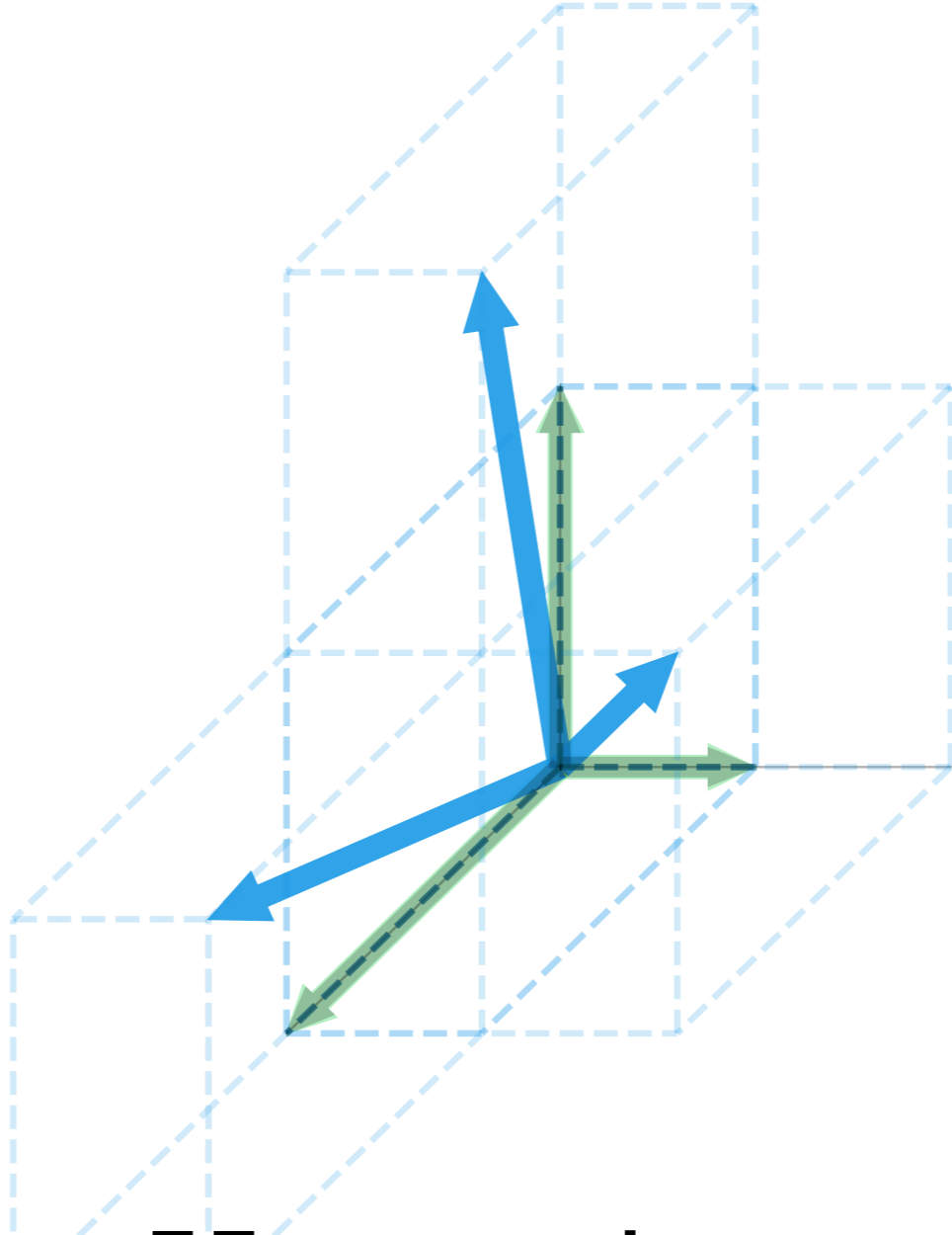
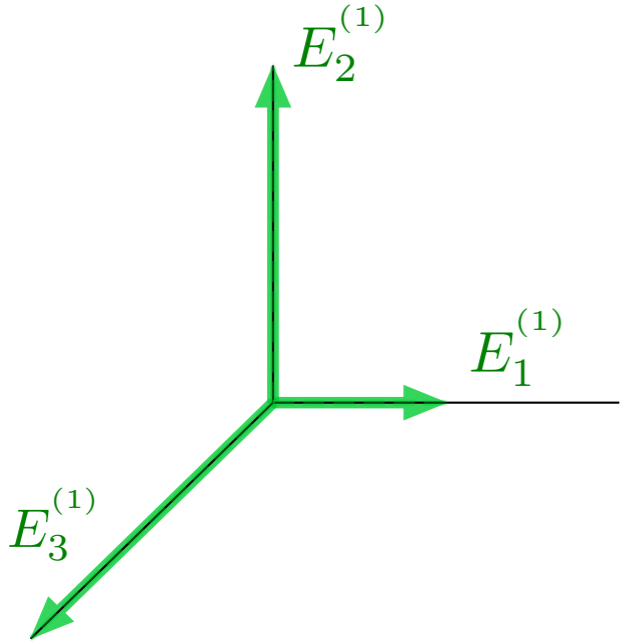
Elementary Matrix



$$E^{(1)} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



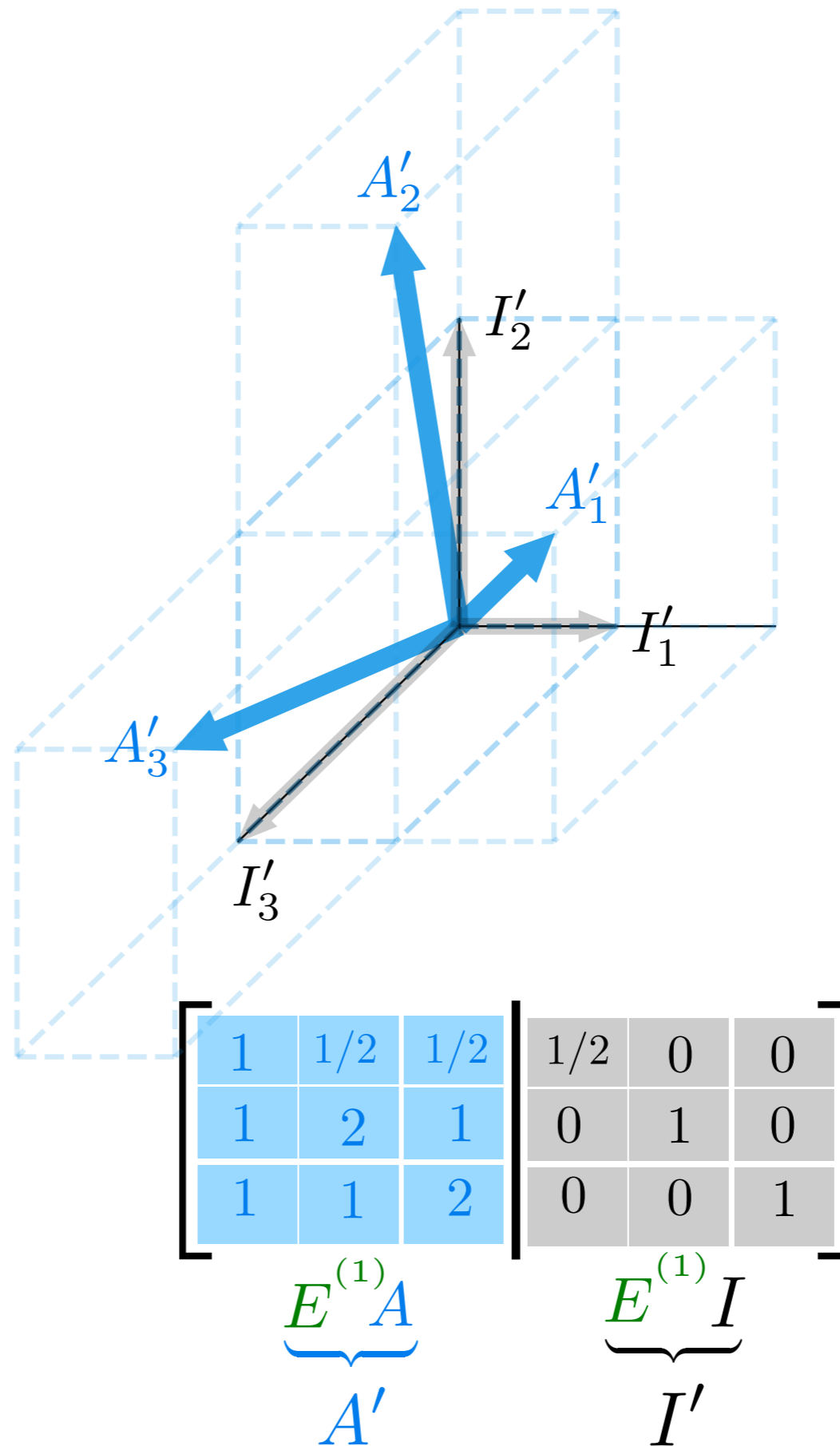
Row-reduction: operation 1...



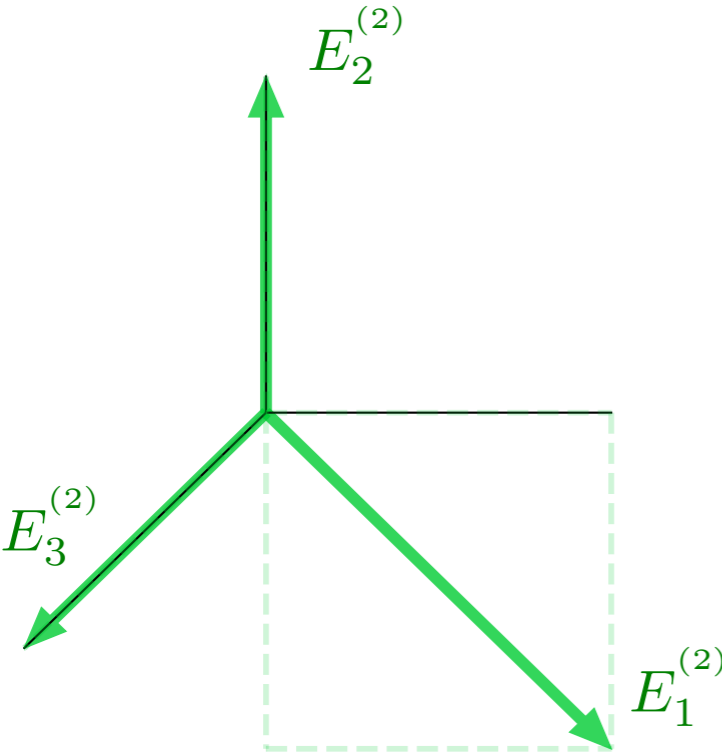
$1/2$	0	0	2	1	1	1	0	0	0
0	1	0	1	2	1	0	1	0	0
0	0	1	1	1	2	0	0	1	1

$E^{(1)}$ A I

Row-reduction:
operation 1...



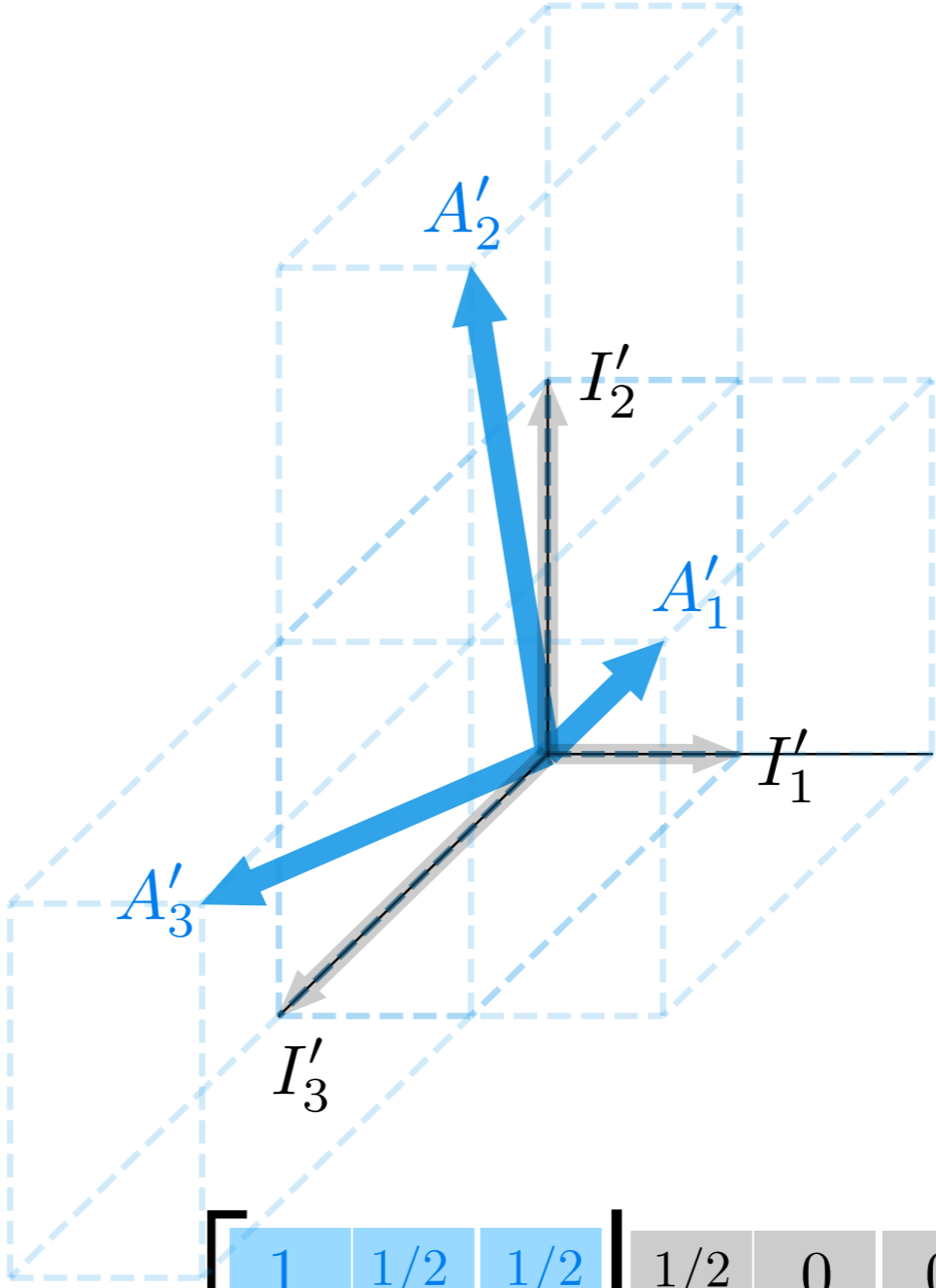
Row-reduction:
operation 2...



$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(2)}$

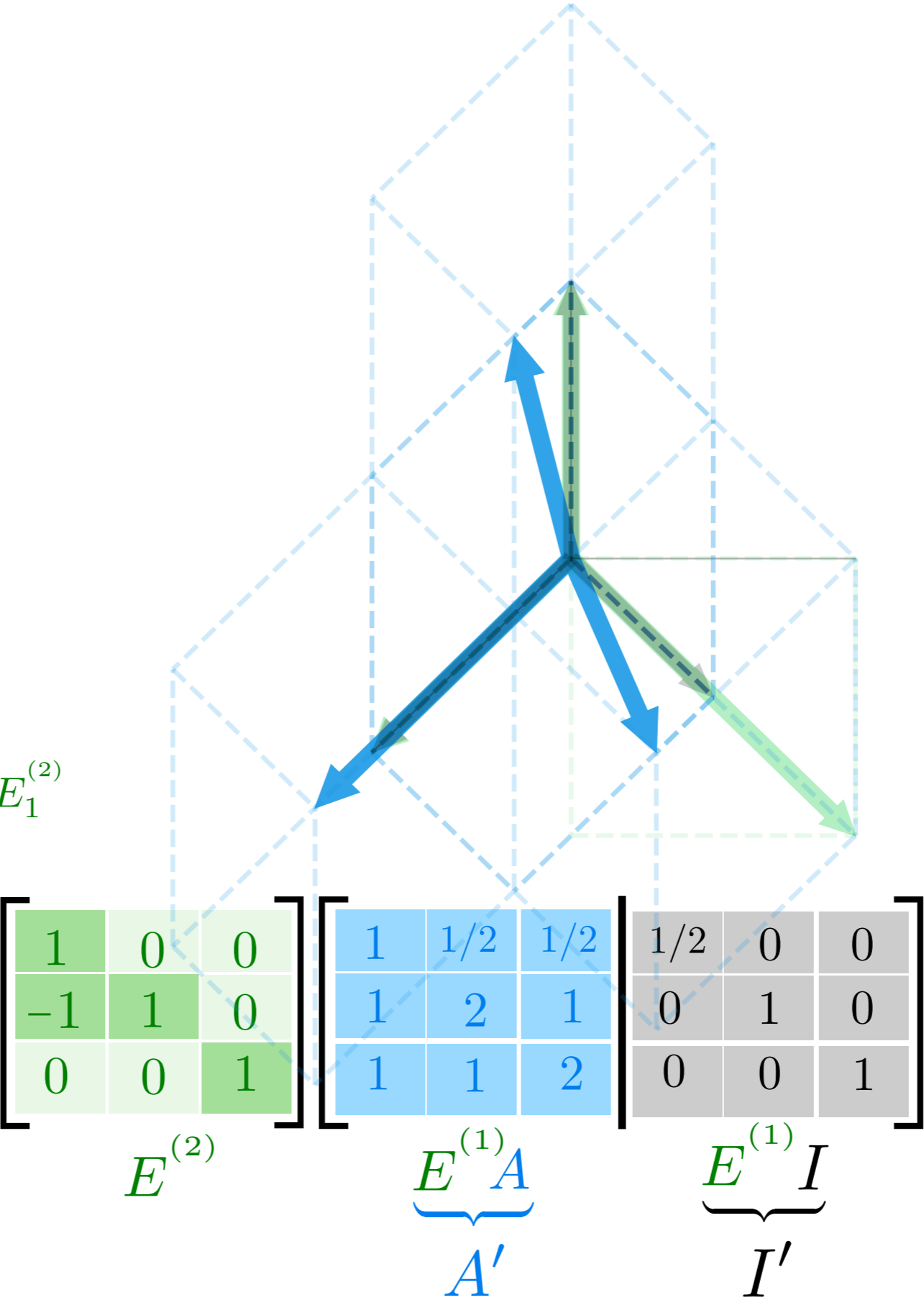
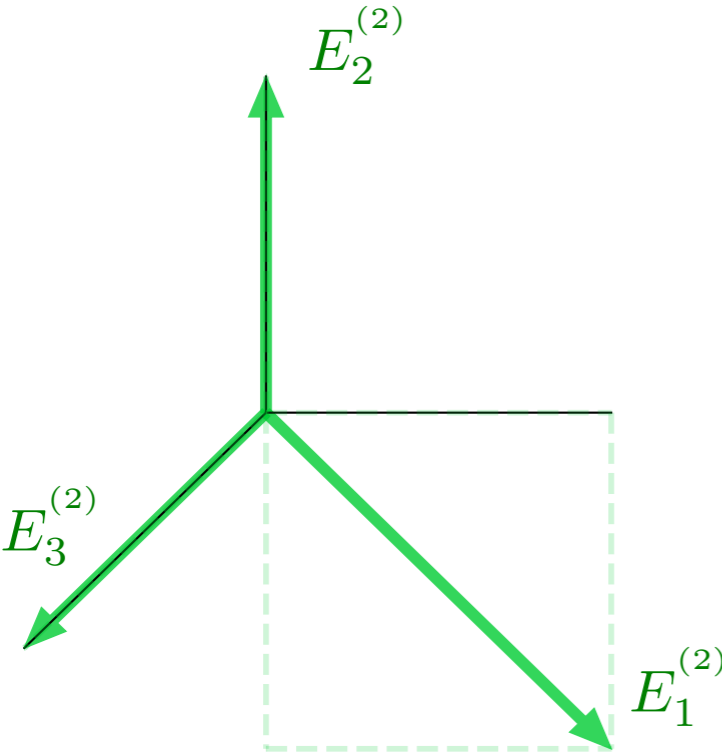
×



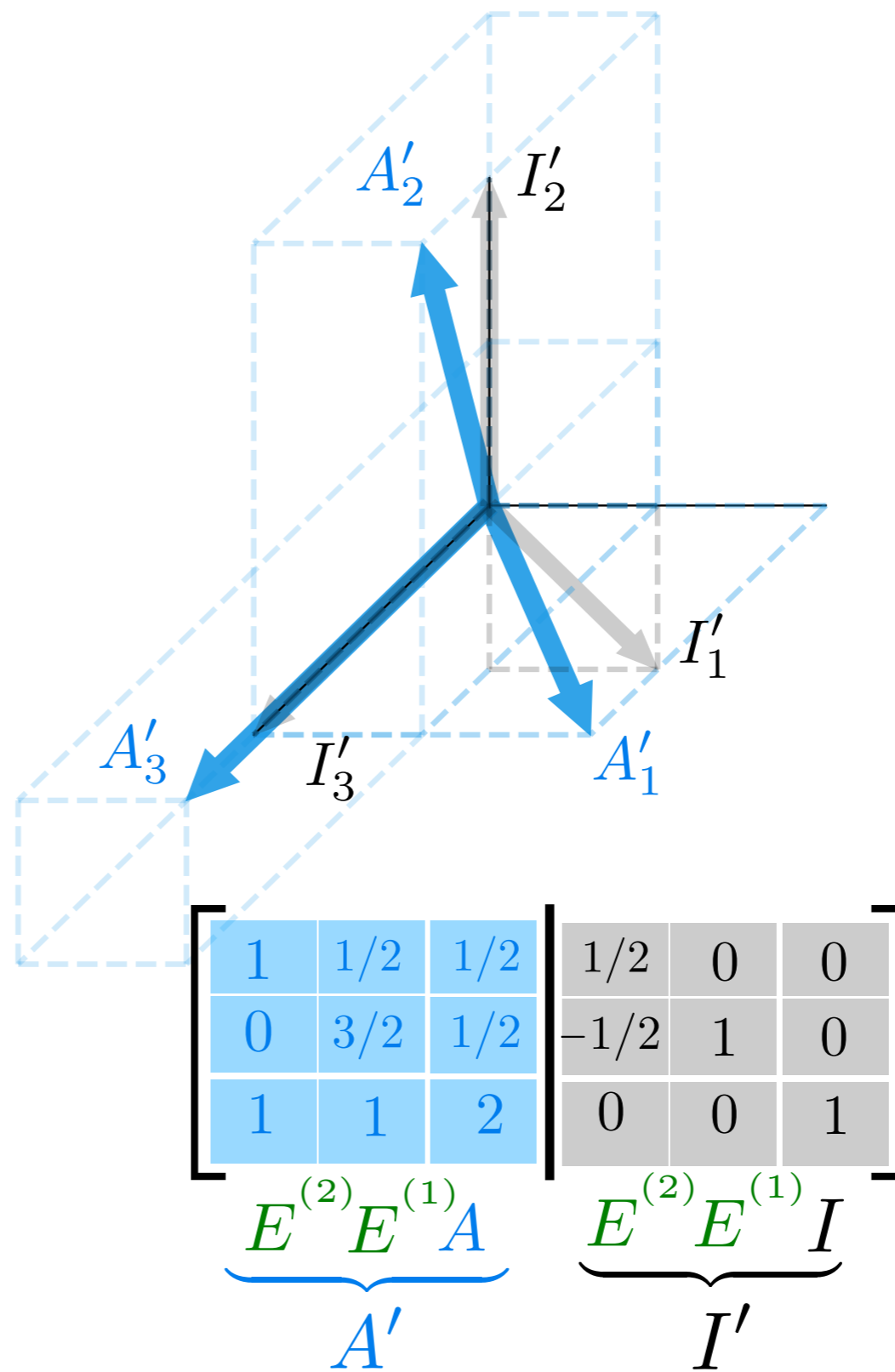
$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \left| \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$

$\underbrace{E^{(1)} A}_{A'}$ $\underbrace{E^{(1)} I}_{I'}$

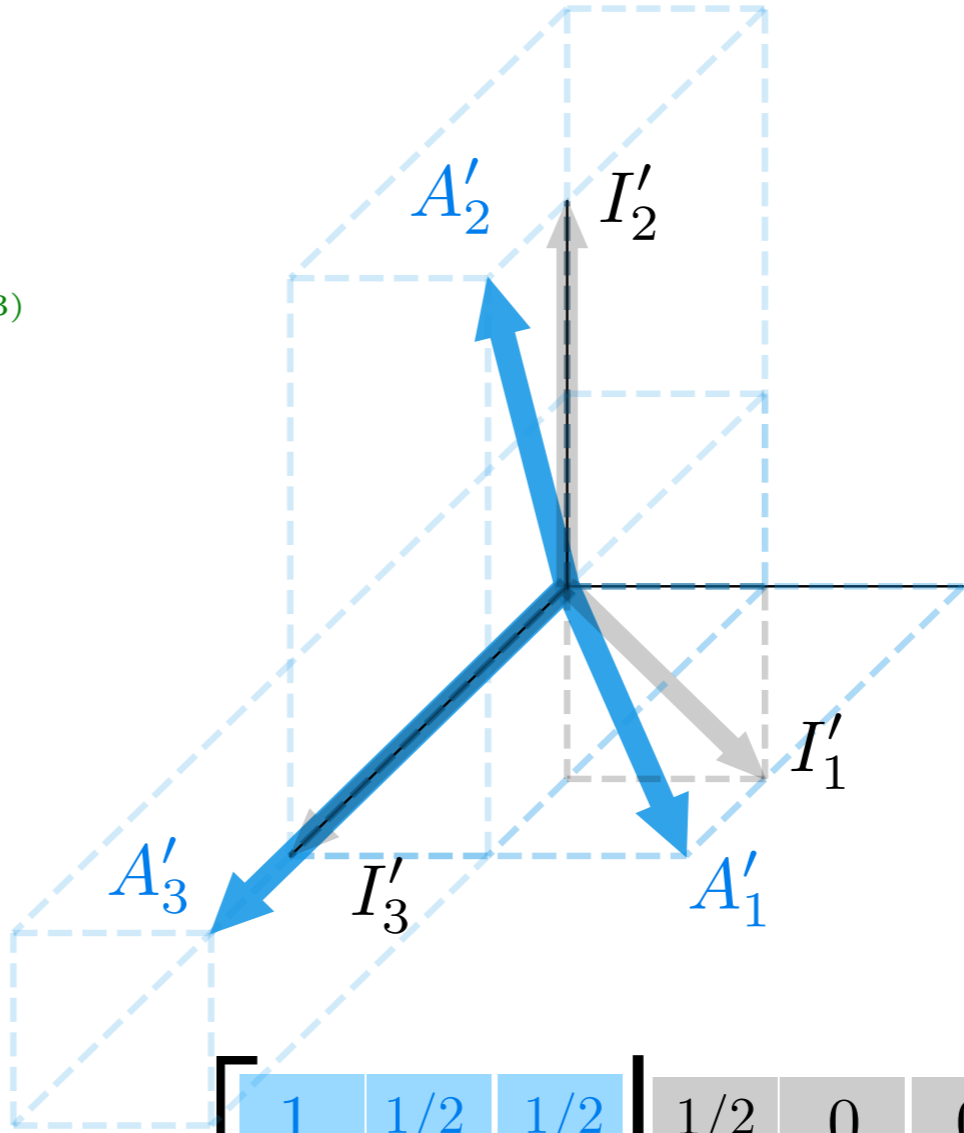
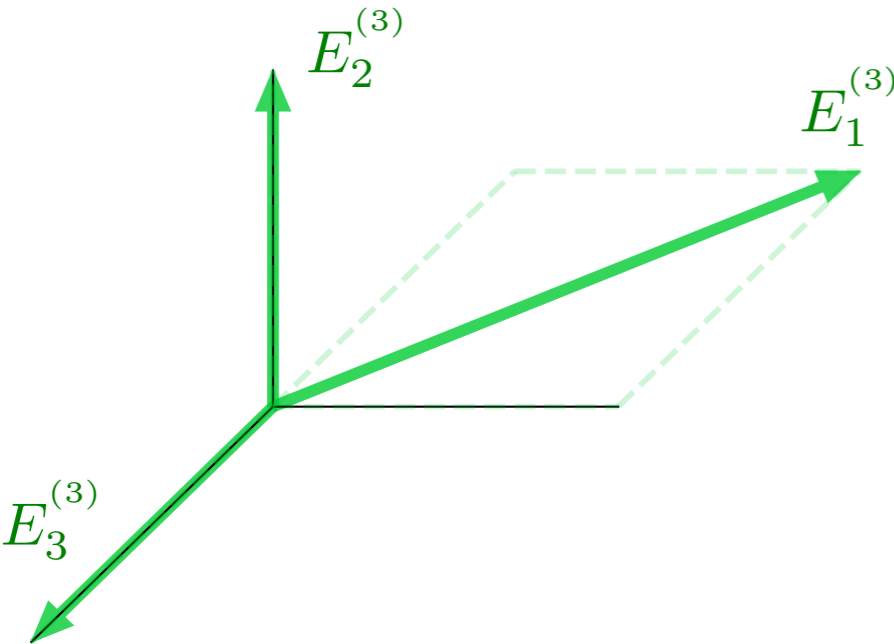
Row-reduction:
operation 2...



Row-reduction: operation 2...



Row-reduction:
operation 3...

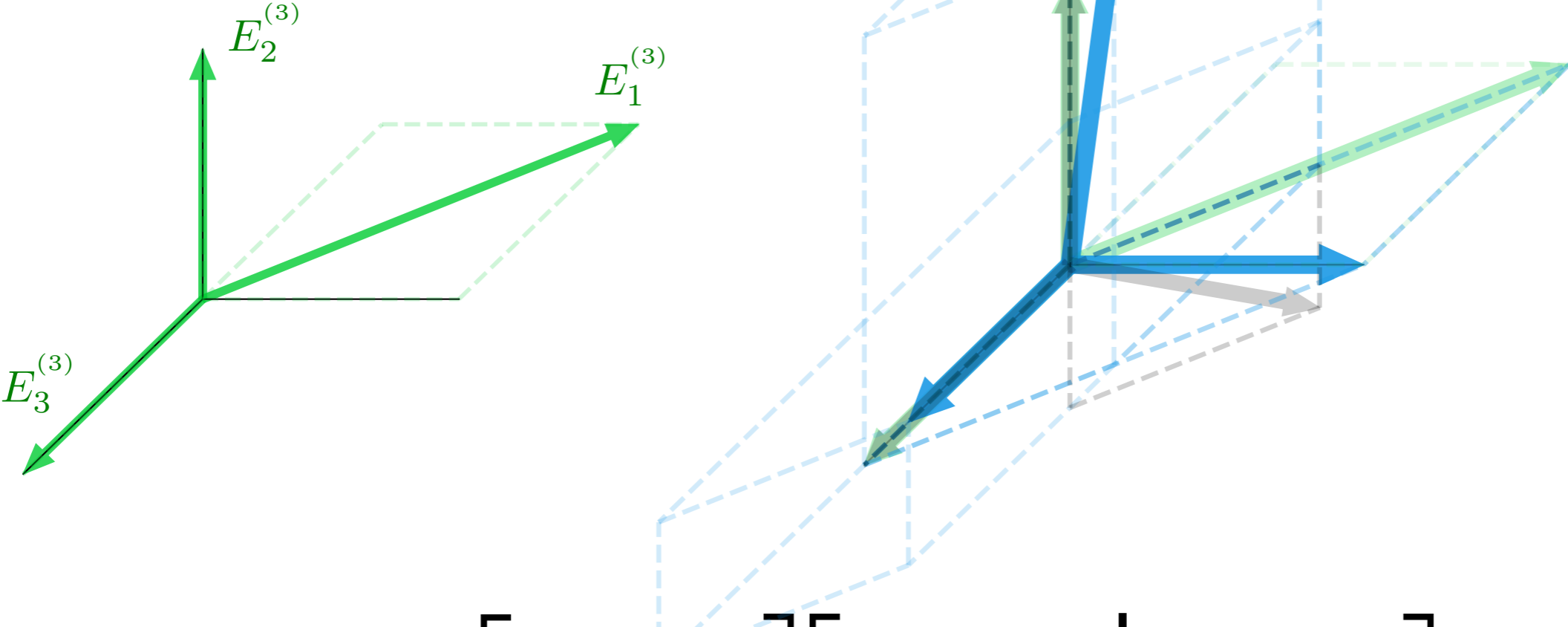


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ E^{(3)}$$

\times

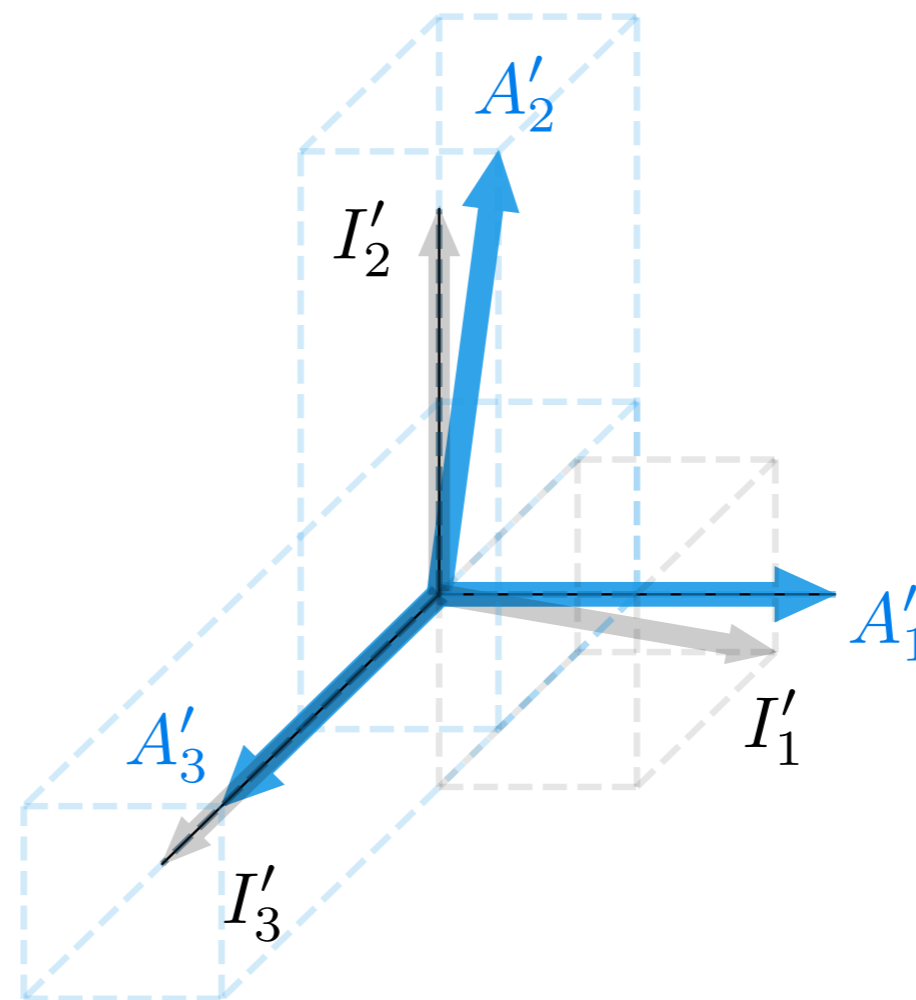
$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 1 & 1 & 2 \end{bmatrix} \left| \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right. \\ \underbrace{\begin{matrix} E^{(2)} & E^{(1)} & A \end{matrix}}_{A'} \quad \underbrace{\begin{matrix} E^{(2)} & E^{(1)} & I \end{matrix}}_{I'}$$

Row-reduction:
operation 3...



1	0	0	1	1/2	1/2	1/2	0	0	0
0	1	0	0	0	3/2	1/2	-1/2	1	0
-1	0	1	1	1	2	0	0	0	1
$E^{(3)}$			$E^{(2)} E^{(1)} A$			$E^{(2)} E^{(1)} I$			
			A'			I'			

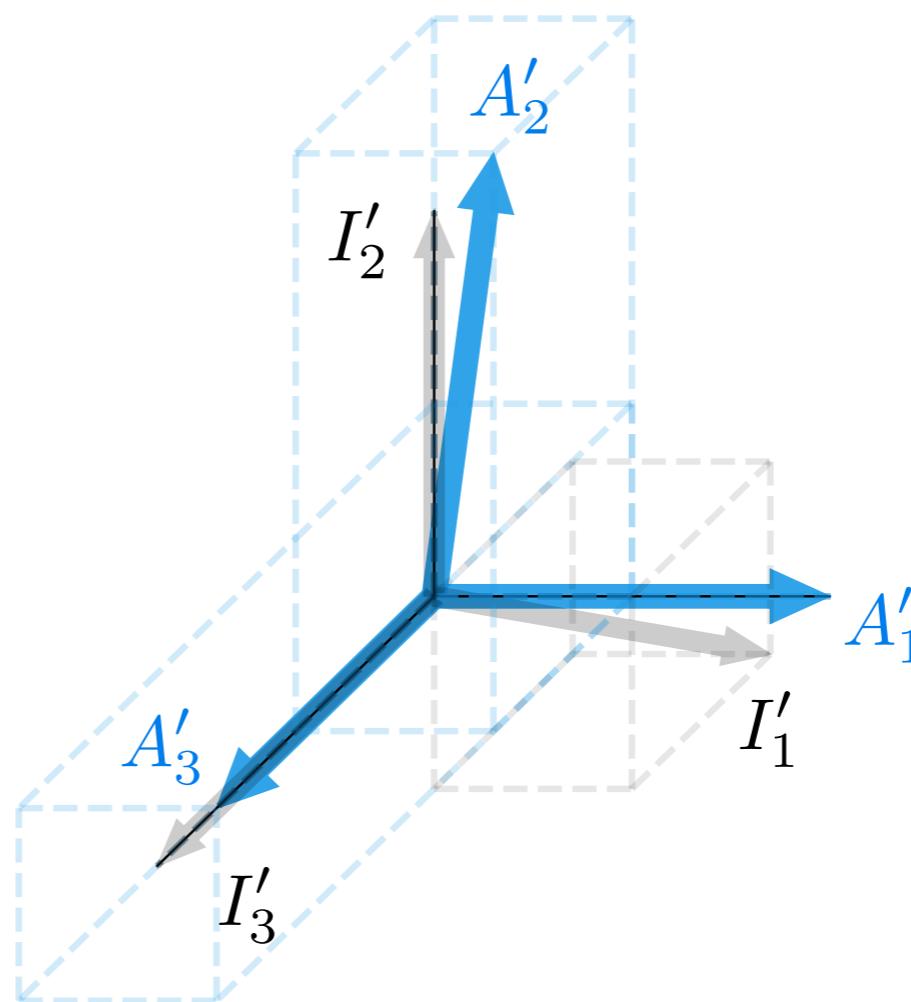
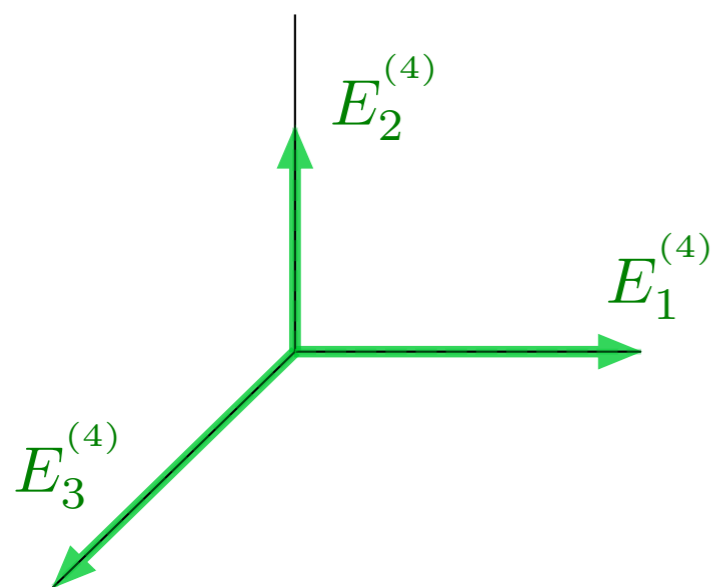
Row-reduction: operation 3...



$$\left[\begin{array}{ccc|ccc}
 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\
 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\
 0 & 1/2 & 3/2 & -1/2 & 0 & 1
 \end{array} \right]$$

$\underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

Row-reduction: operation 4...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

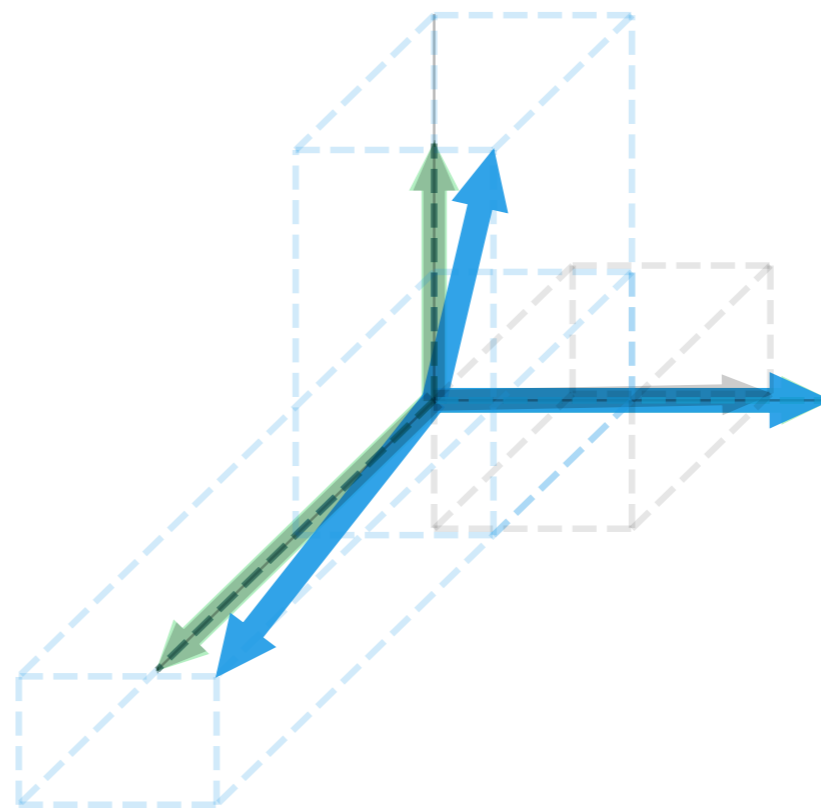
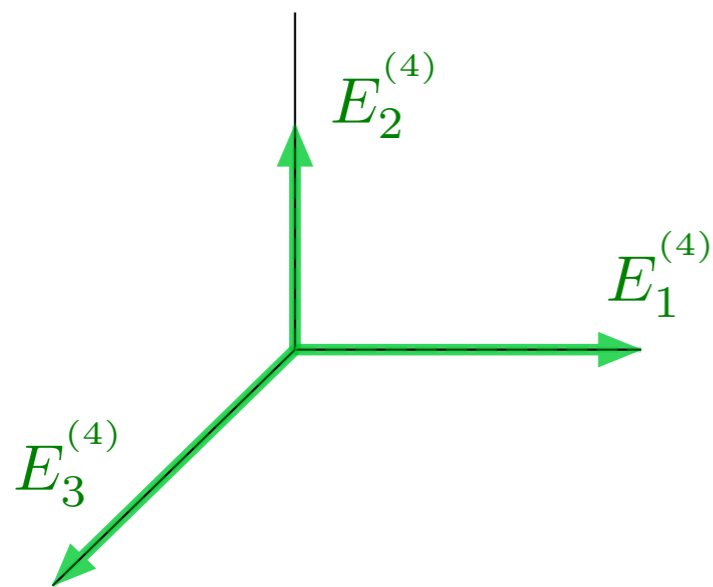
$E^{(4)}$

×

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{array} \right]$$

$\underbrace{E^{(3)} \ E^{(2)} \ E^{(1)} \ A}_{A'}$
 $\underbrace{E^{(3)} \ E^{(2)} \ E^{(1)} \ I}_{I'}$

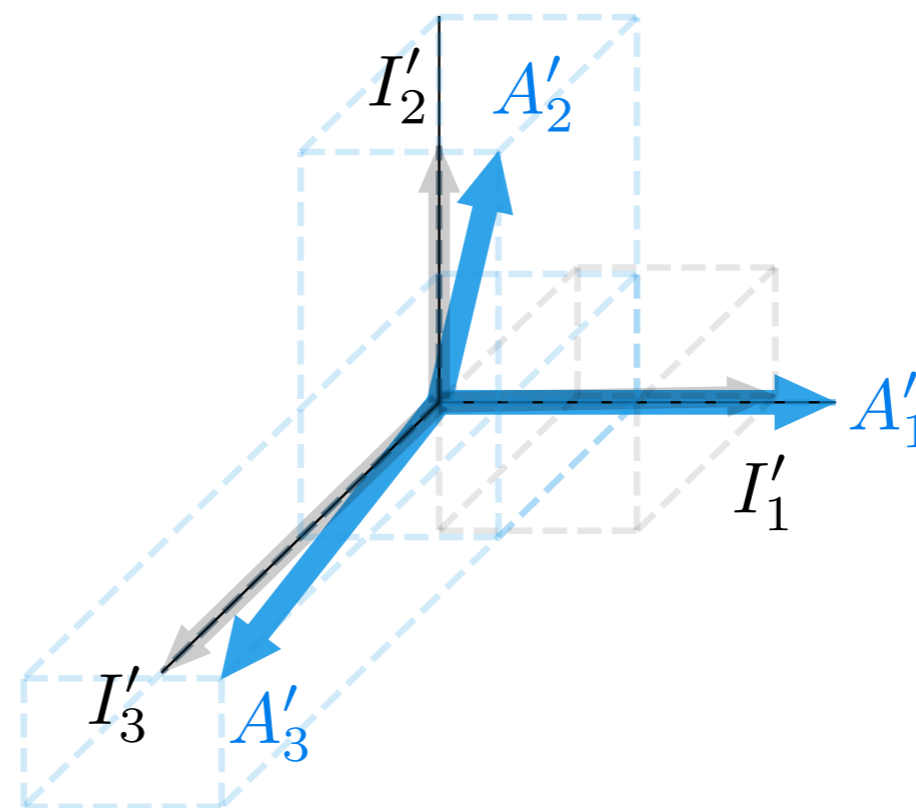
Row-reduction: operation 4...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix} \left| \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \right.$$

$E^{(4)}$
 $\underbrace{E^{(3)} E^{(2)} E^{(1)} A}_{A'}$
 $\underbrace{E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

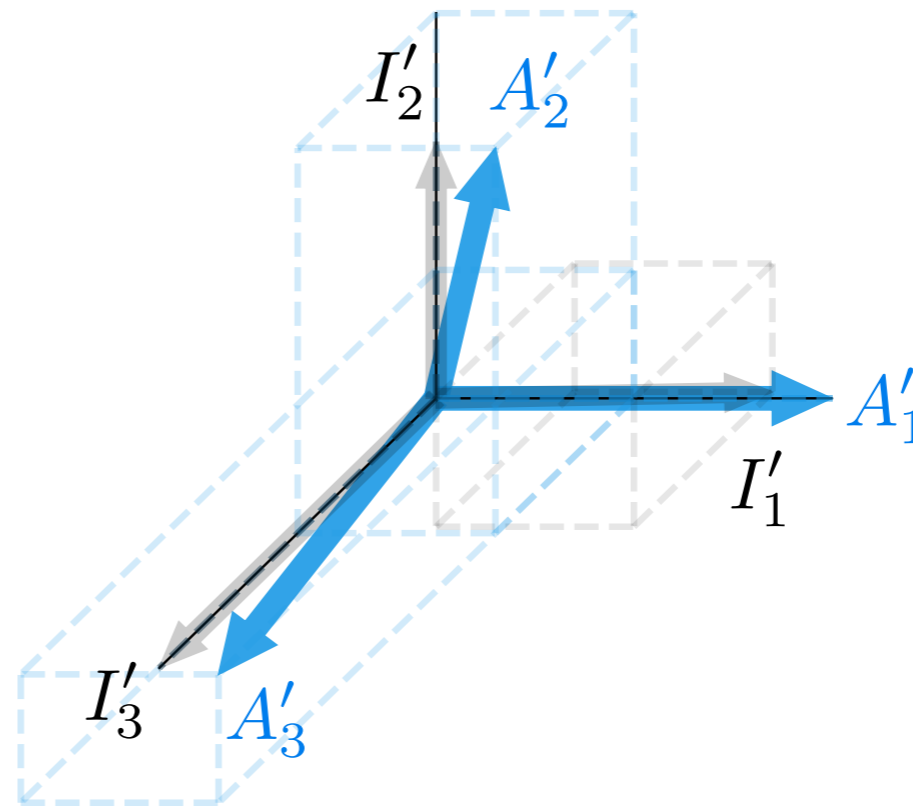
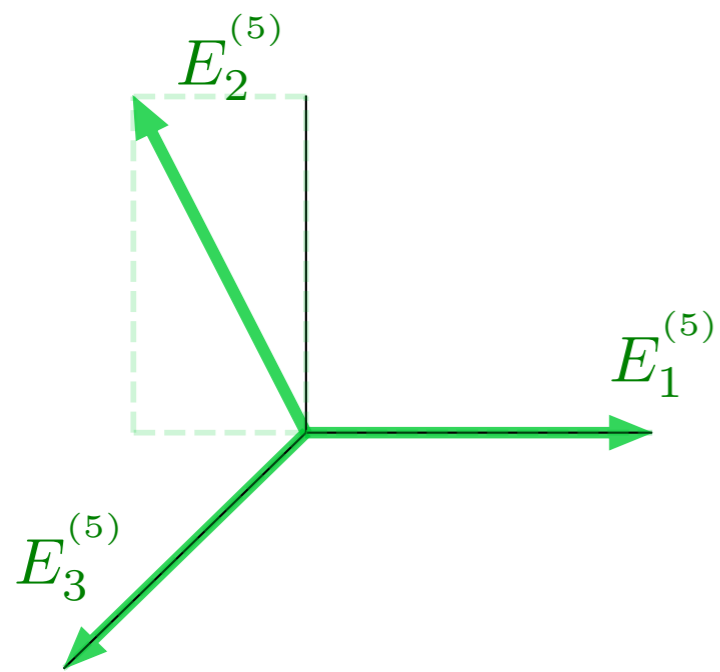
Row-reduction: operation 4...



$$\left[\begin{array}{ccc|ccc}
 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\
 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\
 0 & 1/2 & 3/2 & -1/2 & 0 & 1
 \end{array} \right]$$

$$\underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 5...



$$\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

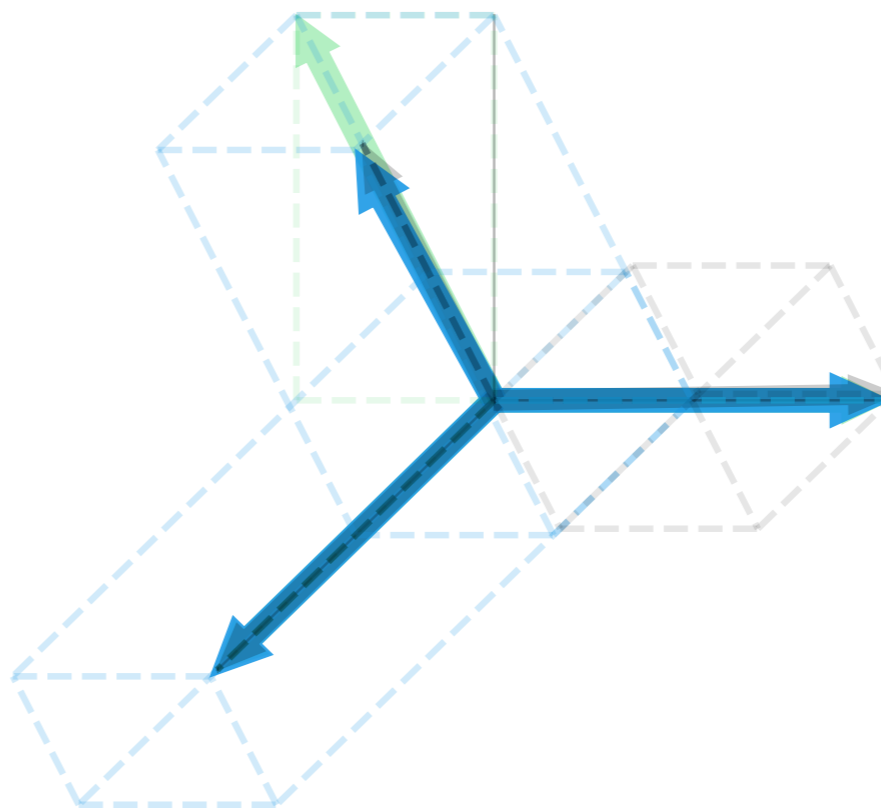
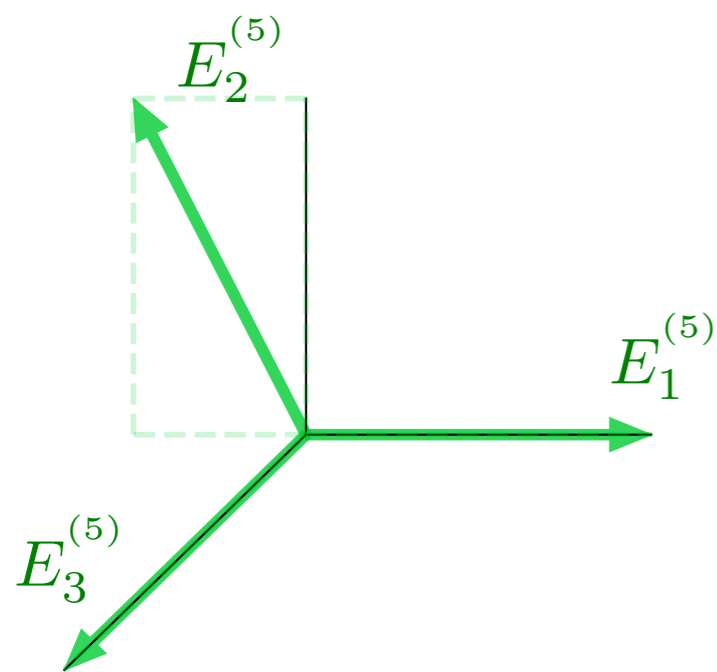
$E^{(5)}$

×

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{array} \right]$$

$\underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

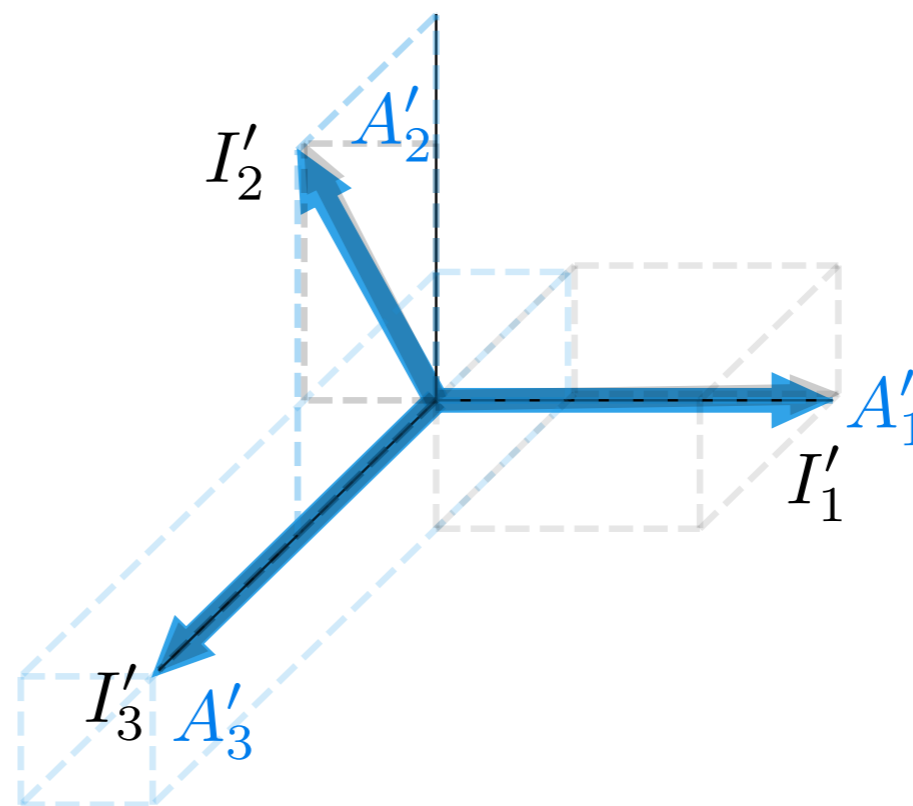
Row-reduction: operation 5...



$$\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix} \left| \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \right.$$

$E^{(5)}$ $\underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'}$ $\underbrace{E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

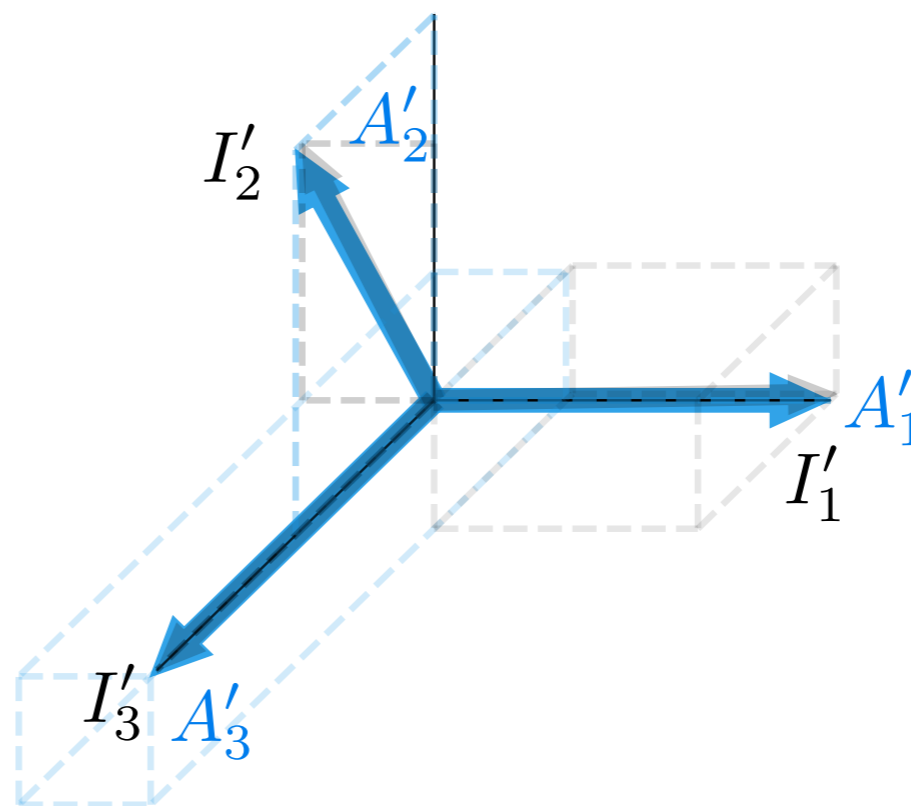
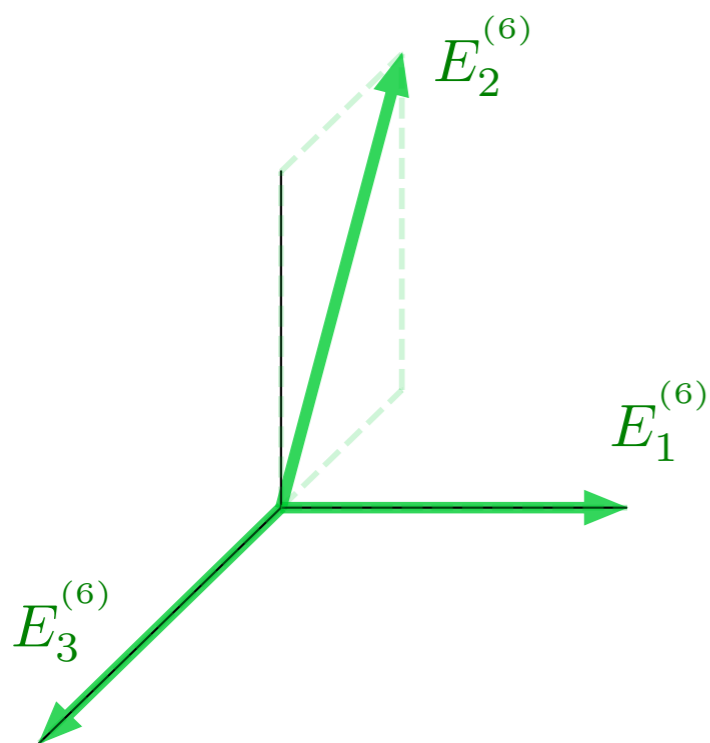
Row-reduction: operation 5...



$$\left[\begin{array}{ccc|ccc}
 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\
 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\
 0 & 1/2 & 3/2 & -1/2 & 0 & 1
 \end{array} \right]$$

$$\underbrace{\begin{array}{c} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A \end{array}}_{A'} \quad \underbrace{\begin{array}{c} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I \end{array}}_{I'}$$

Row-reduction: operation 6...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

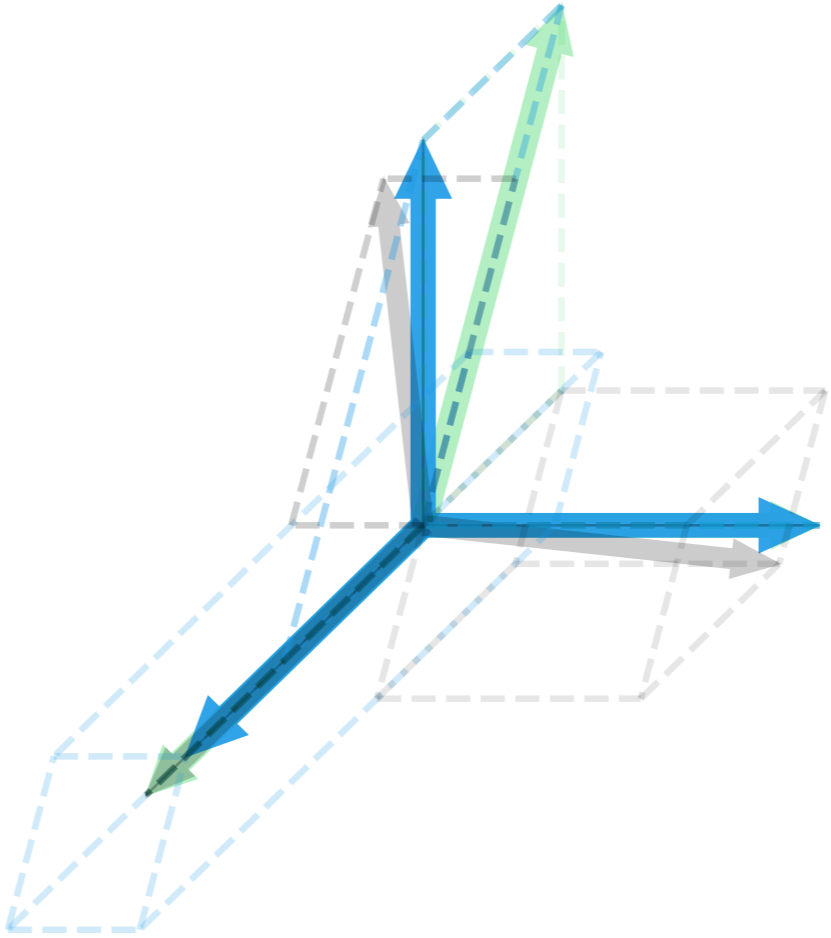
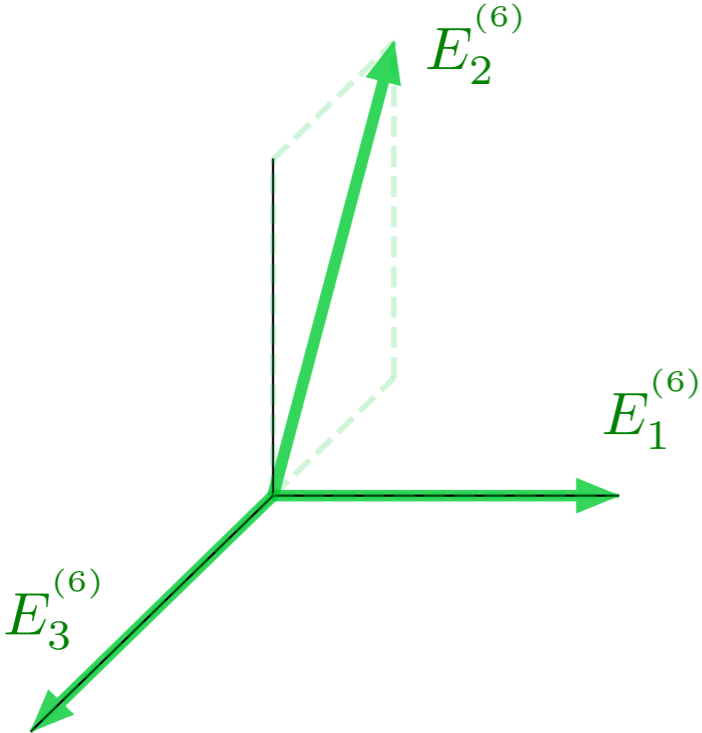
$E^{(6)}$

×

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 1/2 & 3/2 \end{bmatrix} \left| \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \right.$$

$\underbrace{E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'}$
 $\underbrace{E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

Row-reduction:
operation 6...



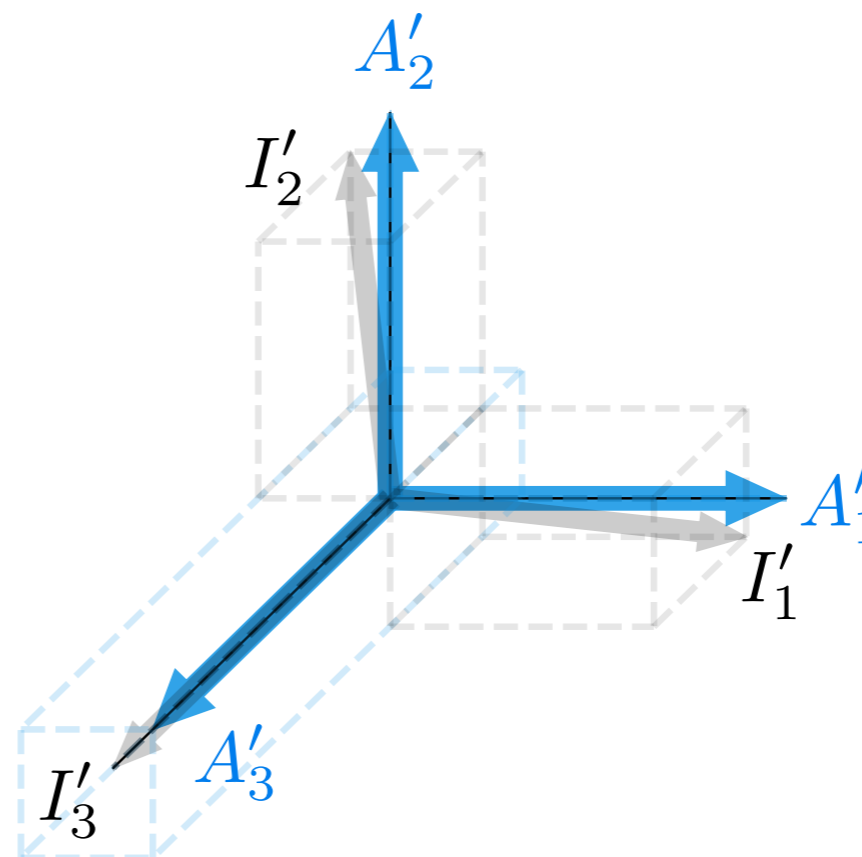
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 1/2 & 3/2 \end{bmatrix} \left| \begin{array}{ccc} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/2 & 0 & 1 \end{array} \right.$$

$E^{(6)}$
 $E^{(5)} E^{(4)}$
 $E^{(5)} E^{(4)}$

$E^{(3)} E^{(2)} E^{(1)} A$
 $E^{(3)} E^{(2)} E^{(1)} I$

A'
 I'

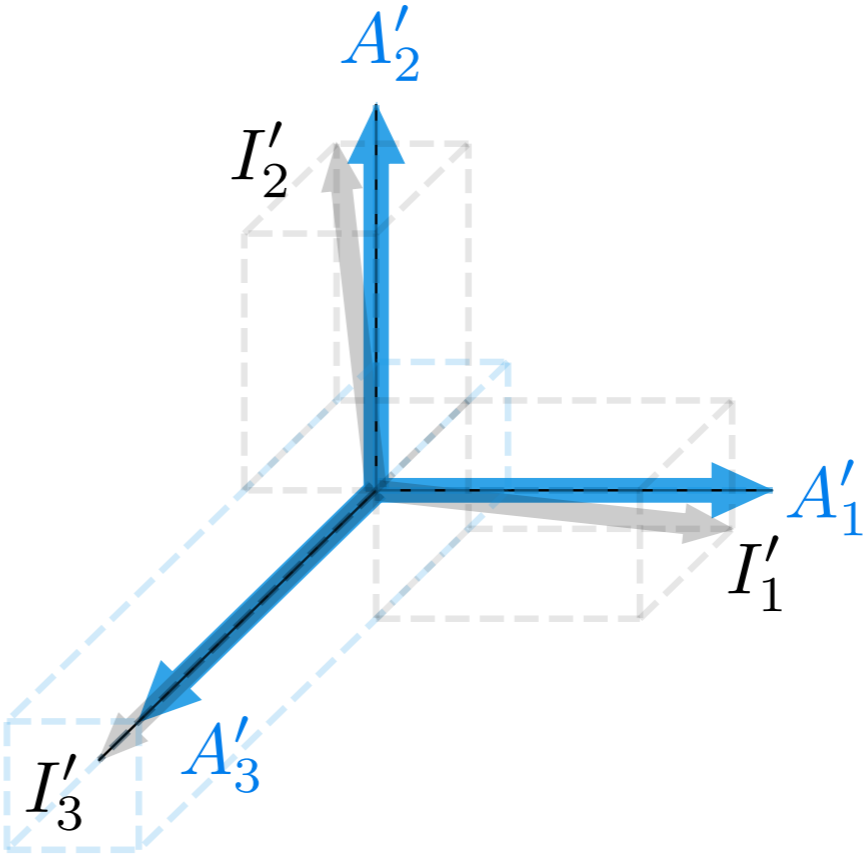
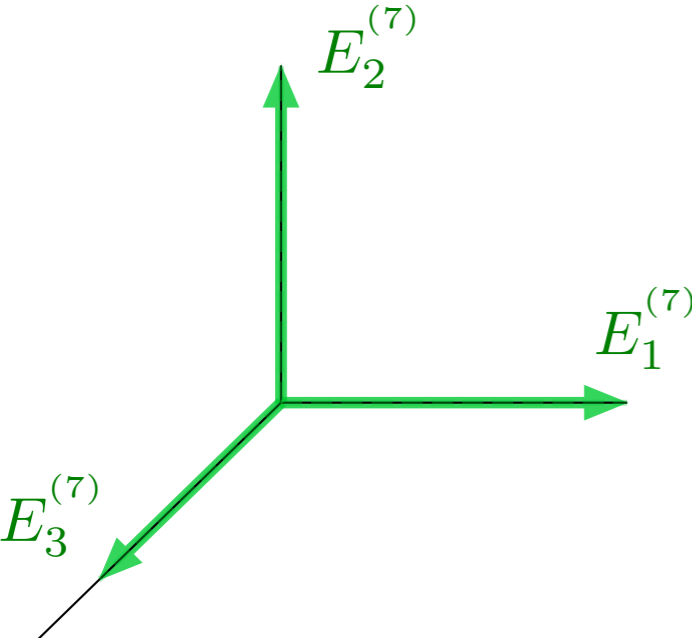
Row-reduction: operation 6...



$$\left[\begin{array}{ccc|ccc}
 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\
 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\
 0 & 0 & 4/3 & -1/3 & -1/3 & 1
 \end{array} \right]$$

$$\underbrace{E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 7...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$E^{(7)}$

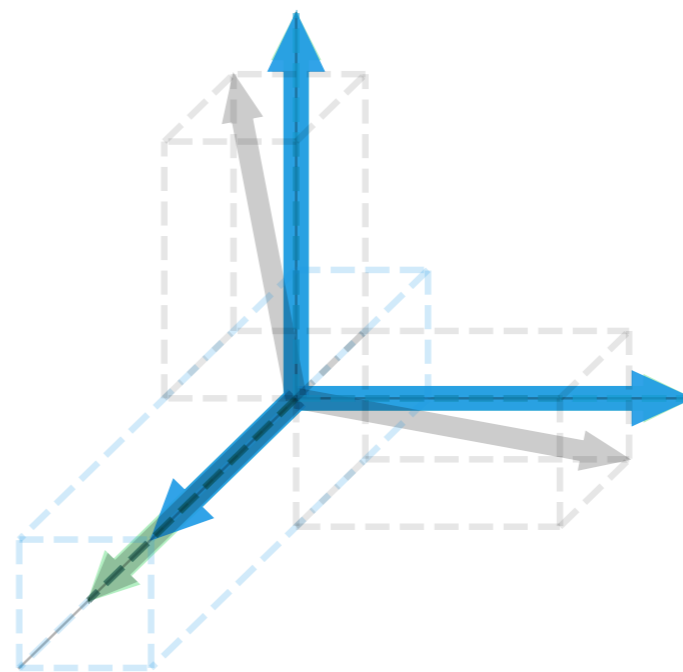
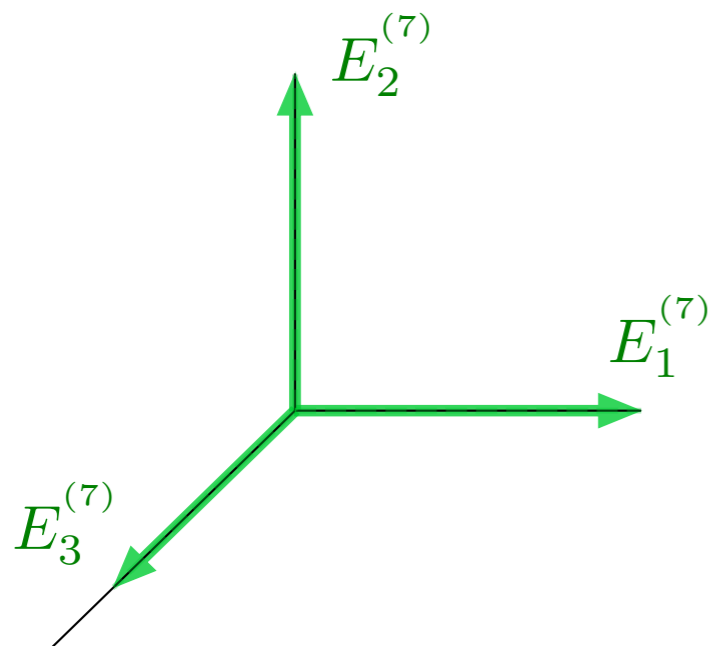
\times

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{array} \right]$$

$\underbrace{E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'}$

$\underbrace{E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

Row-reduction: operation 7...



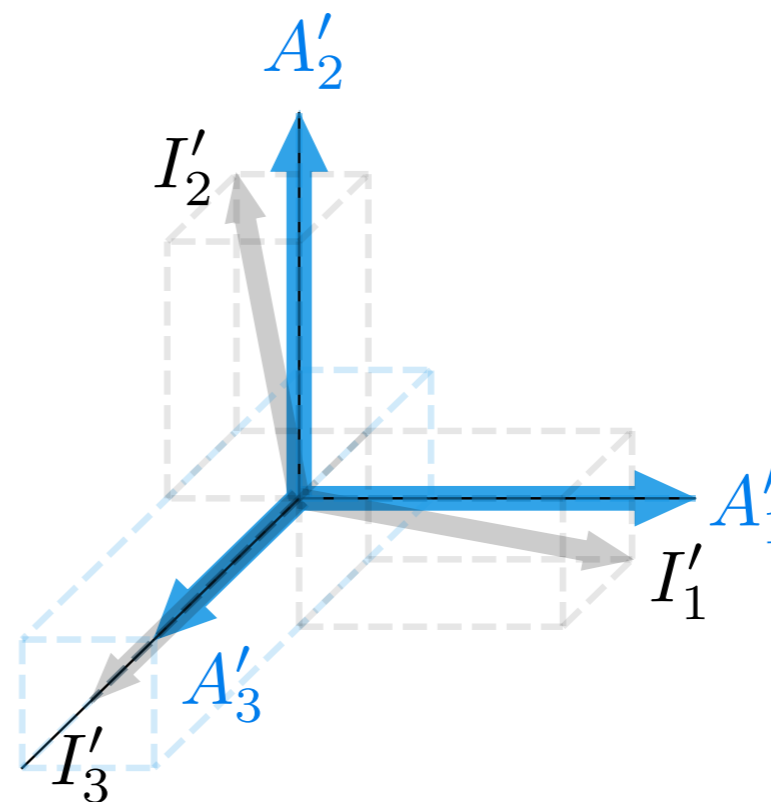
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{bmatrix}$$

$E^{(7)}$
 $E^{(6)} E^{(5)} E^{(4)}$
 $E^{(6)} E^{(5)} E^{(4)}$

$E^{(3)} E^{(2)} E^{(1)} A$
 $E^{(3)} E^{(2)} E^{(1)} I$

A'
 I'

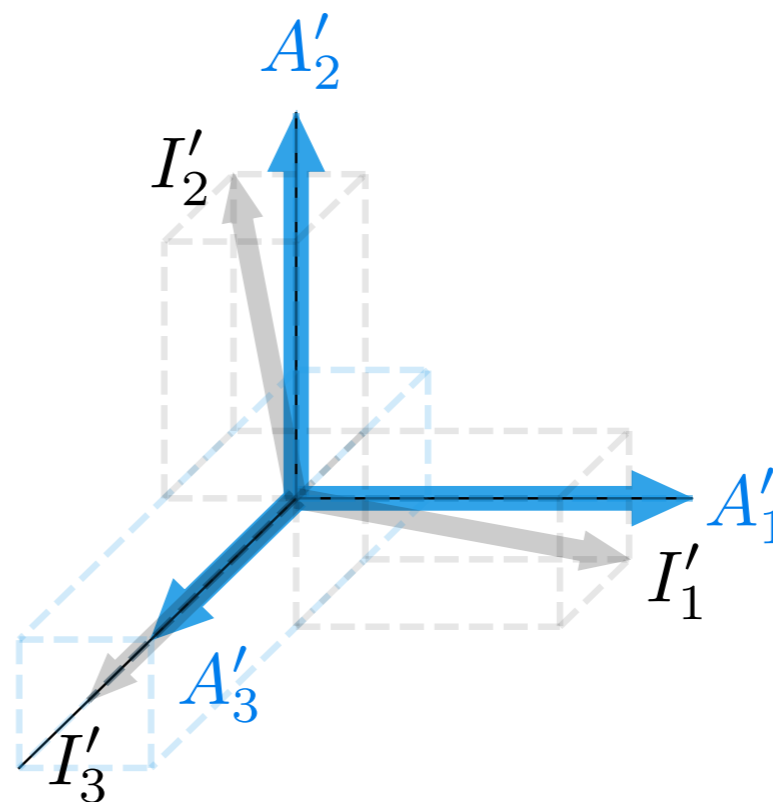
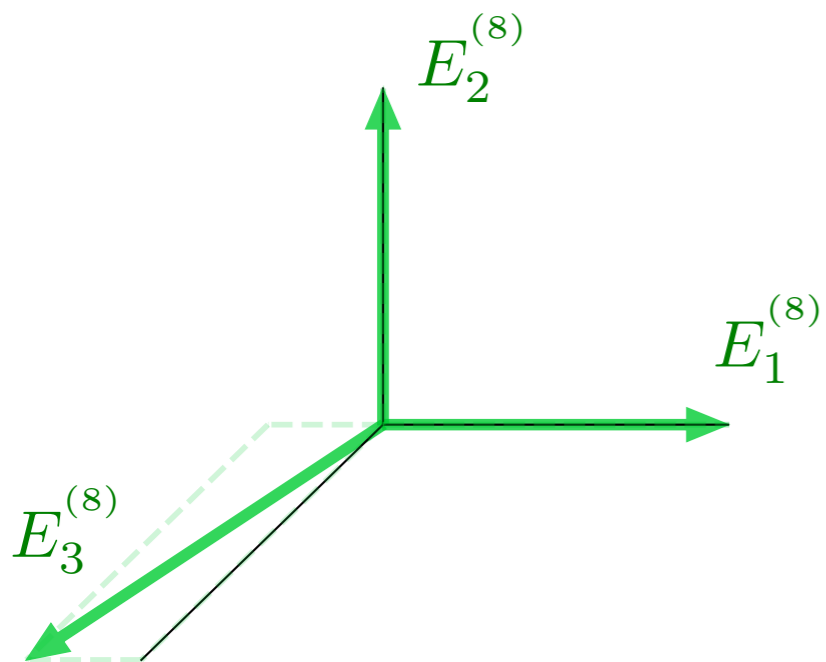
Row-reduction: operation 7...



$$\left[\begin{array}{ccc|ccc}
 1 & 0 & 1/3 & 2/3 & -1/3 & 0 \\
 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\
 0 & 0 & 1 & -1/4 & -1/4 & 3/4
 \end{array} \right]$$

$$\underbrace{E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'} \quad \underbrace{E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$$

Row-reduction: operation 8...



$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

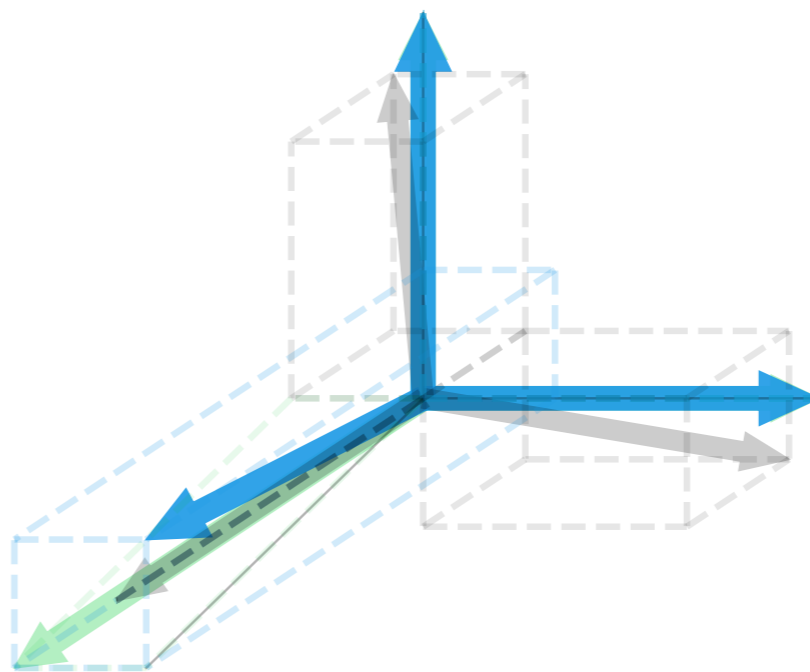
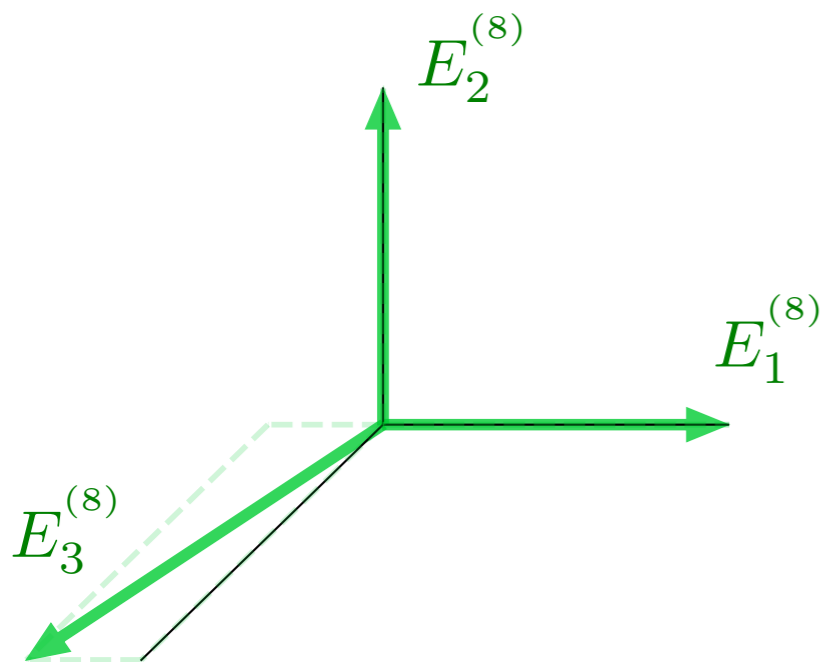
$E^{(8)}$

×

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \left| \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/4 & -1/4 & 3/4 \end{bmatrix} \right.$$

$\underbrace{E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'}$
 $\underbrace{E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

Row-reduction: operation 8...



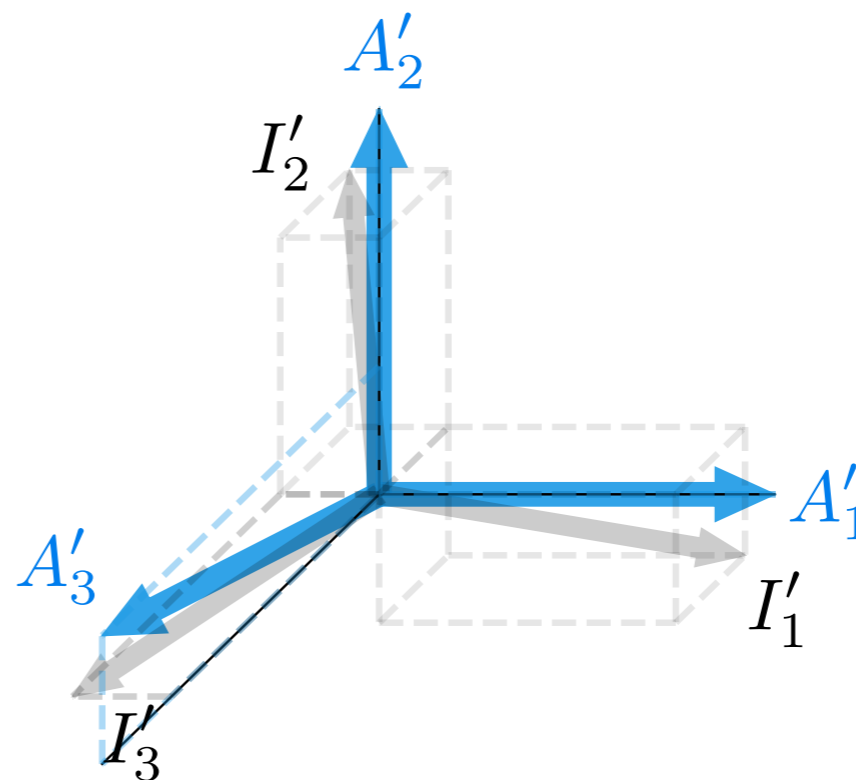
$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$E^{(8)}$
 $E^{(7)} E^{(6)} E^{(5)} E^{(4)}$
 $E^{(7)} E^{(6)} E^{(5)} E^{(4)}$

$E^{(3)} E^{(2)} E^{(1)} A$
 $E^{(3)} E^{(2)} E^{(1)} I$

A'
 I'

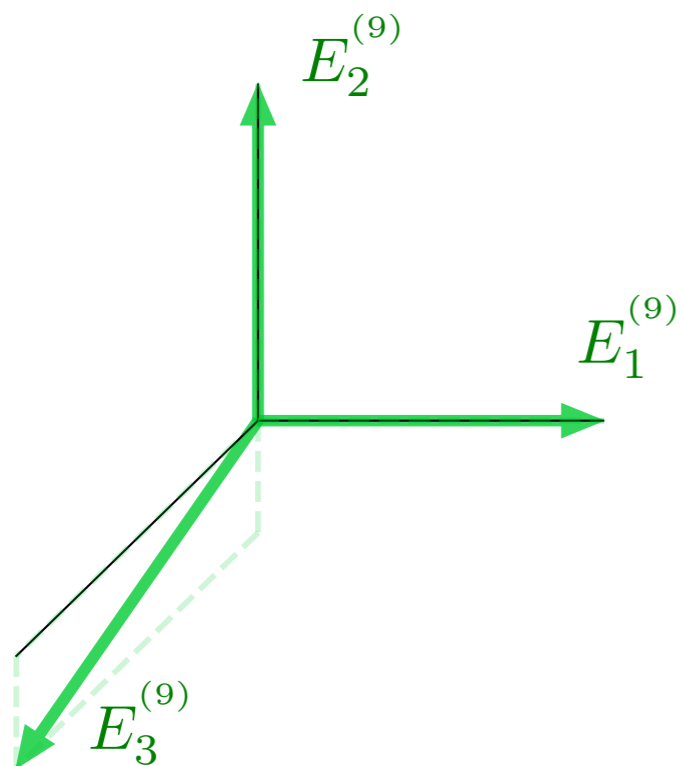
Row-reduction: operation 8...



$$\left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\
 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\
 0 & 0 & 1 & -1/4 & -1/4 & 3/4
 \end{array} \right]$$

$$\underbrace{\begin{array}{l} E^{(8)} E^{(7)} \\ E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A \end{array}}_{A'} \quad \underbrace{\begin{array}{l} E^{(8)} E^{(7)} \\ E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I \end{array}}_{I'}$$

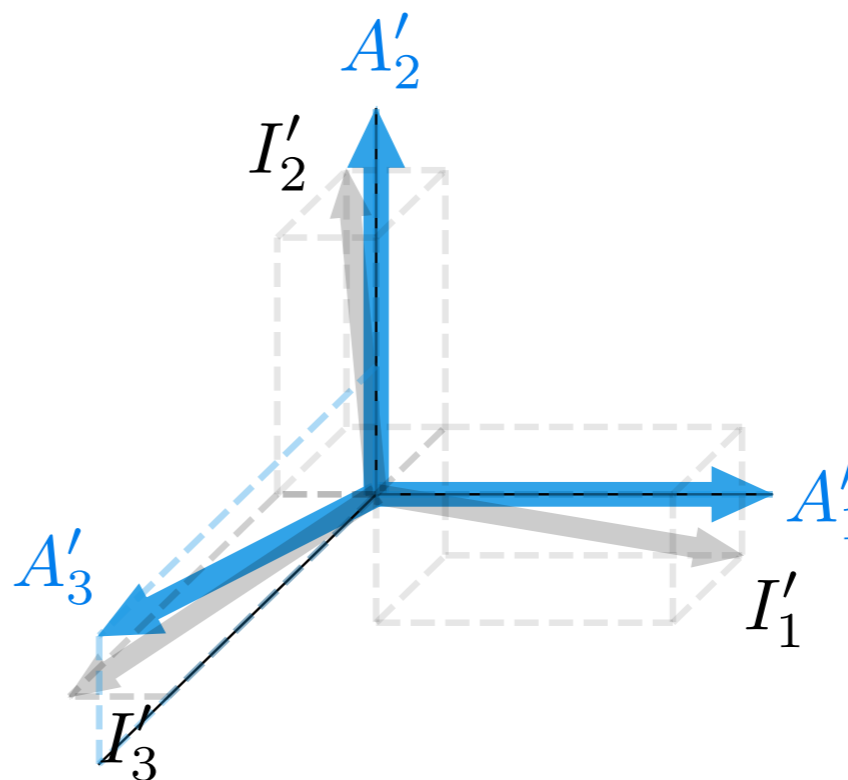
Row-reduction: operation 9...



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(9)}$

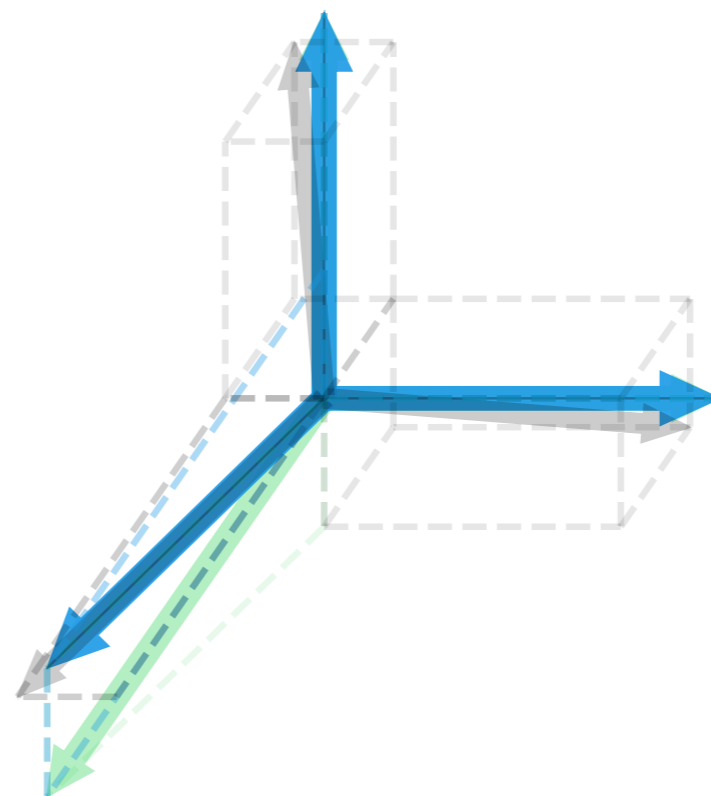
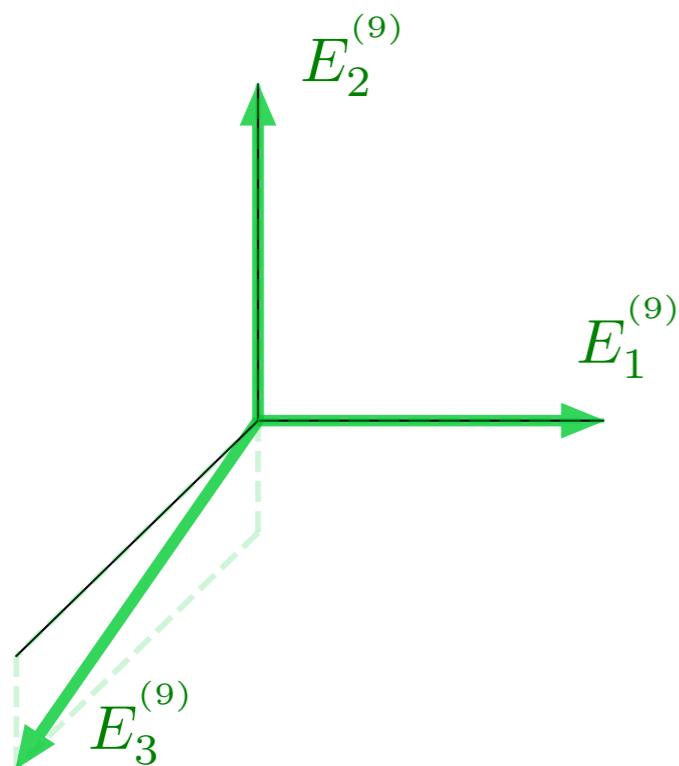
×



$$\begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$\underbrace{\begin{matrix} E^{(8)} & E^{(7)} \\ E^{(6)} & E^{(5)} & E^{(4)} \\ E^{(3)} & E^{(2)} & E^{(1)} & A \end{matrix}}_{A'}$
 $\underbrace{\begin{matrix} E^{(8)} & E^{(7)} \\ E^{(6)} & E^{(5)} & E^{(4)} \\ E^{(3)} & E^{(2)} & E^{(1)} & I \end{matrix}}_{I'}$

Row-reduction: operation 9...



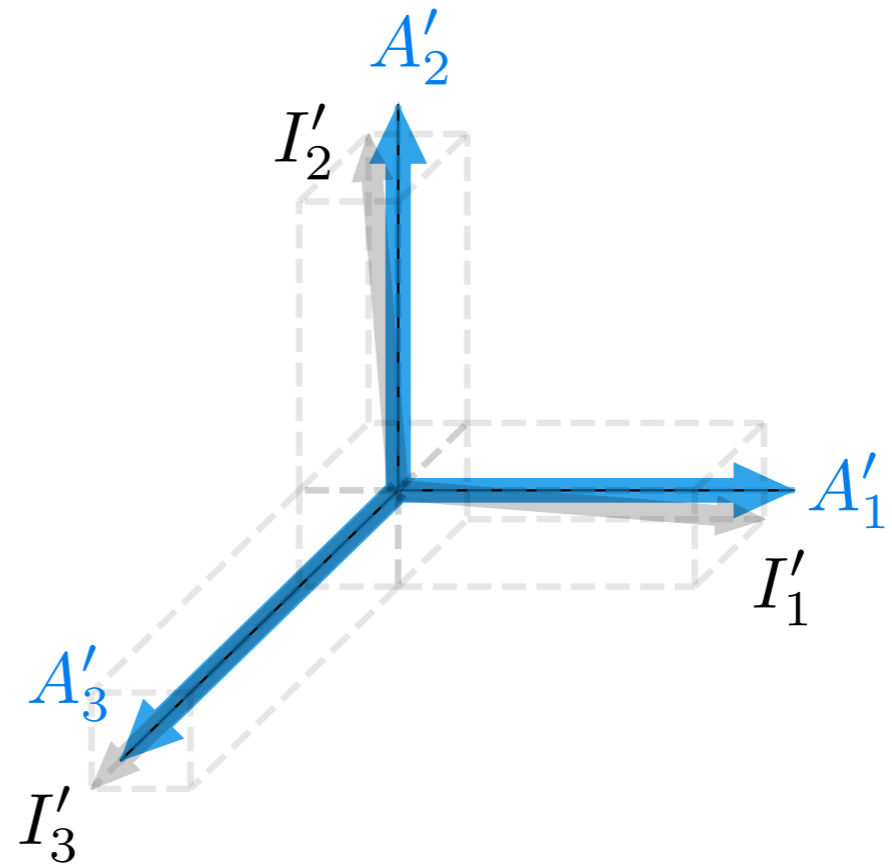
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/3 & 2/3 & 0 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$E^{(9)}$

$\underbrace{E^{(8)} E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} A}_{A'}$

$\underbrace{E^{(8)} E^{(7)} E^{(6)} E^{(5)} E^{(4)} E^{(3)} E^{(2)} E^{(1)} I}_{I'}$

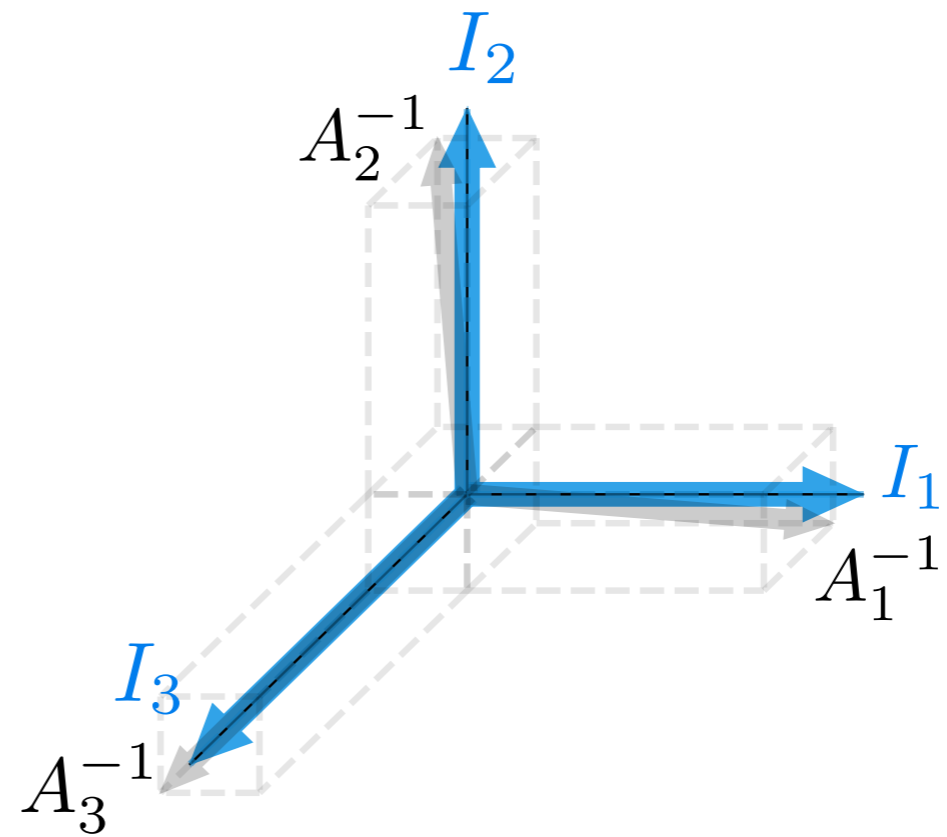
Row-reduction:
operation 9...



final
system...

$$\left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\
 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\
 0 & 0 & 1 & -1/4 & -1/4 & 3/4
 \end{array} \right]$$

$$\underbrace{\begin{array}{c} E^{(9)} E^{(8)} E^{(7)} \\ E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} A \end{array}}_{A'} \quad \underbrace{\begin{array}{c} E^{(9)} E^{(8)} E^{(7)} \\ E^{(6)} E^{(5)} E^{(4)} \\ E^{(3)} E^{(2)} E^{(1)} I \end{array}}_{I'}$$



**final
system...**

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array} \right] \mathbf{A}^{-1}$$