LTI & LTV Systems

Linear System Theory

Major sources:

Winter 2022 - Dan Calderone

CLTI System - Autonomous

LTI = linear time invariant

LTI Scalar ODE $\lambda \in \mathbb{C}$ $x \in \mathbb{R}$

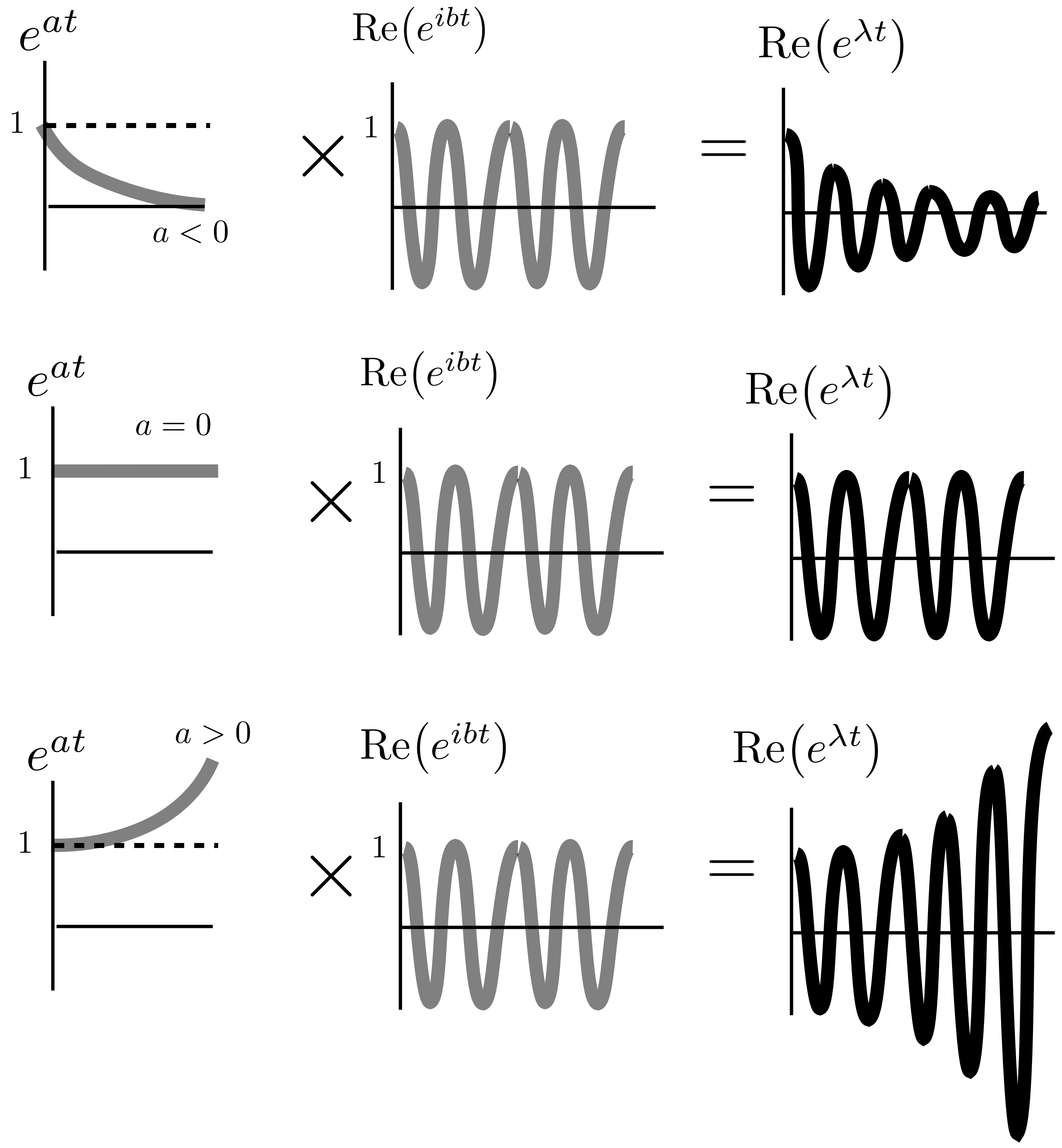
$$\dot{x} = \lambda x \qquad x(t_0) = x_0$$

Solution: $x(t) = e^{\lambda(t-t_0)}x_0$

assume (WLOG) $t_0 = 0$

...for $\lambda = a + bi$

$$e^{\lambda t} = e^{(a+bi)t} = e^{at}e^{ibt}$$

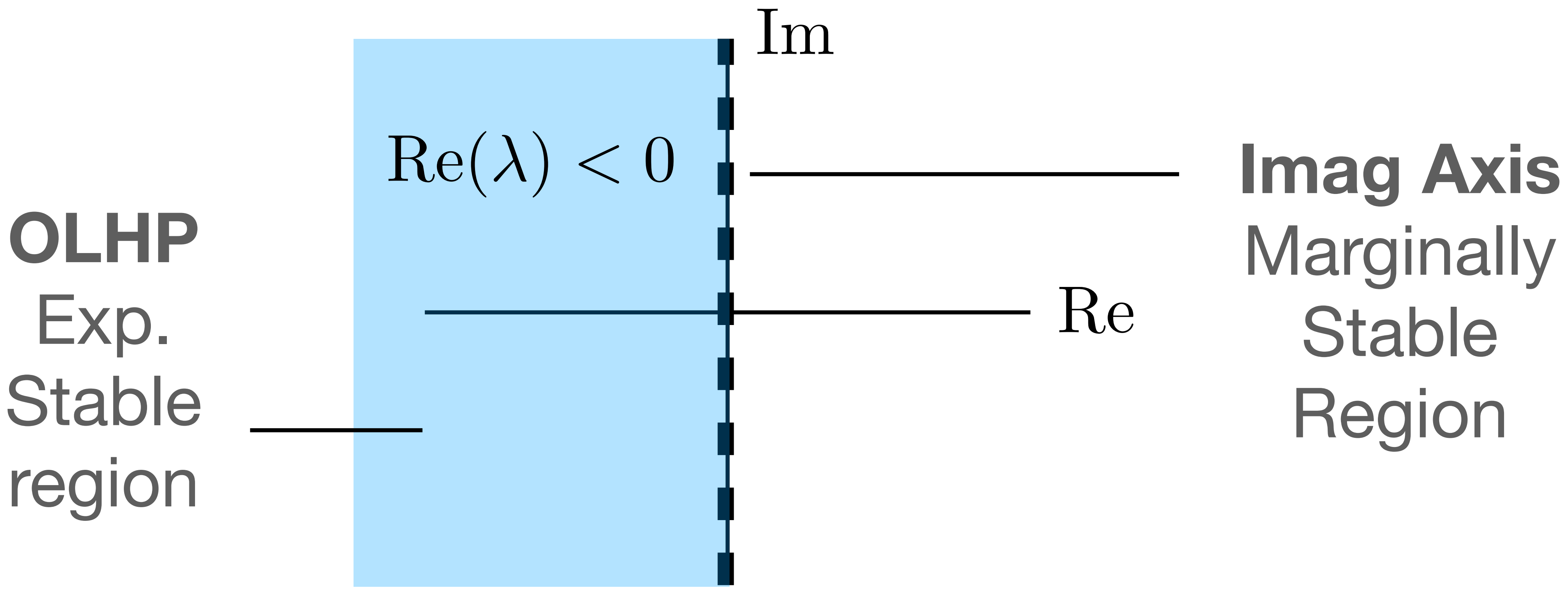


Stability:

Exponentially Stable $\mathrm{Re}(\lambda) = a < 0$

Marginally Stable $\mathrm{Re}(\lambda) = a = 0$

Unstable $\mathrm{Re}(\lambda) = a > 0$



DLTI System - Autonomous

LTI = linear time invariant

LTI Scalar ODE $\lambda \in \mathbb{C}$ $x \in \mathbb{R}$

$$\dot{x} = \lambda x \qquad x(t_0) = x_0$$

…evolution over a time step Δt

$$x(t+\Delta t) = e^{\lambda(t+\Delta t - t)}x(t) = e^{\lambda \Delta t}x(t)$$

…discrete time index $x[k] = x(k\Delta t + t_0) = x(t)$

LTI Discrete Update Eqn $\lambda_\Delta = e^{\lambda \Delta t} = e^{a\Delta t}e^{ib\Delta t}$

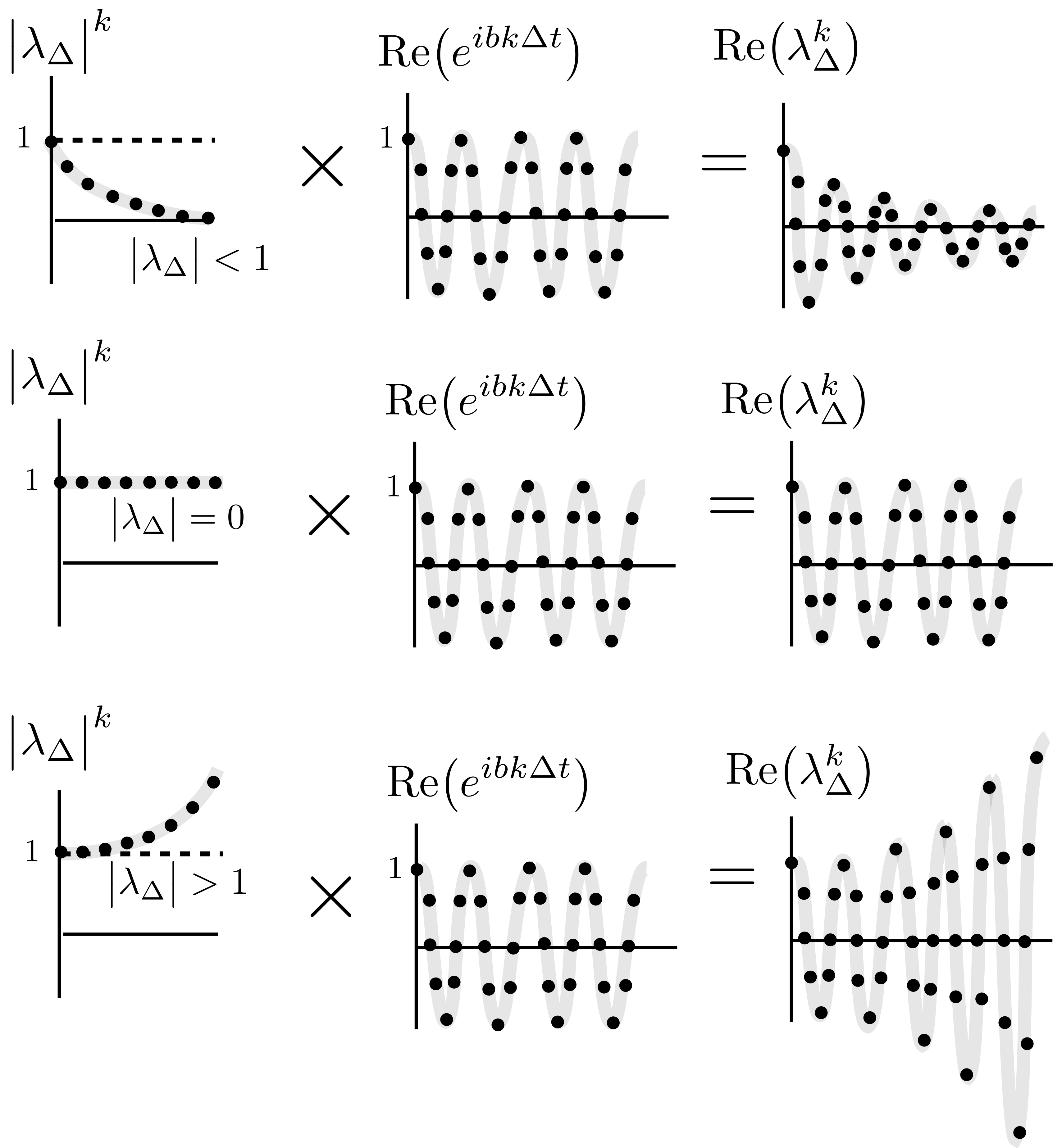
$$x[k+1] = \lambda_\Delta x[k] \qquad x[0] = x_0$$

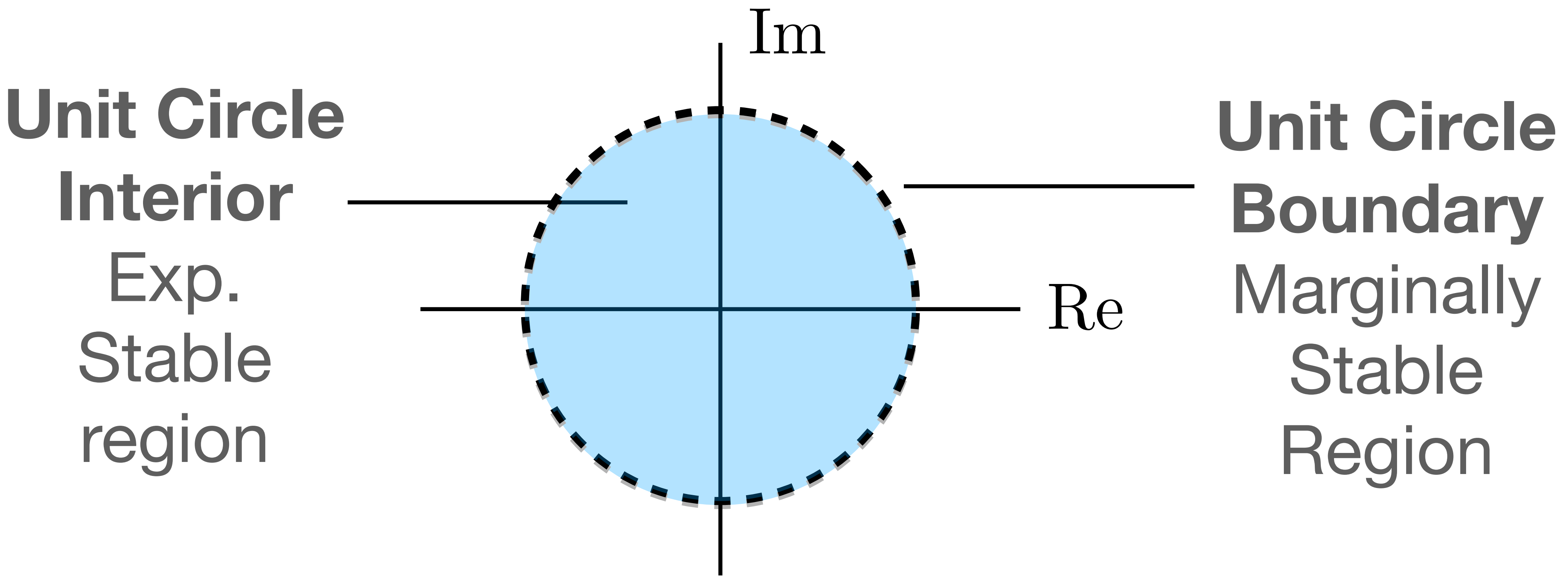
Stability:

Exponentially Stable $|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a\Delta t}| < 1$

Marginally Stable $|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a\Delta t}| = 1$

Unstable $|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a\Delta t}| > 1$





LTI System - CT vs. DT Stability

LTI = linear time invariant

LTI Scalar ODE $\lambda \in \mathbb{C}$ $x \in \mathbb{R}$

$$\dot{x} = \lambda x \qquad x(t_0) = x_0$$

Stability:

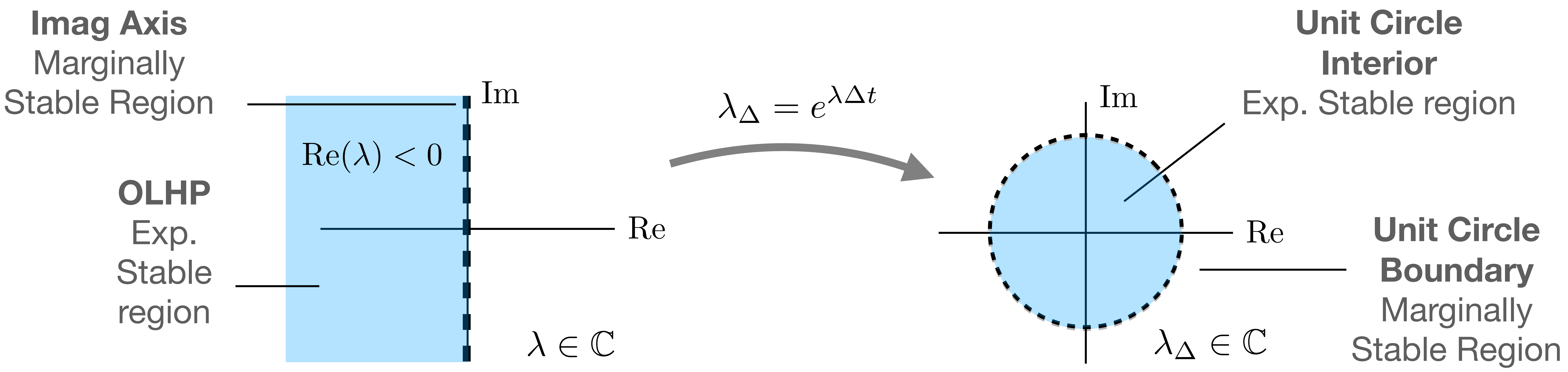
Exponentially Stable	$\mathrm{Re}(\lambda) = a < 0$
Marginally Stable	$\mathrm{Re}(\lambda) = a = 0$
Unstable	$\mathrm{Re}(\lambda) = a > 0$

LTI Discrete Update Eqn $\lambda_\Delta = e^{\lambda \Delta t} = e^{a\Delta t} e^{ib\Delta t}$

$$x[k+1] = \lambda_\Delta x[k] \qquad x[0] = x_0$$

Stability:

Exponentially Stable	$\lvert\lambda_\Delta\rvert = \lvert e^{\lambda\Delta t}\rvert = \lvert e^{a\Delta t}\rvert < 1$
Marginally Stable	$\lvert\lambda_\Delta\rvert = \lvert e^{\lambda\Delta t}\rvert = \lvert e^{a\Delta t}\rvert = 1$
Unstable	$\lvert\lambda_\Delta\rvert = \lvert e^{\lambda\Delta t}\rvert = \lvert e^{a\Delta t}\rvert > 1$



CLTI System - Autonomous

LTI = linear time invariant

LTI Vector ODE $A = \mathbb{R}^{n \times n} \quad x \in \mathbb{R}^n$

$$\dot{x} = Ax \qquad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0$$

matrix exponential

… eigenvalues control decay/expansion and oscillation for eigensubspaces

For diagonalizable A: $A = VDV^{-1}$

$$x(t) = e^{At}x_0 = Ve^{Dt}V^{-1}x_0$$

$$\Rightarrow \qquad V^{-1}x(t) = e^{Dt}V^{-1}x_0 \qquad x = Vx'$$

$$\Rightarrow \qquad x'(t) = e^{Dt}x'_0$$

…in eigenvector coordinates

$$\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x'_1(0) \\ \vdots \\ x'_n(0) \end{bmatrix} \Rightarrow \begin{array}{c} x'_1(t) = e^{\lambda_1 t}x'_1(0) \\ \vdots \\ x'_n(t) = e^{\lambda_n t}x'_n(0) \end{array}$$

Example: $A \in \mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

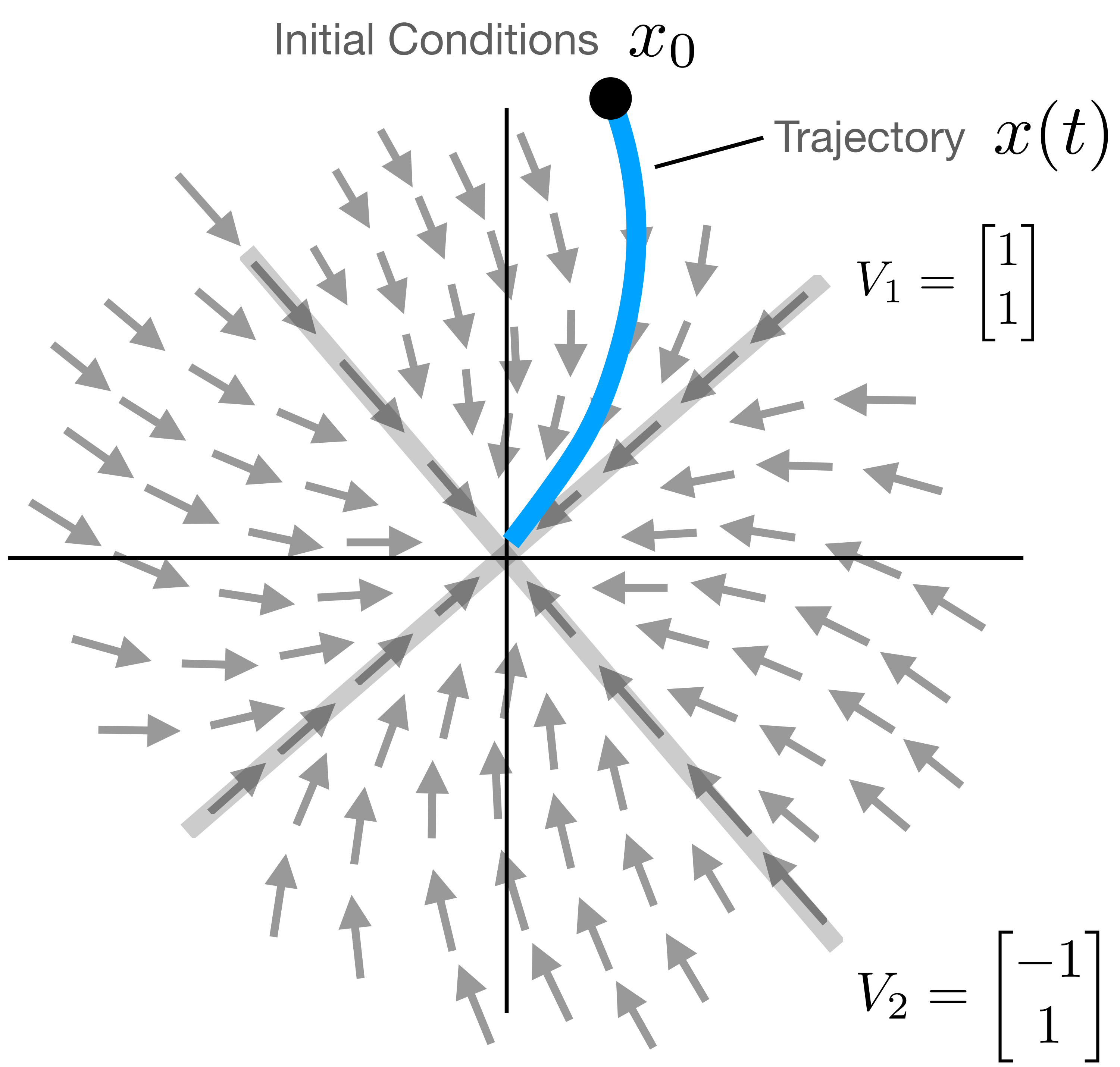
Eigenvalues

$$\lambda_1 = -2, \quad \lambda_2 = -4$$

Eigenvectors

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Quiver Plot & Trajectory:



DLTI System - Autonomous

LTI = linear time invariant

LTI Vector ODE $A = \mathbb{R}^{n\times n}$ $\quad x \in \mathbb{R}^n$

$$\dot{x} = Ax \qquad x(t_0) = x_0$$

…evolution over a time step Δt

$$x(t+\Delta t) = e^{A(t+\Delta t - t)}x(t) \;= e^{A\Delta t}x(t)$$

…discrete time index $\quad x[k] = x(k\Delta t + t_0) = x(t)$

LTI Discrete Update Eqn $\quad A_\Delta = e^{A\Delta t}$

$$x[k+1] = A_\Delta x[k] \qquad x[0] = x_0$$

Solution: $\quad x[k] = A_\Delta^k x_0$

if diagonalizable $\quad x[k] = A_\Delta^k x_0 \;= V D_\Delta^k V^{-1} x_0 \qquad D_\Delta = e^{D\Delta t}$

$$\Rightarrow \qquad V^{-1}x[k] = D_\Delta^k V^{-1} x_0 \qquad x = Vx'$$

$$\Rightarrow \qquad x'[k] = D_\Delta^k x'_0$$

…in eigenvector coordinates

$$\begin{bmatrix} x_1[k] \\ \vdots \\ x_n[k] \end{bmatrix} = \begin{bmatrix} {\lambda_1}_\Delta^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & {\lambda_2}_\Delta^k \end{bmatrix} \begin{bmatrix} x_1[0] \\ \vdots \\ x_n[0] \end{bmatrix} \Rightarrow \begin{matrix} x'_1[k] = {\lambda_1}_\Delta^k x'_1[0] \\ \vdots \\ x'_n[k] = {\lambda_n}_\Delta^k x'_n[0] \end{matrix}$$

$$\lambda_{1\Delta} = e^{\lambda_1 \Delta t} \quad \ldots \quad \lambda_{n\Delta} = e^{\lambda_n \Delta t}$$

Example: $\quad A \in \mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \qquad \begin{matrix} \lambda_1 = -2, \\ \lambda_2 = -4 \end{matrix}$$

Time step $\quad \Delta t = 0.01$

$$A_\Delta = \begin{bmatrix} 0.9705 & 0.0097 \\ 0.0097 & 0.9705 \end{bmatrix}$$

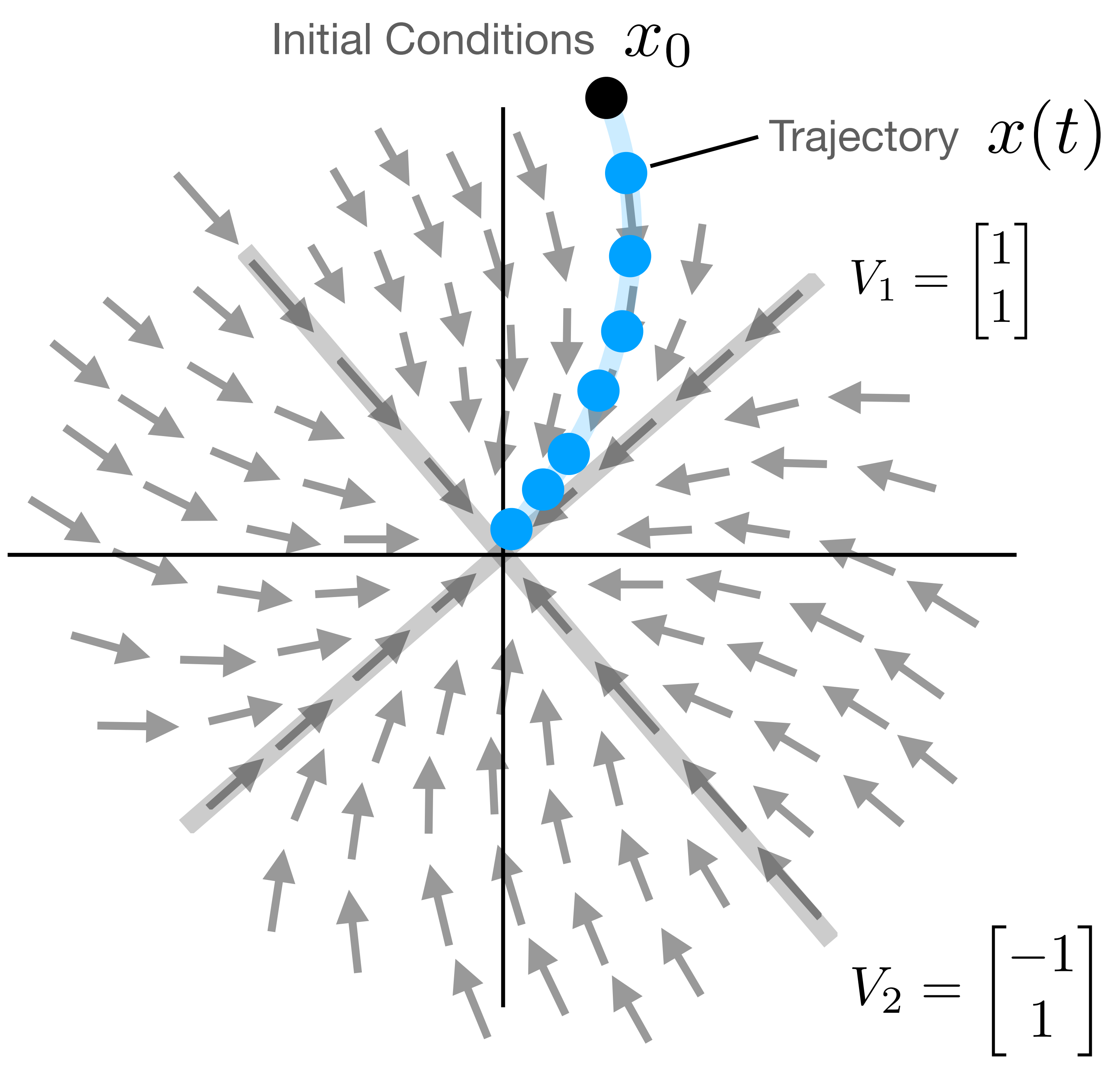
Eigenvalues

$$\lambda_{1\Delta} = 0.9802$$
$$\lambda_{2\Delta} = 0.9608$$

Eigenvectors

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Quiver Plot & Trajectory:



CLTI System - with controls

LTI = linear time invariant

LTI vector ODE

$$A = \mathbb{R}^{n\times n} \quad x \in \mathbb{R}^n$$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

Solution:

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{autonomous “drift” term}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\; d\tau}_{\text{affect of all control inputs}}$$

Checking solution:

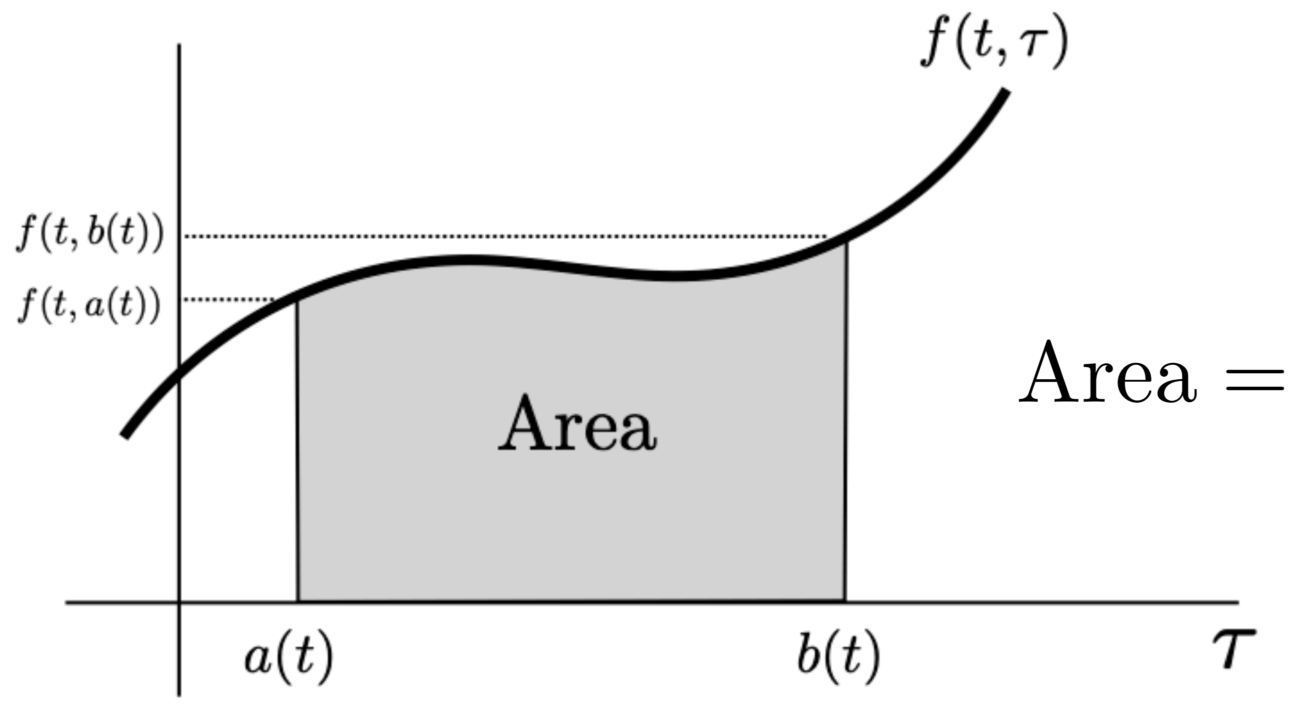
1. Derivative

$$\frac{dx}{dt} = Ae^{A(t-t_0)}x_0 + A\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\; d\tau + e^{A(t-t)}Bu(t)$$

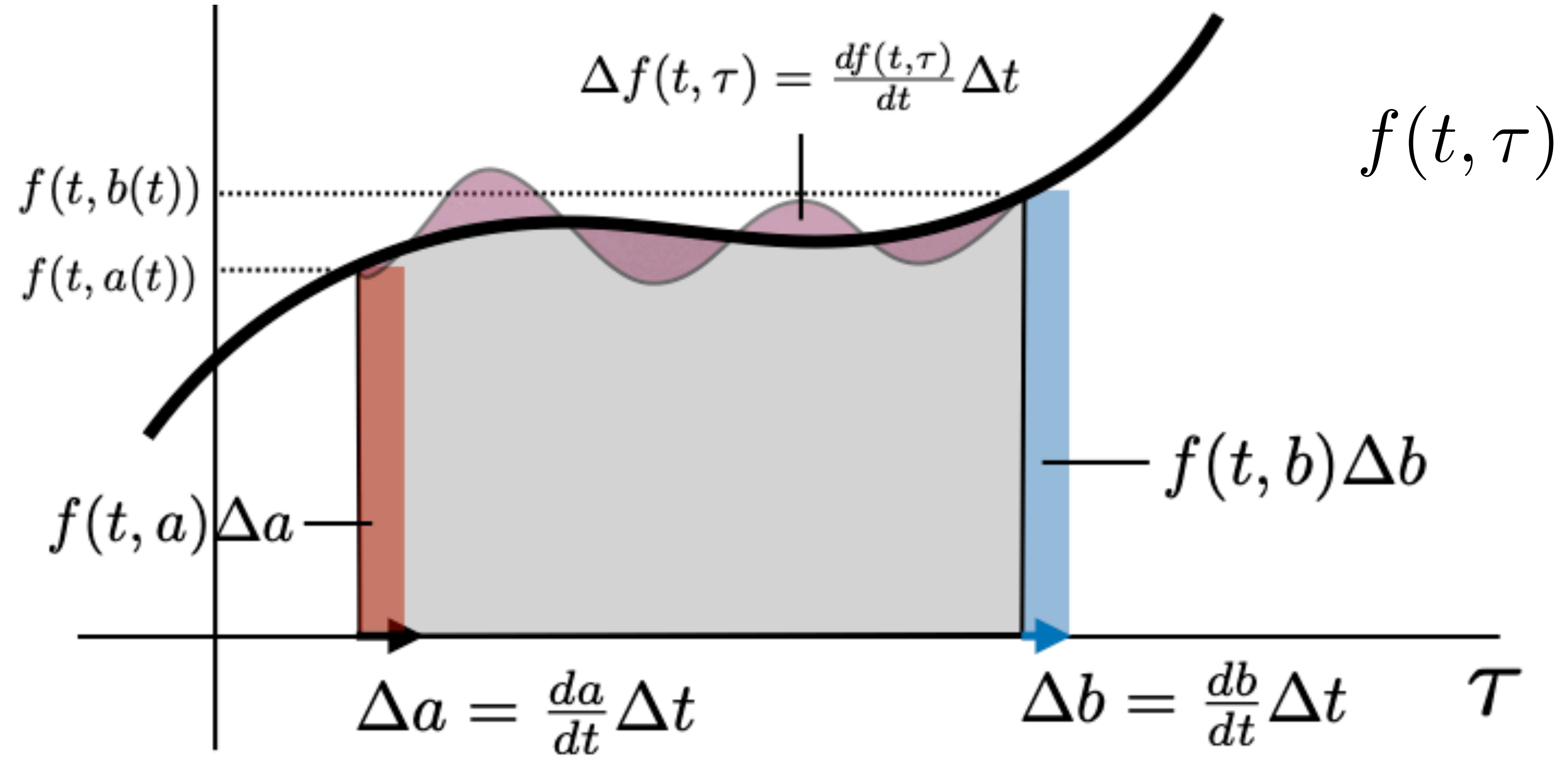
2. Initial Cond.

$$x(t_0) = e^{A(t_0-t_0)}x_0 + \int_{t_0}^{t_0} e^{A(t_0-\tau)}Bu(\tau)\; d\tau$$

Leibniz Rule



$$\text{Area} = \int_{a(t)}^{b(t)} f(t,\tau)\; d\tau$$



$$\frac{d\text{Area}}{dt} = f(t,b)\frac{db}{dt} - f(t,a)\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{df(t,\tau)}{dt}\; d\tau$$

CLTI System - with controls

LTI = linear time invariant

LTI vector ODE $A = \mathbb{R}^{n\times n} \quad x \in \mathbb{R}^n$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\ d\tau$$

autonomous "drift" term

affect of all control inputs

Checking solution:

1. Derivative

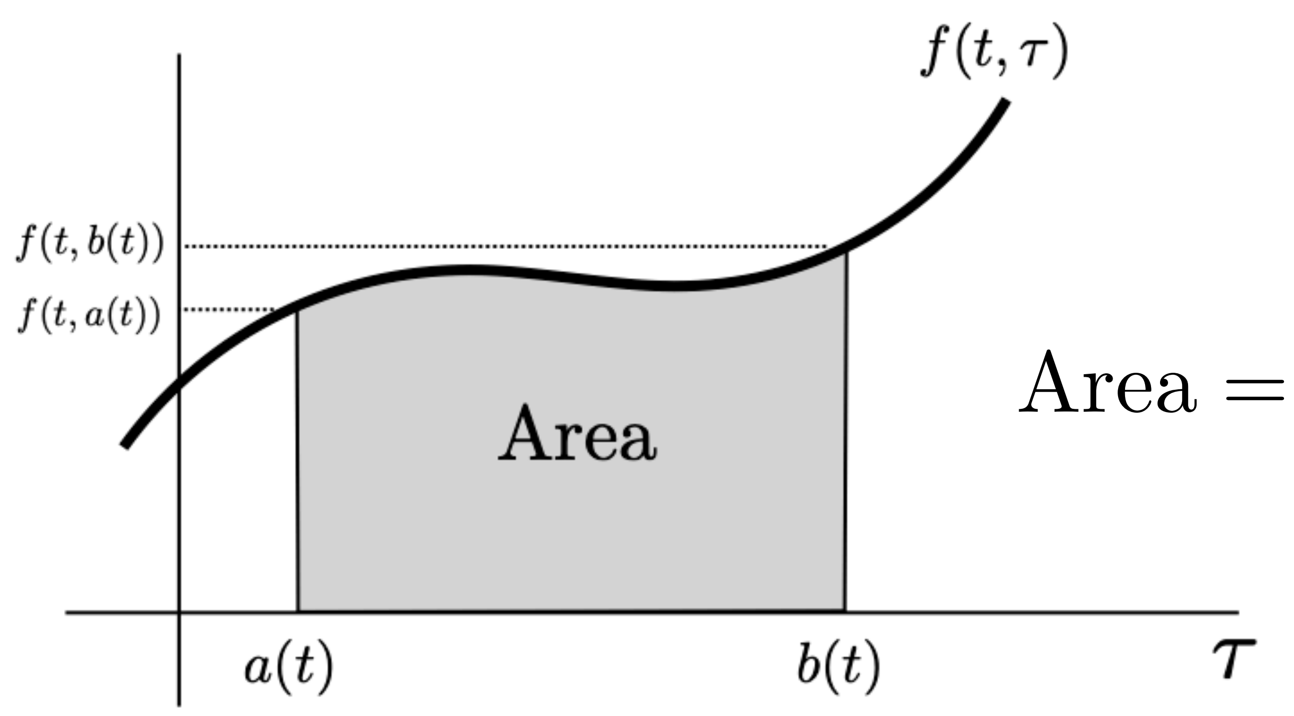
$$\frac{dx}{dt} = Ae^{A(t-t_0)}x_0 + A\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\ d\tau + \overset{I}{\cancel{e^{A(t-t)}}}Bu(t)$$

$$= Ax(t) + Bu(t)$$

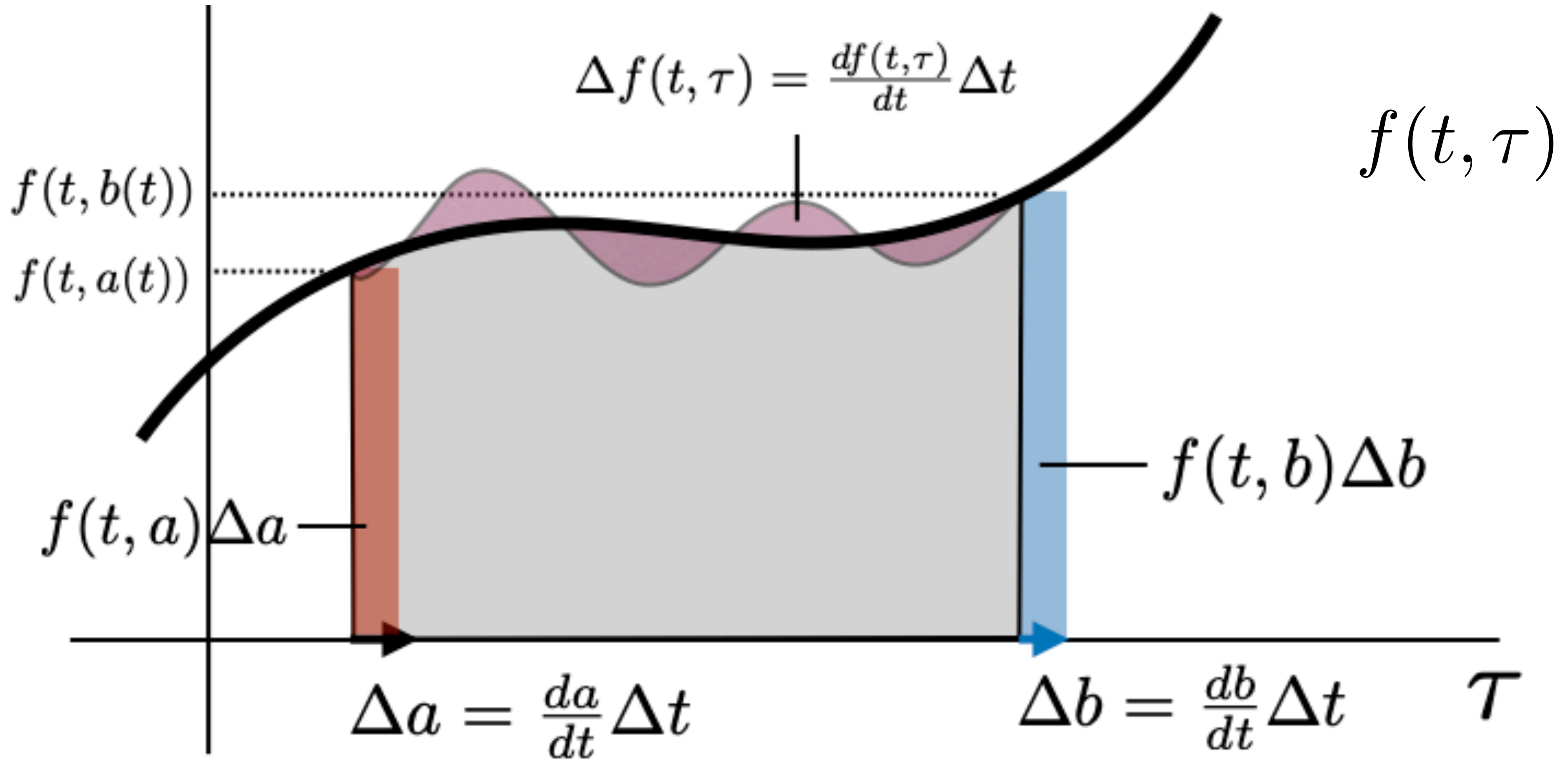
2. Initial Cond.

$$x(t_0) = \overset{I}{\cancel{e^{A(t_0-t_0)}}}x_0 + \int_{t_0}^{t_0} \overset{0}{\cancel{e^{A(t_0-\tau)}}}Bu(\tau)\ d\tau = x_0$$

Leibniz Rule



$$\text{Area} = \int_{a(t)}^{b(t)} f(t,\tau)\ d\tau$$



$$\frac{d\text{Area}}{dt} = f(t,b)\frac{db}{dt} - f(t,a)\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{df(t,\tau)}{dt}\ d\tau$$

CLTI System - with controls

LTI = linear time invariant

LTI vector ODE $A = \mathbb{R}^{n\times n} \quad x \in \mathbb{R}^n$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau$$

autonomous “drift” term

affect of all control inputs

…effect of control input at time $\tau \in [t_0, t]$ on state at time t $x(t)$

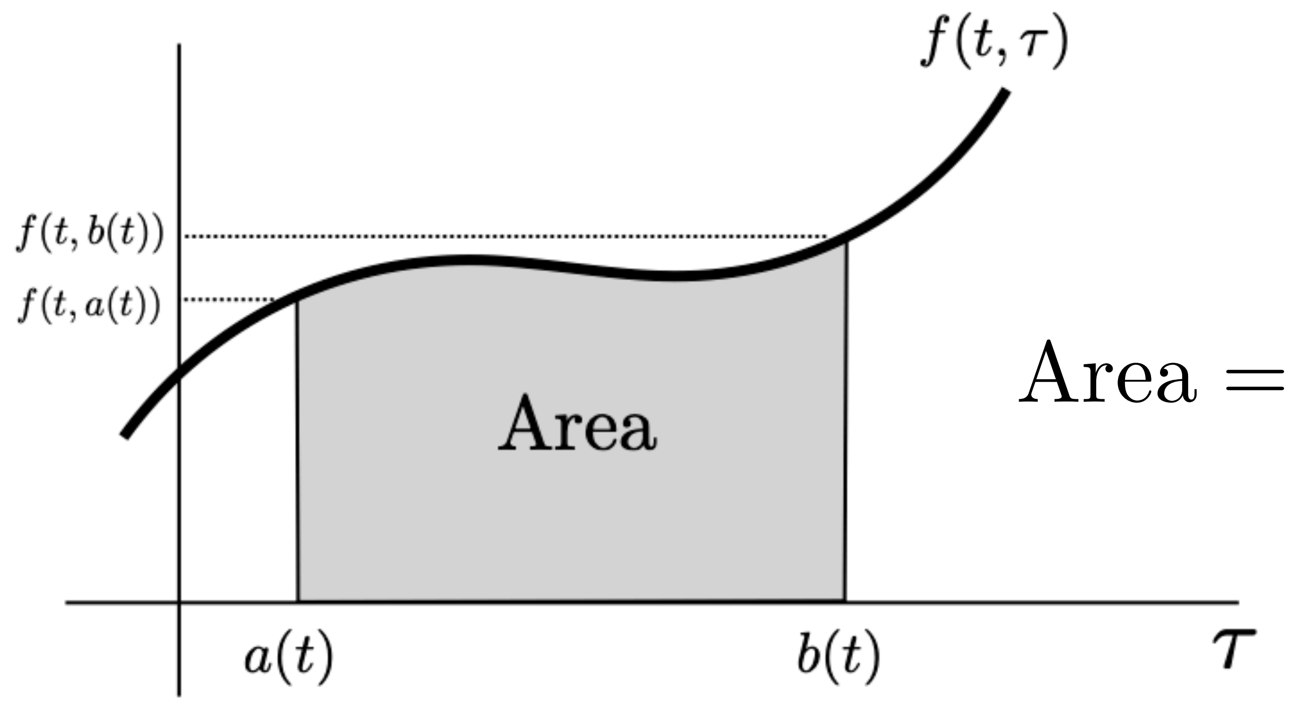
Effect of all inputs added up — $\int_{t_0}^{t}$ $e^{A(t-\tau)}$ $Bu(\tau)$

propagated forward from time τ to t.

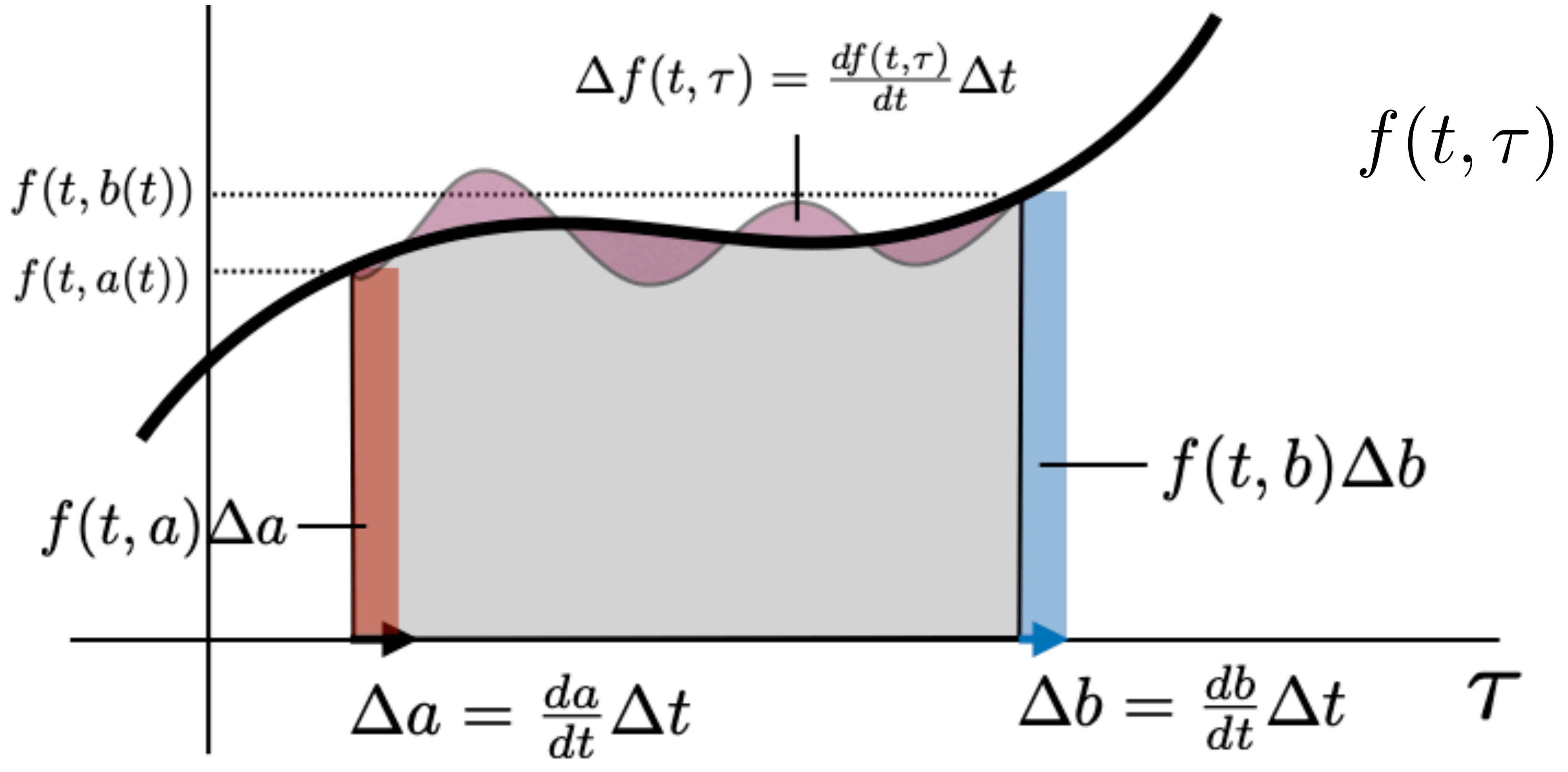
input matrix

input at τ

Leibniz Rule



$$\text{Area} = \int_{a(t)}^{b(t)} f(t,\tau)\, d\tau$$



$$\frac{d\text{Area}}{dt} = f(t,b)\frac{db}{dt} - f(t,a)\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{df(t,\tau)}{dt}\, d\tau$$

DLTI System - with controls

LTI = linear time invariant

LTI Discrete Update Eqn $A_\triangle \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^n$

$$x[k+1] = A_\triangle x[k] + B_\triangle u[k] \qquad x[0] = x_0$$

Discrete Time Matrices

$$A_\triangle = e^{A\Delta t} \qquad B_\triangle = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \, d\tau$$

assuming $u[k]$ constant over Δt

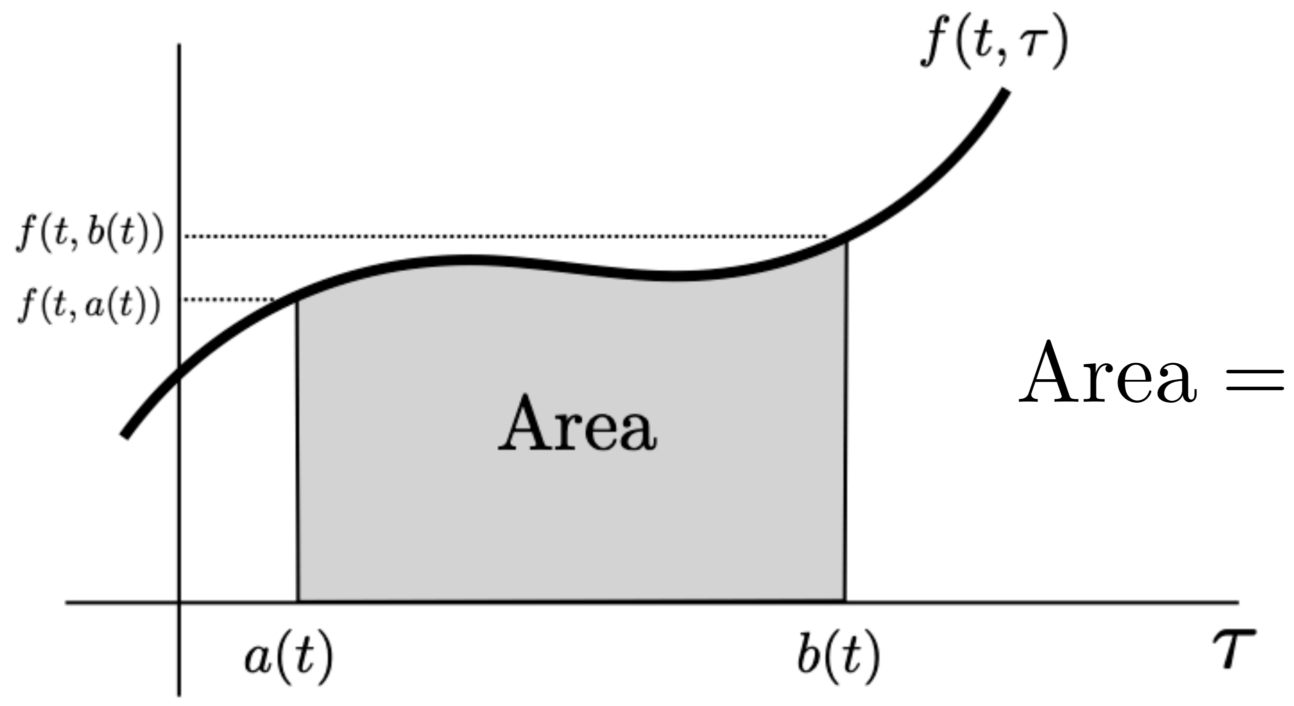
Approximations for small Δt

$$A_\triangle \approx I + A\Delta t \qquad B_\triangle \approx B\Delta t$$

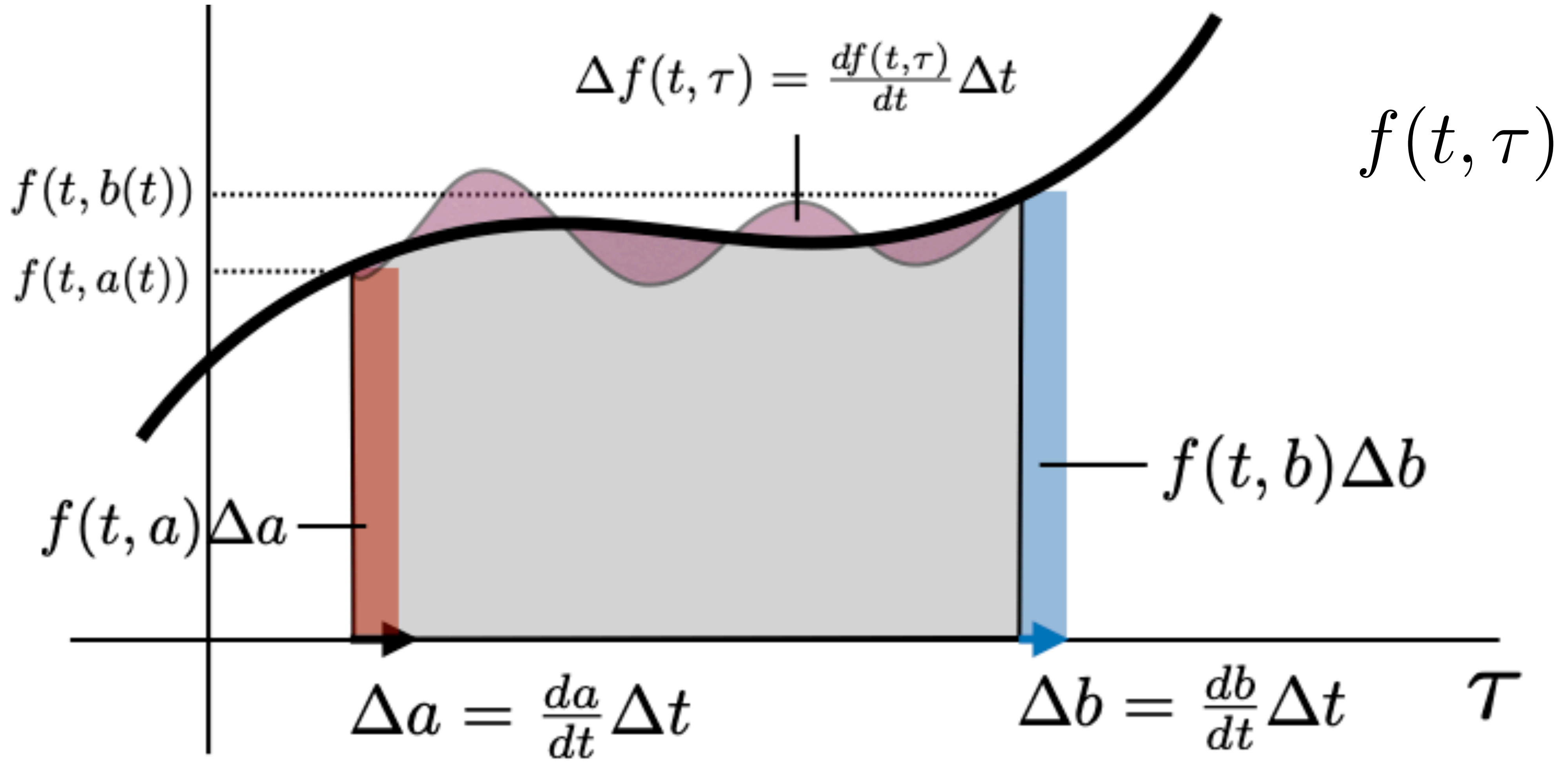
Computation

$$x[0] = x_0$$
$$x[1] = A_\triangle x_0 + B_\triangle u[0]$$
$$x[2] = A_\triangle x[1] + B_\triangle u[1] = A_\triangle^2 x_0 + A_\triangle B_\triangle u[0] + B_\triangle u[1]$$
$$x[3] = A_\triangle x[2] + B_\triangle u[2] = A_\triangle^3 x_0 + A_\triangle^2 B_\triangle u[0] + A_\triangle B_\triangle u[1] + B_\triangle u[2]$$
$$\vdots$$

Leibniz Rule



$$\text{Area} = \int_{a(t)}^{b(t)} f(t,\tau)\, d\tau$$



$$\frac{d\text{Area}}{dt} = \boxed{f(t,b)\frac{db}{dt}} - \boxed{f(t,a)\frac{da}{dt}} + \boxed{\int_{a(t)}^{b(t)} \frac{df(t,\tau)}{dt}\, d\tau}$$

DLTI System - with controls

LTI = linear time invariant

LTI Discrete Update Eqn $A_\Delta \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$

Discrete Time Matrices

$$A_\Delta = e^{A\Delta t} \qquad B_\Delta = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \; d\tau$$

assuming $u[k]$ constant over Δt

Solutions

$$x[k] = A_\Delta^k x_0 + \sum_{k'=0}^{k-1} A_\Delta^{k-1-k'} B_\Delta u[k']$$

$$= A_\Delta^k x_0 + \underbrace{\begin{bmatrix} A_\Delta^{k-1}B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}}_{U}$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of G

Reaching a particular state: x_{des}

…solve $x_{\text{des}} - A_\Delta^k x_0 = GU$ for U

Minimum norm solution:

$$U^* = G^T (GG^T)^{-1} \big(x_{\text{des}} - A_\Delta^k x_0\big)$$

$$= G^T W^{-1} \big(x_{\text{des}} - A_\Delta^k x_0\big)$$

DT Controllability Grammian: $W = GG^T$

$$W = \sum_{k'=0}^{k-1} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'\,T}$$

$$= \begin{bmatrix} A_\Delta^{k-1}B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} B_\Delta^T A_\Delta^{k-1\,T} \\ \vdots \\ B_\Delta^T A_\Delta^T \\ B_\Delta^T \end{bmatrix}$$

if G is fat, then W is invertible
if and only if G has full row rank

DLTI System - Reachability

LTI = linear time invariant

LTI Discrete Update Eqn $A_\Delta \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^n$

$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$

Discrete Time Matrices

$$A_\Delta = e^{A\Delta t} \qquad B_\Delta = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \; d\tau$$

assuming $u[k]$ constant over Δt

Solutions

$$x[k] = A_\Delta^k x_0 + \sum_{k'=0}^{k-1} A_\Delta^{k-1-k'} B_\Delta u[k']$$

$$= A_\Delta^k x_0 + \underbrace{\begin{bmatrix} A_\Delta^{k-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}}_{U}$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of G

Reaching a particular state: x_{des}

…solve $x_{\text{des}} - A_\Delta^k x_0 = GU$ for U

Minimum norm solution:

$$U^* = G^T (GG^T)^{-1} \big(x_{\text{des}} - A_\Delta^k x_0\big)$$

$$= G^T W^{-1} \big(x_{\text{des}} - A_\Delta^k x_0\big)$$

by Cayley-Hamilton

$$\mathcal{R}(G) = \mathcal{R}\Big(\begin{bmatrix} A_\Delta^{n-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}\Big)$$

…since $A_\Delta^{k'} = \beta_{n-1} A_\Delta^{n-1} + \cdots + \beta_1 A_\Delta^1 + \beta_0 I$

for $k' > n-1$

CLTI System - Reachability

LTI = linear time invariant

LTI vector ODE $A = \mathbb{R}^{n\times n} \quad x \in \mathbb{R}^n$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\ d\tau$$

$$= e^{A(t-t_0)}x_0 + \tilde{G}(u[t_0,t])$$

…recall

Operator

- infinite-dimensional input $u[t_0,t]$
- n dimensional output

$$\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)}B(\cdot)\ d\tau$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of $\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)}B(\cdot)\ d\tau$

Reaching a particular state: x_{des} at time t

…solve $x_{\text{des}} - e^{A(t-t_0)}x_0 = \tilde{G}(u)$ for u

Minimum norm solution:

…works with infinite-dimensional operators too!

DT $U^* = G^T W^{-1}\big(x_{\text{des}} - A_\Delta^k x_0\big) \qquad W = \sum_{k'=0}^{k-1} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'T}$

CT **Controllability Grammian:**

$$\tilde{W} = \int_{t_0}^{t} e^{A(t-\tau)}BB^Te^{A^T(t-\tau)}\ d\tau \in \mathbb{R}^{n\times n}$$

Solution:

$$u^*(\tau) = B^Te^{A^T(t-\tau)}\tilde{W}^{-1}\big(x_{\text{des}} - e^{A(t-t_0)}x_0\big)$$